SOLUTIONS

Question 1

(a) The table of standard integrals gives
\[ \int \frac{1}{\sqrt{4-x^2}} \, dx = \sin^{-1}\left(\frac{x}{2}\right) + c \]

(b) The domain of \( f(x) = \cos^{-1}(x) \) is \(-1 \leq x \leq 1\).
Hence, the domain of \( f(x) = \cos^{-1}\left(\frac{x}{2}\right) \) is
\[ -1 \leq \frac{x}{2} \leq 1 \]
\[ \therefore -2 \leq x \leq 2 \]

(c) The table of standard integrals gives
\[ \ln(x + 6) = 2 \ln(x) \]
\[ \ln(x + 6) = \ln(x)^2 \]
Now, \( x + 6 \) must be positive as well as \( x > 0 \).
Then \( x + 6 = x^2 \)
\[ \therefore x^2 - x - 6 = 0 \]
\[ \therefore (x - 3)(x + 2) = 0 \]
\[ \therefore x = 3 \text{ or } -2 \text{ but, because } x > 0, \]
\[ x = 3. \]

(d) \( \frac{3}{x + 2} < 4 \) with \( x \neq -2 \)

On the graph axes just shown, both \( y = \frac{3}{x + 2} \) and \( y = 4 \) are sketched.
The intersection at \( y = 4 \) is investigated:
\[ \therefore 4 = \frac{3}{x + 2} \]
\[ \therefore 4x + 8 = 3 \]
\[ \therefore x = -\frac{5}{4} \]
\[ \therefore x = -1 \frac{1}{4} \]
Using the graph, the hyperbola is below the line \( y = 4 \) when \( x < -2 \) and \( x > -1 \frac{1}{4} \) and is shown in bold.
Solution: \( x > -1 \frac{1}{4} \)

(e) Let \( u = 1 - x \)
\[ \therefore \frac{du}{dx} = -1 \text{ or } dx = -du \]
Also, when \( x = 0, u = 1, \) and when \( x = 1, u = 0. \)
Thus,
\[ \int_{0}^{1} x\sqrt{1-x} \, dx \]
\[ = \int_{0}^{1} (1-u)\sqrt{u}(-du) \]
\[ = \int_{0}^{1} (1-u)u^{\frac{1}{2}} du \]
\[ = \int_{0}^{1} u^{\frac{3}{2}} - u^{\frac{5}{2}} du \]
\[ = \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right]_{0}^{1} \]
\[ = \left[ \frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5} \right]_{0}^{1} \]
\[ = \left( \frac{2}{3} - \frac{2}{5} \right) - (0 - 0) \]
\[ = \frac{4}{15} \]
(f) This is a binomial probability question.
\[ P(4) = \frac{1}{6} \] on one die.
\[ P(\text{not } 4) = \frac{5}{6} \] on one die.
\[ P(\text{exactly } 2 \text{ fours on } 5 \text{ dice}) = \binom{5}{2} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3. \]

Question 2

(a) \[ f''(x) = \sin^2 x \]
\[ \therefore f(x) = \int \sin^2 x \, dx \]
To integrate this, we will need double angles:
Use: \[ \cos 2x = 1 - 2\sin^2 x \]
\[ \therefore \sin^2 x = \frac{1}{2} (1 - \cos 2x) \]
\[ \therefore f(x) = \frac{1}{2} \int (1 - \cos 2x) \, dx \]
\[ = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + c \]
Now, we know that \( f(0) = 2 \), so
\[ 2 = \frac{1}{2} \left( 0 - \frac{1}{2} \sin 0 \right) + c \]
\[ \therefore c = 2 \]
Thus,
\[ f(x) = \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + 2 \]
\[ = \frac{1}{2} x - \frac{1}{4} \sin 2x + 2 \]

