Mathematics  
2010 exam solutions

Question 1

(a) \( x^2 = 4x \)
\[
\therefore x^2 - 4x = 0
\]
\[
\therefore x(x - 4) = 0
\]
\[
\therefore x = 0 \text{ or } 4
\]
(b) \[
\frac{1}{\sqrt{5} - 2}
\]
\[
= \frac{1}{\sqrt{5} - 2} \times \frac{\sqrt{5} + 2}{\sqrt{5} + 2}
\]
\[
= \frac{\sqrt{5} + 2}{\sqrt{5^2 - 2^2}}
\]
\[
= \frac{\sqrt{5} + 2}{1}
\]
\[
= \sqrt{5} + 2
\]
We are told this is equal to \( a + b\sqrt{5} \), so \( a = 2 \) and \( b = 1 \).

(c) The equation required is
\[
(x + 1)^2 + (y - 2)^2 = 5^2
\]

(d) Solve \(|2x + 3| = 9\)

Either \( 2x + 3 = 9 \)
\[
\therefore 2x + 3 = 9
\]
\[
\therefore x = 3
\]
Or \( -2x - 3 = 9 \)
\[
\therefore -2x - 3 = 9
\]
\[
\therefore x = -6
\]
Answer: \( x = -6 \) or \( 3 \).

(e) We use the product rule.
If \( y = x^2 \tan x \)
\[
u = x^2 \text{ and } v = \tan x.
\]
\[
\therefore \frac{dy}{dx} = uv' + vu'
\]
\[
= (\tan x)(2x) + (x^2 + x^2 \sec^2 x
\]
\[
= 2x \tan x + x^2 \sec^2 x.
\]

(f) This is a geometric progression with \( a = 1 \) and \( r = -\frac{1}{3} \).
The limiting sum (sum to infinity) is
\[
S = \frac{a}{1 - r}
\]
\[
= \frac{1}{1 - (-\frac{1}{3})}
\]
\[
= \frac{3}{4}
\]

(g) \( f(x) = \sqrt{x} - 8 \)
For real values of \( f(x) \).
\[
\therefore x - 8 \geq 0
\]
So the domain is \( x \geq 8 \).

Question 2

(a) Use the quotient rule.
\[
y = \frac{\cos x}{x}
\]
Let \( u = \cos x \) and \( v = x \)
\[
y = \frac{u}{v}
\]
\[
\frac{dy}{dx} = \frac{vu' - uv'}{v^2}
\]
\[
= \frac{x(-\sin x) - (\cos x)(1)}{x^2}
\]
\[
= \frac{-x\sin x - \cos x}{x^2}
\]

(b) Solve the inequality \( x^2 - x - 12 < 0 \)
Preferred method: Solve with an equal sign and then resolve the inequality on a graph:

Let \( x^2 - x - 12 = 0 \)
\[
\therefore (x - 4)(x + 3) = 0
\]
\[
\therefore x = 4 \text{ or } -3.
\]
Now the graph of \( y = (x + 3)(x - 4) \)
The graph of \( y = (x + 3)(x - 4) \) is below the \( x \)-axis between \( x = -3 \) and \( x = 4 \). That section is shown darker here.

Answer: \(-3 < x < 4\).

(c) \( y = \ln (3x) \)
\[ \frac{dy}{dx} = \frac{3}{3x} = \frac{1}{x} \]
At the point where \( x = 2 \).
\[ \frac{dy}{dx} = \frac{1}{2} \]

(d) (i) \[ \int \sqrt{5x + 1} \, dx \]
\[ = \int (5x + 1)^{1/2} \, dx \]
\[ = \frac{1}{15/2} \int (5x + 1)^{1/2} \, dx \]
\[ = \frac{2}{15} \int (5x + 1)^{3/2} + c \]
\[ = \frac{2}{15} (5x + 1)^{3/2} + c \]

(ii) \[ \int \frac{x}{4 + x^2} \, dx \]
\[ = \frac{1}{2} \int \frac{2x}{4 + x^2} \, dx \]
\[ = \frac{1}{2} \ln(4 + x^2) + c \]