(b) i. \( M = 36 - 35.5e^{-kt} \)
\[ \therefore \frac{dM}{dt} = 35.5ke^{-kt} \]
\[ = k(35.5e^{-kt}) \]
Now, \( M = 36 - 35.5e^{-kt} \)
\[ \therefore 35.5e^{-kt} = 36 - M. \]
\[ \therefore \frac{dM}{dt} = k(36 - M) \]

ii. When \( t = 10 \), \( M = 20 \) tonnes.
\[ M = 36 - 35.5e^{-kt} \]
\[ \therefore 20 = 36 - 35.5e^{-k(10)} \]
\[ \therefore \quad e^{-k(10)} = \frac{16}{35.5} \]
\[ \therefore -10k = \ln \left( \frac{16}{35.5} \right) \]
\[ = 0.0796943... \]
\[ = 0.080 \text{ correct to } 3 \text{ decimal places.} \]

iii. Since a limiting mass exists,
\[ as \ t \to \infty, \quad \frac{dM}{dt} \to 0. \]
\[ \therefore k(36 - M) \to 0 \]
\[ \therefore 36 - M \to 0 \]
\[ \therefore M \to 36 \]
According to this model, the limiting mass of the whale is 36 tonnes.

(c) i. \( P(x) = (x + 1)(x - 3)Q(x) + ax + b \)
We are told that \((x - 3)\) is a factor of \( P(x) \).
\[ \therefore P(3) = 0 \]
\[ \therefore P(3) = (3 + 1)(3 - 3)Q(x) + 3a + b \]
\[ = 3a + b \]
\[ \therefore 3a + b = 0 \quad ... 1. \]
When \( P(x) \) is divided by \((x + 1)\), the remainder is 8.
\[ \therefore P(-1) = 8 \]
\[ \therefore P(-1) = (-1 + 1)(-1 - 3)Q(x) + 3a + b \]
\[ = -a + b \]
\[ \therefore -a + b = -8 \quad ... 2. \]
Solving equations 1. and 2. simultaneously,
\[ 3a + b = 0 \quad 1. \]
\[ -a + b = -8 \quad 2. \]
1. subtract 2,
\[ 4a = -8 \]
\[ a = -2 \text{ and thus } b = 6. \]
Solution: \( a = -2, \quad b = 6 \)

ii. \( P(x) = (x + 1)(x - 3)Q(x) + ax + b \)
When \( P(x) \) is divided by \((x + 1)(x - 3)\), the remainder is \( ax + b \)
\[ = -2x + 6 \]
(d) We need to find \( \frac{dr}{dt} \).

The car has speed 100 km/h
\[
\frac{dx}{dt} = 100
\]
Using Pythagoras’ theorem,
\[
r^2 = x^2 + 6^2
\]
\[
\therefore r = \sqrt{x^2 + 6^2}
\]
(Knowing that \( r > 0 \))
\[
\therefore r = (x^2 + 6^2)^{\frac{1}{2}}
\]
\[
\therefore \frac{dr}{dx} = \frac{1}{2} (x^2 + 6^2)^{-\frac{1}{2}} \times (2x)
\]
\[
= \frac{x}{\sqrt{x^2 + 6^2}}
\]
Now,
\[
\frac{dr}{dt} = \frac{dr}{dx} \times \frac{dx}{dt}
\]
\[
= \frac{x}{\sqrt{x^2 + 6^2}} \times 100
\]
\[
= \frac{100x}{\sqrt{x^2 + 6^2}}
\]

**Question 3**

(a) i. The number of ways of arranging 5 doors is 5!. Since there are 2 red doors, the number of arrangements is
\[
\frac{5!}{2!} = 60 \text{ ways.}
\]

ii. Having 2 red doors together means there are 4 distinct items. For this, the number of arrangements is
\[
4! = 24 \text{ ways.}
\]

(b) i. \( f(x) = e^{-x^2} \)
\[
f'(x) = -2x \times e^{-x^2}
\]
\[
f''(x) = -2x \times (-2x)e^{-x^2} + (-2e^{-x^2})
\]
\[
= e^{-x^2} (4x^2 - 2).
\]
The second derivative will be zero at a point of inflection \( f''(x) = 0 \).
\[
\therefore e^{-x^2} (4x^2 - 2) = 0
\]
\[
\therefore 4x^2 - 2 = 0 \text{ (since } e^{-x^2} \neq 0\)
\]
\[
\therefore x = \pm \frac{1}{\sqrt{2}}.
\]
Since we are told of two points of inflection, they will be at \( x = -\frac{1}{\sqrt{2}} \) and
\[
x = \frac{1}{\sqrt{2}}.
\]

ii. For the graph of \( y = f(x) \), a horizontal line can be drawn that will interest the graph in more than one place. That would mean that the inverse will be a non-function. Restricting the domain of \( f(x) \) (for example, to \( x \geq 0 \)) would solve this problem.

iii. Let \( y = e^{-x^2} \). The inverse will be
\[
x = e^{-y^2}
\]
\[
\therefore e^{-y^2} = x
\]
\[
\therefore \ln (e^{-y^2}) = \ln (x)
\]
\[
\therefore -y^2 = \ln(x)
\]
\[
\therefore y^2 = \ln(\frac{1}{x})
\]
\[
\therefore y = \pm \sqrt{\ln(\frac{1}{x})}
\]
Now, for \( f(x) \), \( x \geq 0 \).
So, for \( f^{-1}(x) \), \( y \geq 0 \).
\[
\therefore y = \sqrt{\ln(\frac{1}{x})}
\]
\[
\therefore f^{-1}(x) = \sqrt{\ln(\frac{1}{x})}.
\]
iv. The domain of \( f^{-1}(x) \) is the range of the function \( f(x) \). Here is the restricted graph of \( f(x) \):

![Graph of f(x)](image)

The range of \( f(x) \) is \((0, 1]\) or \(0 < f(x) \leq 1\). The domain of \( f^{-1}(x) \) is \(0 < x \leq 1\).

v. The inverse graph is most easily found by considering the graph just shown.

![Graph of f^{-1}(x)](image)

The graph of \( y = f^{-1}(x) \) is a reflection of \( y = f(x) \) in the line \( y = x \). Showing both graphs on the same axes is useful here:

![Both graphs on same axes](image)

Note that the x-axis is an asymptote for the graph of \( y = f^{-1}(x) \).

vi. 1. \( x = e^{-x^2} \)

\[
\therefore \quad x - e^{-x^2} = 0
\]

Let \( g(x) = x - e^{-x^2} \).

When \( x = 0.6 \), \( g(x) = 0.6 - e^{-0.6^2} \)

\[= -0.09767...\]

Since \( g(0.6) < 0 \) and \( g(0.7) > 0 \), then a solution for \( g(x) = 0 \) lies between \( x = 0.6 \) and \( x = 0.7 \).

2. \( g\left(\frac{0.6 + 0.7}{2}\right) = g(0.65) \)

\[= 0.65 - e^{-0.65^2}\]

\[= -0.005406...\]

Since \( g(0.65) < 0 \), the solution must lie between \( x = 0.65 \) and \( x = 0.7 \). Thus, \( x = 0.7 \), correct to one decimal place.

![Graph of y = x](image)

Question 4

(a) i. \( v^2 = 24 - 8x - 2x^2 \)

The particle is at rest when \( v = 0 \).

\[
\therefore \quad 0 = 24 - 8x - 2x^2
\]

\[
\therefore \quad 2x^2 + 8x - 24 = 0
\]

\[
\therefore \quad x^2 + 4x - 12 = 0
\]

\[
\therefore \quad (x + 6)(x - 2) = 0
\]

\[
\therefore \quad x = -6, \; \text{or} \; 2.
\]

So the particle is at rest at \( x = -6, \; \text{or} \; 2 \) seconds.

ii. Acceleration is \( a = \frac{d}{dx}\left(\frac{1}{2} v^2\right)\)

\[
= \frac{d}{dx}\left(\frac{1}{2} (24 - 8x - 2x^2)\right)
\]

\[
= \frac{d}{dx} (12 - 4x - x^2)
\]

So \( a = -4 - 2x \).

iii. The maximum speed will occur when \( a = 0 \).

\[
\therefore \quad -4 - 2x = 0
\]

\[
\therefore \quad -2x = 4
\]

\[
\therefore \quad x = -2
\]
At \( x = -2, v^2 = 24 - 8(-2) - 2(-2)^2 \)
\[
= 24 + 16 - 8
\]
\[
= 32
\]
\[
\therefore v = \pm \sqrt{32}
\]
\[
\therefore v = \pm 4\sqrt{2}
\]
Speed is \(|\text{velocity}|\).
The maximum speed is \(4\sqrt{2}\) speed units.