(e) \[ \int_0^6 (x + k) \, dx = 30 \]
\[ \therefore \left[ \frac{x^2}{2} + kx \right]_0^6 = 30 \]
\[ \therefore \left( \frac{36}{2} + 6k \right) - (0) = 30 \]
\[ \therefore 18 + 6k = 30 \]
\[ \therefore 6k = 12 \]
\[ \therefore k = 2 \]

Question 3

(a) (i) The midpoint is found as follows:
\[ A = (-2, -4) \quad B = (12, 6) \]
If \( M \) is the midpoint of \( AB \), then
\[ M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]
\[ \therefore M = \left( \frac{-2 + 12}{2}, \frac{-4 + 6}{2} \right) \]
\[ = (5, 1) \]

(ii) The gradient of \( BC \), with \( B(12, 6) \) and \( C(6, 8) \):
\[ m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ = \frac{8 - 6}{6 - 12} \]
\[ = \frac{2}{-6} \]
\[ = -\frac{1}{3} \]

(iii) We first need the gradient of \( MN \), with \( M(5, 1) \) and \( N(2, 2) \):
\[ m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ = \frac{2 - 1}{2 - 5} \]
\[ = \frac{1}{-3} \]
\[ = -\frac{1}{3} \]

Thus, \( MN \) and \( BC \) are parallel.
In triangles $ABC$ and $AMN$,
\[ \angle NAM \text{ is common.} \]
\[ \angle ABC = \angle AMN \]
(corresponding angles (NM / CB)
\[ \angle ACB = \angle ANM \]
(corresponding angles (NM / CB)
\[ \therefore \Delta ABC \parallel \Delta ANM \]
(equiangular).

(iv) The gradient of line $MN$ is \(-\frac{1}{3}\).

Point $N(2, 2)$ is on that line. Using the general equation
\[ y - y_1 = m(x - x_1) \]
\[ y - 2 = -\frac{1}{3}(x - 2) \]
\[ y = -\frac{1}{3}x + \frac{8}{3} \]
or \[ x + 3y = 8 = 0. \]

(v) $B$ is (12, 6) and $C$ is (6, 8).
The distance $(D)$ from $B$ to $C$ is
\[ D = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} \]
\[ = \sqrt{(8 - 6)^2 + (6 - 12)^2} \]
\[ = \sqrt{4 + 36} \]
\[ = \sqrt{40} \]
\[ = 2\sqrt{10} \text{ units.} \]

(vi) The perpendicular distance from $A$ to $BC$ is needed. Now, the area:
\[ \frac{1}{2} \times \text{base} \times \text{height} = 44 \]
\[ \therefore \frac{1}{2} \times 2\sqrt{10} \times \text{height} = 44 \]
\[ \therefore \text{height} = \frac{44}{\sqrt{10}} \]
\[ = \frac{22\sqrt{10}}{5} \text{ units. (Perpendicular distance)}. \]

(b) (i) The graph of $y = \ln x$.
Facts: Only positive $x$-values allowed.
As $x$ becomes larger, the $y$ increase but the graph 'flattens out'.
Note that when $x = e$,
\[ y = \ln x \]
\[ \therefore y = \ln_e e \]
\[ = 1. \text{ Hence label } f(e) = 1 \]

(ii) Here are the trapezoids. The first is rather like a triangle. Three are only two because there are three function values here:
\[ f(1) = \log_e (1) = 0 \]
\[ f(2) = \ln(2) \]
\[ f(3) = \ln(3) \]
Thus the sum of these areas is
\[ \frac{1}{2}[2\ln(2) + \ln(2) + \ln(3)] \times 1 \]
\[ = \ln(2) + \frac{1}{2} \ln 3 \]
\[ \approx 1.24 \text{ units}^2. \]

(iii) This answer is less than the required area. The trapezia in the diagram shown above, are both below the log graph. Hence that approximate will be smaller.
Question 4

(a) The sequence of numbers here is 1, 1.75, 2.50 and so on. That is, an arithmetic progression with \( a = 1 \) and \( d = 0.75 \)
The sequence is
\[
T_n = a + (n - 1)d
\]
\[
= 1 + (n - 1) \times 0.75
\]
The ninth term is
\[
= 1 + (9 - 1) \times 0.75
\]
\[
= 7
\]
Answer: Susannah runs 7 km in the 9th week.