(b) i. \(2\cos \theta + 2\cos(\theta + \frac{\pi}{3})\)
\[
= 2\cos \theta + 2(\cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3})
\]
\[
= 2\cos \theta + 2\cos \theta \times \frac{1}{2} - \sin \theta \times \frac{\sqrt{3}}{2}
\]
\[
= 2\cos \theta + \cos \theta - \sqrt{3} \sin \theta
\]
\[
= 3\cos \theta - \sqrt{3} \sin \theta.
\]
Now, \(R \cos(\theta + \alpha) = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha\).
For all \(\theta\),
\[
3\cos \theta - \sqrt{3} \sin \theta - R \cos \theta \cos \alpha
\]
\[
\therefore R \cos \alpha = 3
\]

and \(R \sin \alpha = \sqrt{3}\)
\[
\therefore (R \cos \alpha)^2 + (R \sin \alpha)^2 = 3^2 + \sqrt{3}^2
\]
\[
\therefore R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 12
\]
\[
\therefore R^2 (\cos^2 \alpha + \sin^2 \alpha) = 12
\]
\[
\therefore R^2 = 12
\]
\[
\therefore R = \pm \sqrt{12}
\]
But, since \(R > 0\),
\[
\therefore R = \sqrt{12}
\]
\[
\therefore R = 2\sqrt{3}.
\]
That means
\[
2\sqrt{3} \cos \alpha = 3
\]
\[
\therefore \cos \alpha = \frac{3}{2\sqrt{3}}
\]
\[
\therefore \cos \alpha = \frac{\sqrt{3}}{2}
\]
And
\[
2\sqrt{3} \sin \alpha = \sqrt{3}
\]
\[
\therefore \sin \alpha = \frac{\sqrt{3}}{2\sqrt{3}}
\]
\[
\therefore \sin \alpha = \frac{1}{2}.
\]
The signs on these answers means that \(\alpha\) is on the first quadrant.

\[
\therefore \alpha = \frac{\pi}{6}.
\]
So, \(2\cos \theta + 2\cos(\theta + \frac{\pi}{3}) = 2\sqrt{3}\cos(\theta + \frac{\pi}{6})\).

ii. \(2\cos \theta + 2\cos(\theta + \frac{\pi}{3}) = 3\)
\[
\therefore 2\sqrt{3}\cos(\theta + \frac{\pi}{6}) = 3
\]
\[
\therefore \cos(\theta + \frac{\pi}{6}) = \frac{3}{2\sqrt{3}}
\]
\[
\therefore \cos(\theta + \frac{\pi}{6}) = \frac{3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}
\]
\[
\therefore \cos(\theta + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}
\]
So,
\[
\theta + \frac{\pi}{6} = \frac{11\pi}{6} \quad \text{or} \quad 2\pi - \frac{\pi}{6} = \frac{3\pi}{2}.
\]
\[
\therefore \theta = 0, \frac{5\pi}{3}.
\]
But the domain is \(0 < \theta < 2\pi\).
\[
\therefore \theta = 0, \frac{5\pi}{3}.
\]

(c) A good starting point here is the fact that a quadrilateral whose diagonals bisect each other at right angles will be a rhombus.
From the diagram, the two diagonals are \(SM\) and \(PL\).
\(S = (0, a)\)
\(P = (2ap, ap^2)\)
\(M = (2ap, -a)\)
\(L\) lies on the \(y\)-axis \((x = 0)\) and is on the tangent line at \(P\) \((y = px - ap^2)\)
\[
\therefore x = 0
\]
\[
\therefore y = p(0) - ap^2
\]
\[
\therefore y = -ap^2
\]
\[
\therefore L = (0, ap^2).
\]
Midpoint of \(SM = \left(\frac{0 + 2ap}{2}, \frac{a + -a}{2}\right) = (ap, 0)\)
Midpoint of \( PL = \left( \frac{2ap + 0}{2}, \frac{ap^2 - ap^2}{2} \right) \)
\[ = (ap, 0) \]
Midpoint of \( SM = \) Midpoint of \( PL \).
Thus the two diagonals bisect each other.

The gradient of \( SM \)
\[ = \frac{a - a}{q - 2ap} = \frac{2a}{-2ap} = -\frac{1}{p} \]
The gradient of \( PL \)
\[ = \frac{ap^2 + ap^2}{2ap - 0} = \frac{2ap^2}{2ap} = p \]
The gradient of \( SM \) multiplied by the gradient if \( PL \) is
\[ = -\frac{1}{p} \times p = -1 \]
Thus, \( SLMP \) is a rhombus since its diagonals bisect each other at right angles.

Note: an alternative proof exists. You could use the fact that a quadrilateral with all 4 sides having equal length, will be a rhombus.

**Question 5.**

(a) i. From triangle \( BTP \),
\[ \tan 30^\circ = \frac{PT}{BP} \]
\[ \therefore \frac{1}{\sqrt{3}} = \frac{PT}{BP} \]
\[ \therefore BP = \sqrt{3} \times PT \quad ...1. \]
From triangle \( ATP \),
\[ \tan 3^\circ = \frac{PT}{AP} \]
\[ \therefore \frac{1}{\sqrt{3}} = \frac{PT}{BP} \]
\[ \therefore \]

\( \therefore BP = \sqrt{3} \times PT. \)
Using 1.
\[ BP = \sqrt{3} \times \tan 3^\circ \times AP \quad ...2. \]
From triangle \( APL \),
\[ \tan 20^\circ = \frac{1}{AP} \]
\[ \therefore AP = \frac{1}{\tan 20^\circ} \]
Using 2.
\[ BP = \sqrt{3} \times \tan 3^\circ \times \frac{1}{\tan 20^\circ} \]
\[ \therefore BP = \frac{\sqrt{3} \times \tan 3^\circ}{\tan 20^\circ} \]
As required.

ii. Consider the line \( AP \).
\[ AB = AP - BP \]
\[ = \frac{1}{\tan 20^\circ} - \sqrt{3} \tan 3^\circ \]
\[ = 1 - \sqrt{3} \tan 3^\circ \]
\[ = \frac{2.49808...}{2.5 \text{ km correct to 1 decimal place.}} \]

(b) i. \[ \frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1 + x^2} \]
\[ \therefore (\tan^{-1}(\frac{1}{x})) = \frac{1}{1 + (\frac{1}{x})^2} \times (-x^{-2}) \]
\[ = \frac{1}{x^2 + 1} \]
\[ = \frac{1}{x^2 + 1} \times \frac{1}{x^2} \]
\[ \therefore f'(x) = \frac{1}{1 + x^2} - \frac{1}{x^2 + 1} \]
\[ = 0 \quad \text{for} \ x \neq 0. \]
For \( x > 0 \), \( f(x) \) is a constant.
Choose \( x = 1 \),
\[ f(1) = \tan^{-1}(1) + \tan^{-1}(\frac{1}{1}) \]
\[ = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \]
\[ \therefore f(x) = \frac{\pi}{2} \text{ when} \ x > 0. \]
The function $f(x)$ sketched over the domain $x > 0$ looks like this:

Since $f(x)$ is an odd function, $f(-x) = -f(x)$.
This means the graph of $f(x)$ when $x > 0$ will be reflected across the $x$-axis and the result reflected across the $y$-axis. Given that $f(x)$ is an odd function, the sketch becomes:

(C) i. $\angle AXD$ is an exterior angle to $\Delta XBD$ and the exterior angle is equal to the sum of the opposite interior angles.
   \[ \angle AXD = \angle ABD + \angle XDB \]
   \[
   \therefore \angle AXD = \alpha + \beta 
   \]
   From part i, $\angle AXD = \angle ABD + \angle XDB$
   \[ \therefore \angle AXD = \alpha + \beta 
   \]
   And from part ii,
   \[ \angle AXD = \angle TAC + \angle CAD 
   \]
   \[ \therefore \angle TAC + \angle CAD = \alpha + \beta 
   \]
   \[ \therefore \angle XAD = \angle CAD = \beta 
   \]
   Thus, $AD$ bisects $\angle BAC$.

Question 6

(a) i. The LHS is
   \[ \cos A \cos B \left( 1 + \frac{\sin A \sin B}{\cos A \cos B} \right) \]
   \[ = \cos A \cos B \left( 1 + \tan A \tan B \right) \]
   \[ = \cos A \cos B + \cos A \frac{\sin A}{\sin B} + \cos B \frac{\sin B}{\cos B} \]
   \[ = \cos A \cos B + \sin A \sin B \]
   \[ = \cos (A - B) \]
   \[ = \text{RHS} \]

ii. Using the result from part i,
   \[ \text{If } \tan A \tan B = -1 \]
   \[ \text{then } \cos(A - B) = \cos A \cos B (1 + 1) \]
   \[ \therefore \cos(A - B) = 0 \]
   \[ \therefore A - B = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \]
   Since $B < A < \pi$
   \[ \text{and } 0 < B < \frac{\pi}{2} \]
   \[ \therefore \text{The solution is } A - B = \frac{\pi}{2} \]