(ii) We require \( t_n = 10 \).
The sequence is
\[
t_n = a + (n - 1)d
\]
\[
\therefore 1 + (n - 1) \times 0.75 = 10
\]
\[
\therefore (n - 1) \times 0.75 = 9
\]
\[
\therefore (n - 1) = 12
\]
\[
\therefore n = 13.
\]
Answer: In week 13, she first runs 10 km.

(iii) In weeks 1 to 13 weeks the distance is
\[
S_n = \frac{n}{2}(a + l)
\]
\[
= \frac{13}{2}(1 + 10)
\]
\[
= 71.5 \text{ km.}
\]
(iii) In weeks 14 to 26 (13 weeks) she runs 10 km per week.
That is \( 13 \times 10 = 130 \text{ km.} \)
In 26 weeks,
\[
S_{26} = \frac{26}{2}(2(1) + 25(0.75))
\]
\[
= 130 + 71.5 = 201.5 \text{ km.}
\]
Susannah runs 269.75 km in 26 weeks.

(b) The area is given by
\[
\int_{0}^{2} (\text{top graph} - \text{lower graph}) \, dx
\]
\[
= \int_{0}^{2} (e^{2x} - e^{-x}) \, dx
\]
\[
= \left[ \frac{e^{2x} - e^{-x}}{2} \right]_{0}^{2}
\]
\[
= \left( \frac{e^{4} + e^{-2}}{2} - \frac{1 + 1}{2} \right)
\]
\[
= \frac{e^{4} + e^{-2} - 1}{2} \text{ units}^2.
\]

(c) (i) \( P(\text{First chocolate is mint}) = \frac{4}{12} \)
\( P(\text{Second chocolate is mint}) = \frac{3}{11} \)
\( P(\text{first 2 mint}) = \frac{4}{12} \times \frac{3}{11} = \frac{1}{11} \)

(ii) The initial situation is identical for all 3 chocolate colours. It is therefore 3 times the probability in part (i).
\( P(\text{2 same centre}) = 3 \times \frac{1}{11} = \frac{3}{11} \)

(iii) The probability that the two chocolates have different centres is the 'complement situation' from part ii.
That is, \( 1 - \frac{3}{11} = \frac{8}{11} \).

(d) \( f(x) = 1 + e^x \)
LHS of \( f(x) \times f(-x) = f(x) + f(-x) \)
\[
= (1 + e^x)(1 + e^{-x})
\]
\[
= 1 + e^x + e^{-x} + e^0
\]
\[
= 2 + e^x + e^{-x}
\]
RHS of \( f(x) \times f(-x) = f(x) + f(-x) \)
\[
= 1 + e^x + 1 + e^{-x}
\]
\[
= 2 + e^x + e^{-x}
\]
= LHS.
Hence true as required.
Question 5

(a) (i) The cylinder:

\[ A = 2\pi r^2 + 2\pi rh \]

Now, the volume \( V \) is given to us

\[ V_{cyl} = \pi r^2 h \]

\[ \therefore h = \frac{V}{\pi r^2} \]

\[ \therefore A = 2\pi r^2 + 2\pi r \times \frac{V}{\pi r^2} \]

\[ \therefore A = 2\pi r^2 + 2V \]

\[ \therefore A = 2\pi r^2 + 2\sqrt{r^{-1}} \]

\[ \therefore A = 2\pi r^2 + \frac{20}{r} \text{ when } R = 10. \]

As required.

(ii) We wish to minimise \( A \). First find the stationary points

\[ \frac{dA}{dr} = 4\pi r - 2V r^{-2} = 0 \text{ for a stationary point.} \]

\[ \therefore 4\pi r = 2 \times 10 \times r^{-2} \]

\[ \therefore 4\pi = 20 \times r^{-3} \]

\[ \therefore r^3 = \frac{5}{\pi} \]

Consider the graph of \( A = 2\pi r^2 + 2\sqrt{r^{-1}} \). It has a minimum value at \( r = \frac{\sqrt{5}}{\pi} \). To prove that it is a minimum rather than a maximum when \( r = \frac{\sqrt{5}}{\pi} \), consider the second derivative,

\[ \frac{d^2A}{dr^2} = 4\pi + 40r^{-3} \]

\[ = 4\pi + \frac{40}{r^3} \]

Note that this expression is positive for all \( r > 0 \). This indicates a curve that is concave upwards. Area is a minimum when \( r = \frac{\sqrt{5}}{\pi} \).

(b) (i) \( \sec^2 x + \sec x \tan x = \frac{1 + \sin x}{\cos^2 x} \)

The LHS

\[ = \frac{\sec^2 x + \sec x \tan x}{\cos^2 x} \]

\[ = \frac{1}{\cos^2 x} + \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \]

\[ = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \]

\[ = \frac{1 + \sin x}{\cos^2 x} \]

\[ = \text{ RHS} \]

As required.

(ii) The LHS

\[ = \frac{\sec^2 x + \sec x \tan x}{\cos^2 x} \]

\[ = \frac{1 + \sin x}{\cos^2 x} \text{ from part (i).} \]

Now,

\[ = \frac{1 + \sin x}{\cos^2 x} \]

\[ = \frac{1 + \sin x}{1 - \sin^2 x} \]

\[ = \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)} \]

\[ = \frac{1}{1 - \sin x} \]

\[ = \text{ RHS as required.} \]

(iii) \[ \int_0^\pi \frac{1}{\sin x} \, dx \]

\[ = \int_0^\pi (\sec^2 x + \sec x \tan x) \, dx \]

Using standard integrals,

\[ = \left[ \tan x + \sec x \right]_0^\pi \]

\[ = \left( \tan \frac{\pi}{4} + \sec \frac{\pi}{4} \right) - (\tan 0 + \sec 0) \]
\[
\begin{aligned}
&\left(1 + \frac{1}{\cos \frac{\pi}{4}}\right) - 1 \\
&= \sqrt{2}
\end{aligned}
\]

(c) Consider area \(A_1\). Using the information and graph given to us

\[
A_1 = \int_a^1 \frac{1}{x} \, dx = 1 \quad \ldots \quad 1
\]

and

\[
A_1 = \int_a^b \frac{1}{x} \, dx = 1 \quad \ldots \quad 2
\]

From these:

1. \[
\int_a^1 \frac{1}{x} \, dx = 1
\]
   \[
\therefore \, [\ln x]_a^1 = 1
\]
   \[
\therefore \, (\ln 1) - (\ln a) = 1
\]
   \[
\therefore \, \log_e a = -1
\]
   \[
\therefore \, a = e^{-1}
\]

2. \[
\int_a^b \frac{1}{x} \, dx = 1
\]
   \[
\therefore \, [\ln x]_a^b = 1
\]
   \[
\therefore \, (\ln b) - (\ln 1) = 1
\]
   \[
\therefore \, \ln b = 1
\]
   \[
\therefore \, b = e
\]

Answer: \(a = e^{-1}\) and \(b = e\).

**Question 6**

(a) (i) \(f(x) = (x + 2)(x^2 + 4)\).

\[\therefore\text{ Using the product rule:}\]
\[
\begin{aligned}
f(x) &= u \times v \\
f'(x) &= vu' + uv' \\
&= (x^2 + 4)(1) + (x + 2)(2x) \\
&= x^2 + 4 + 2x^2 + 4x \\
&= 3x^2 + 4x + 4
\end{aligned}
\]

For this quadratic, \(b^2 - 4ac\)
\[
\begin{aligned}
b^2 &= -4ac \\
&= 16 - 4(3)(4) \\
&= -32
\end{aligned}
\]

Thus there are no solutions to the equation \(3x^2 + 4x + 4 = 0\)

That means there are no stationary points to the graph of \(f(x) = (x + 2)(x^2 + 4)\).