(b) i. $x = vt \cos \theta$
   \[ y = vt \sin \theta - 5t^2 \]
   First, we wish to find an equation with parameter $t$.
   \[ x = vt \cos \theta \]
   \[ \therefore t = \frac{x}{v \cos \theta} \]
   So \[ y = vt \sin \theta - 5t^2 \]
   becomes
   \[ y = \frac{x}{v \cos \theta} \sin \theta - 5 \left( \frac{x}{v \cos \theta} \right)^2 \]
\[
x = \frac{x \sin \theta - 5x^2}{v^2 \cos^2 \theta}.
\]
The ball passes through \((d, h)\).
\[
\therefore h = \frac{d \sin \theta - 5d^2}{\cos \theta - v^2 \cos^2 \theta}
\]
\[
\therefore h = \frac{\sin \theta - v^2}{\cos \theta - v^2 \cos^2 \theta}
\]
From the diagram we know that \(\tan \alpha = \frac{h}{d}\).
\[
\therefore \tan \alpha = \frac{\sin \theta - 5d}{\cos \theta - v^2 \cos^2 \theta}
\]
\[
\therefore \tan \alpha = \frac{\cos^2 \theta - v^2 \cos^2 \theta}{\cos \theta - v^2 \cos^2 \theta}
\]
Multiply both sides by \(v^2 \cos^2 \theta\)
\[
\therefore v^2 \cos^2 \theta \tan \alpha = v^2 \sin \theta \cos \theta - 5d
\]
\[
\therefore 5d = v^2 \sin \theta \cos \theta - v^2 \cos^2 \theta \tan \alpha
\]
\[
\therefore v^2 = \frac{5d}{\cos \theta \sin \theta - \cos^2 \theta \tan \alpha}
\]
Thus \(v \to \infty\).

\[\text{ii. 1.}\]
As \(\theta \to \alpha\), the expression
\[
\cos \theta \sin \theta - \cos^2 \theta \tan \alpha
\]
approaches
\[
\cos \alpha \sin \alpha - \cos^2 \alpha \sin \alpha \cos \alpha
\]
Thus, \(v^2 = \frac{5d}{\cos \theta \sin \theta - \cos^2 \theta \tan \alpha}\) \(\to \infty\).

\[\text{ii. 2.}\]
As \(\theta \to \frac{\pi}{2}\)
\[
\cos \theta \sin \theta - \cos^2 \theta \tan \alpha
\]
\[
= 0 \times 1 - 0^2 \tan \alpha
\]
\[
= 0
\]
\[
\therefore v^2 = \frac{5d}{\cos \theta \sin \theta - \cos^2 \theta \tan \alpha} \to \infty
\]
\[
\therefore v \to \infty.
\]

\[\text{iii.}\]
Since \(\sin \alpha\) is a fixed value, \(\alpha\) and \(\tan \alpha\) are constants.
\[
F(\theta) = \cos \theta \sin \theta - \cos^2 \theta \tan \alpha
\]
\[
\therefore F'(\theta) = (\cos \theta \times \cos \theta + \sin \theta \times (-\sin \theta))
\]
\[
= (\cos^2 \theta - \sin^2 \theta) + (2 \cos \theta \sin \theta \tan \alpha)
\]
\[
= \cos 2\theta + \sin 2\theta \tan \alpha.
\]
Now, \(F' \theta = 0\) when
\[
\cos 2\theta + \sin 2\theta \tan \alpha = 0
\]
\[
\therefore \sin 2\theta \tan \alpha = -\cos 2\theta
\]
Divide by \(\cos 2\theta\),
\[
\therefore \tan 2\theta \tan \alpha = -1
\]
\[\text{iv.}\]
Part (a) ii. shows that if
\[
\tan A \tan B = l, \text{ then } A - b = \frac{\pi}{2}
\]
So, if \(\tan 2\theta \tan \alpha = -1\),
then \(\theta - \alpha = \frac{\pi}{2}\)
\[
\therefore \theta = \alpha + \frac{\pi}{4}
\]
\[\text{v.}\]
When \(\theta = \alpha \frac{\pi}{2} + \frac{\pi}{4}\) then \(F'(\theta) = 0\).
\[
\therefore F(\theta) \text{ is stationary at } \theta = \frac{\alpha}{2} + \frac{\pi}{4}.
\]
Now, \(v^2 = \frac{5d}{F(\theta)}\)
so that this is also a stationary point for \(v^2\).
From (b) ii, parts 1 and 2,
\[
v^2 \to \infty \text{ when } \theta \to \frac{\pi}{2}
\]
and \(\theta \to \frac{\pi}{2}\),

So the stationary point at \(\theta = \alpha \frac{\pi}{2} + \frac{\pi}{4}\) must
be a minimum stationary point.
Question 7.