(ii) To investigate concavity, first find \(f''(x)\).

\[
f''(x) = 3x^2 + 4x + 4
\]

\[
\therefore \, f''(x) = 6x + 4
\]

Concave down means \(f''(x) < 0, \quad 6x + 4 < 0\)

\[
\therefore \, x < -\frac{2}{3}
\]

Concave down means \(f''(x) < 0, \quad 6x + 4 < 0\)

\[
\therefore \, x > -\frac{2}{3}
\]

(iii) The graph. There are no stationary points and the shape alters at \(x = -\frac{2}{3}\).
(b) (i) The length of an arc is given by
\[ l = r\theta \]
\[ 9 = 5 \times \frac{\pi}{5} \]
\[ \therefore \theta = \frac{9}{5} \text{ radians.} \]
That is, \( \angle POQ = 18 \text{ radians.} \)

(ii) The length \( OT \) is common to the two triangles.
\[ \angle OPT = \angle OQT \]
\[ OP = OQ = 5 \]
\[ \therefore \text{By Pythagoras' theorem, } PT = QT \]
The triangles are congruent because there are three equal length sides.

(iii) To find the length of \( PT \), we will use \( \triangle OPT \), noting that the angle \( TOP \) is one half of the angle found in part i:
\[ \tan \left( \frac{9}{10} \right) = \frac{PT}{5} \]
\[ \therefore PT = 5 \tan \left( \frac{9}{10} \right) \text{ (angle in radians)} \]
= 6.3 cm
(Correct to 1 decimal place).

(iv) The area of parallelogram \( OPTQ \) is \( 5 \times 6.3 = 31.5 \).
The area of the sector \( OPY \) is \( \frac{\theta}{2\pi} \times \pi r^2 \)
Where \( \theta = 18 \) from part i.
\[ = \frac{18 \times 5^2}{2} \]
\[ = 22.5 \]
The shaded area is \( 31.5 - 22.5 = 9 \text{ cm}^2 \).

**Question 7**

(a) (i) \( a = 4 \cos 2t \)
\[ \therefore \dot{x} = \frac{4}{2} \sin 2t + c \]
Initially, the velocity is zero,
\[ \therefore 1 = \frac{4}{2} \sin 2(0) + c \]
\[ \therefore c = 1 \]
\[ \therefore \dot{x} = \frac{4}{2} \sin 2t + 1 \]
As required.

(ii) The particle comes to rest when \( \dot{x} = 0 \).
\[ \therefore 0 = 2 \sin 2t + 1 \]
\[ \therefore 2 \sin 2t = -1 \]
\[ \therefore \sin 2t = -\frac{1}{2} \]
Since \( t \geq 0 \), the first solution is in the 3rd quadrant, \( 2t = \frac{7\pi}{6} \)
\[ \therefore 2t = \frac{7\pi}{6} \]
\[ \therefore t = \frac{7\pi}{12} \]
The particle first comes to rest at \( \frac{7\pi}{12} \) seconds.

(iii) The displacement is given by
\[ \int_{0}^{t} \dot{x} \, dt \]
\[ = \int_{0}^{t} (2 \sin 2t + 1) \, dt \]
\[ = \left[ \frac{2}{2} \cos 2t + t \right]_{0}^{7\pi} \]
\[ = \left[ \cos 2t + t \right]_{0}^{7\pi} \]
\[ = (\cos 2t + t) - (-1 + 0) \]
\[ = 1 - \cos 2t + t \]

(b) \( y = x^2 \)
\[ \therefore \frac{dy}{dx} = 2x \]
At the point \((-1,1), x = -1\), so
\[
\frac{dy}{dx} = -2
\]

The equation of the tangent line is
\[
y = mx + c \text{ at } (-1, 1) \implies y - y_1 = m(x - x_1) \]
\[
\therefore y - 1 = -2(x - (-1)) \\
\therefore y - 1 = -2x - 2 \\
\therefore y = -2x - 1 \\
\text{(or } y + 2x = -1).)
\]

(ii) Let \(M\) be the midpoint of a \(AB\). The coordinates of \(M\):
\[
\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)
\]
\[
= \left(\frac{-1 + 2}{2}, \frac{1 + 4}{2}\right)
\]
\[
= \left(\frac{1}{2}, \frac{5}{2}\right)
\]

The gradient of \(AB\) must be found next.
\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]
\[
= \frac{4 - 1}{2 - (-1)}
\]
\[
= \frac{3}{3}
\]
\[
= 1.
\]

C is a point on the parabola. We know that \(\frac{dy}{dx} = 2x\) so at that point,
\[
\frac{dy}{dx} = 2x
\]
\[
= 1
\]

(Because the tangent is parallel to \(AB\).)

This, at \(C, x = \frac{1}{2}\).

Note that this has the same \(x\)-value as the point \(\left(\frac{1}{2}, \frac{5}{2}\right)\) found in part ii. Thus, The midpoint of \(AB\) and the point \(C\) are vertically aligned.

(iii) Here is a reasonable representation of the graphical structure being asked about:

Start by finding the coordinates of point \(T\).
The tangent line at \(A\): \(y = -2x - 1\).

From earlier, the \(x\)-value at \(T\) is \(\frac{1}{2}\). Thus,

the \(y\)-value at \(T\) is
\[
y = -2x - 1
\]
\[
y = -2\left(\frac{1}{2}\right) - 1
\]
\[
= -2
\]

Thus, \(T\) is \(\left(\frac{1}{2}, -2\right)\).

Now, that means the gradient \(TB\) is
\[
\frac{y_2 - y_1}{x_2 - x_1} \text{ of }
\]
\[
= \frac{-2 - 4}{\frac{1}{2} - 2}
\]
\[
= \frac{-6}{-\frac{3}{2}} = 4
\]

The tangent line at \(B\) was also just found to have a gradient of 4. Thus, the line \(BT\) is a tangent to the parabola at \(B\).

**Question 8**

(a) Given: \(\frac{dP}{dt} = kP\)

Where \(k\) is a positive constant.
\[
\frac{dt}{dP} = \frac{1}{k \times \frac{1}{P}}
\]
\[ t = \frac{1}{k} \int \frac{f}{P} \, dP \]
\[ = \frac{1}{k} \ln P + c' \]
\[ \therefore kt = \ln P + c \quad \text{...1.} \]

Noting units carefully, we let 1935 be when \( t = 0 \). The population at that time is 100 million.

Using 1.,
\[ 0 = \ln (0.000102) + c \]
\[ \therefore c = -\ln (0.000102) \]
\[ \therefore kt = \ln P - \ln(0.000102) \]
\[ \therefore kt = \ln \frac{P}{0.000102} \quad \text{...2.} \]

In 2010, \( t = 2010 - 1935 = 75 \)
And, the population is 200.

Using 2.,
\[ 75 \times k = \ln \frac{200}{0.000102} \]
\[ \therefore k = \frac{1}{75} \times \ln \frac{200}{0.000102} \]
By calculator, \( k = 0.193185 \)
Equation 2. becomes
\[ \therefore 0.193185t = \ln \frac{P}{0.000102} \]

Or
\[ \therefore P = 0.000102 \times e^{0.193185t} \]

In the year 2035, \( t \) will be 100
\[ \therefore P = 0.000102 \times e^{0.193185 \times 100} \]
\[ = 25 \text{ 033.} \]

There will be approximately 25 billion cane toads by 2035.

(b) Let \( p = P(\text{landing heads on one coin}) \)
\[ p(\text{two coins, heads}) = p^2 \]
\[ \therefore p^2 = 0.36 \]
\[ \therefore p = 0.6 \]
Thus \( P(\text{one coin is tails}) = 0.4 \)
And \( P(\text{both landing tails}) = 0.4^2 \)
\[ = 0.16. \]

(c) (i) The graph shows the amplitude is 4 units. Thus, (A) is 4.