(a) \(47^n + 53 \times 147^{n-1}\)

Let \(n = 1\).

\[
47^n + 53 \times 147^{n-1} \\
= 47 + 53 \times 147^0 \\
= 47 + 53 \\
= 100
\]

which is divisible by 100.

Now assume that it is true for \(n = k\).
That is, \(47^k + 53 \times 147^{k-1}\) is divisible by 100. \(\ldots\)

Now we need to show that if \(n = k + 1\) that the result is also divisible by 100.

\[
47^{k+1} + 53 \times 147^k \\
= 47(47^k) + 53 \times 147(147^{k-1}) \\
= 47(47^k) + 53 \times (100 + 47) \times 147^{k-1} \\
= 47(47^k) + 53 \times 147^{k-1} + 100(53 \times 147^{k-1})
\]

Note that 100\(M\) is an expression that is divisible by 100 for any positive integer value of \(M\).

\[
= 47(100M) + 100(53 \times 147^{k-1}) \\
= 100[47M + 53 \times 147^{k-1}]
\]

Which is divisible by 100.

Hence when the statement is true for \(n = k\), it is also true of \(n = k + 1\).

\(\therefore\) Since it is true for \(n = 1\), it is also true for \(n = 1, n = 2, \ldots\)

Thus it is true for all integer values of \(n \geq 1\).

(b) i. \((1 + x)^n\)

\[
= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n
\]

Let \(x = 1\),

\[
(1 + 1)^n \\
= \binom{n}{0} + \binom{n}{1} + \binom{n}{2}(1^2) + \ldots + \binom{n}{n}(1^n) \\
\therefore 2^n = \sum_{k=0}^{n} \binom{n}{k}
\]

ii. \[
\sum_{k=0}^{n} \binom{n}{k} = \frac{n!}{(n-r)!(n-(n-r))!} \\
= \frac{n!}{(n-r)!r!} = \binom{n}{r}
\]

From part (i),

\[
= 2^{100}.
\]

iii. \((1 + x)^n\)

\[
= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n
\]

Now differentiate both sides:

\[
n(1 + x)^{n-1} \\
= \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}
\]

Substitute \(x = 1\)

\[
\sum_{k=1}^{n} \binom{n}{k}
\]

As required.

(e) i. There are \(n\) red balls and \(n\) blue balls.

A selection of \(r\) balls could contain:

- \(r\) red balls and 0 blue balls
- \((r - 1)\) red balls and 1 blue ball
- \((r - 2)\) red balls and 2 blue ball

\(\ldots\)

0 red balls and \(r\) blue balls.
This is a total of \(r - 1\) combinations.

ii. Selecting \((n - r)\) balls from \(n\) white balls:

\[
\binom{n}{n-r} \\
= \frac{n!}{(n-r)!(n-(n-r))!} \\
= \frac{n!}{(n-r)!r!} = \binom{n}{r}
\]
iii. The selection of $n$ balls is made up of selecting $r$ balls from red and blue, and selecting $(n - r)$ balls from the white balls. Number of selections is 

$$(r + 1) \times \binom{n}{r} \quad \text{for each value of } r$$

(from 0 to $n$).

Total selections $= \sum_{r=0}^{n} (r + 1) \binom{n}{r}$

$= \binom{n}{0} + 2 \binom{n}{1} + 3 \binom{n}{2} + \ldots + (n + 1) \binom{n}{n}$

$= \left[ \binom{n}{0} + 2 \binom{n}{1} + 3 \binom{n}{2} + \ldots + (n + 1) \binom{n}{n}\right]$ 

$+ \left[ \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}\right]$ 

$= n(2)^{n-1} + 2^n$ 

$= (2)^{n-1} (n + 2)$. 