(ii) From the graph, the period is \( \pi \) units.
From the equation, the period is \( \frac{2\pi}{b} \)
\[ \therefore \frac{2\pi}{b} = \pi \]
\[ \therefore b = 2. \]

(iii) The graph given to us has equation \( y = 4 \sin 2x \)
Now we will sketch \( y = 3 \sin x + 1 \). This will have amplitude 3 and period \( 2\pi \). It is shifted up’ by 1 unit.
On the same set of axes:
Be sure to note the domain restrictions carefully.

(d) An increasing function \( f(x) \) will have \( f'(x) > 0 \). So, if
\[ f(x) = x^3 - 3x^2 + kx + 8 \]
\[ \therefore f'(x) = 3x^2 - 6x + k \]
For this to be positive, we require \( b^2 - 4ac < 0 \), since the graph must not cross the \( x \)-axis,
\[ \therefore b^2 - 4ac < 0 \]
\[ \therefore (-6)^2 - 4(3)(k) < 0 \]
\[ \therefore 36 - 12k < 0 \]
\[ \therefore k > 3 \]
Since \( a = 3 \) and thus \( a > 0 \), the function is above the \( x \)-axis and positive.

Question 9

(a) (i) The equation for calculating the amount of money in the account is
\[ P = \frac{Q(R^x - 1)}{R - 1} \]
The time unit is the month and $R = (1 + \frac{0.5}{100})$ which is 1.005.

$Q$, the monthly payment, is $500.

$n$, the number of payments is 240.

\[
\therefore P = \frac{Q(R^n - 1)}{R - 1} = \frac{500(1005^{240} - 1)}{1005 - 1} = 231020.4475
\]

With this model, at the end of the 240th payment, the beginning of the month, the money would deposited and then withdrawn immediately. More likely it would beat the end of the month when it is withdrawn. Thus, it would increase by a factor of 1.005:

\[
231020.4475 \times 1005 = 232175.55
\]

As required.

Alternative answer.
Consider each $500 in turn.

The first $500 will grow to

\[
A_1 = 500(1 + 0.005)^{240}
\]

The second $500 would grow to

\[
A_2 = 500(1 + 0.005)^{239}
\]

The this would grow to

\[
A_3 = 500(1 + 0.005)^{238}
\]

And so on until the last payment which would grow to

\[
A_{240} = 500(1 + 0.005)^1
\]

Adding these amounts:

\[
P = 500(1.005 + 1.005^2 \ldots 1.005^{240})
\]

The series in brackets is geometric with $a = 1.005, r = 1.005$ and $n = 240.$

\[
S_n = \frac{a(r^n - 1)}{r - 1} = \frac{1005(1005^{240} - 1)}{1005 - 1}
\]

Thus, $P = \frac{1005(1005^{240} - 1)}{1005 - 1}$

\[
= 232175.55
\]

As required.

(ii) Consider the equation

\[
A = PR^n - \frac{Q(R^n - 1)}{R - 1}
\]

$A$ is the amount left in an account, where, in this case, $Q = 2000$ and $R = 1.005.$

\[
\therefore A = PR^n - \frac{2000(1005^n - 1)}{0.005} = P \times 1005^n - \frac{2000(1005^n - 1)}{0.005}
\]

\[
= P \times 1005^n - 400000(1005^n - 1) = 1005^n(P - 400000) + 400000
\]

As required.

(2) Look for when the remaining retirement money is zero.

\[
1.005^n(P - 400000) + 400000 = 0
\]

But we know that the amount will be $232175.55 from part i.

\[
\therefore 1.005^n(232175.55 - 400000) + 400000 = 0
\]

\[
1.005^n = \frac{-400000}{232175.55 - 400000}
\]

\[
\therefore 1.005^n = \frac{-400000}{232175.55 - 400000}
\]

\[
1.005^n = 238345
\]

Take the log of both sides:

\[
\ln[1.005^n] = \ln[238345]
\]

\[
\therefore n \ln[1.005] = \ln[238345]
\]

\[
\ln(238345) = \ln(1.005)
\]

\[
= 174.143
\]

After 174 months, there is money left. After 175 months, there is nothing.

(b) (i) The function $f(x)$ is not shown in the graph. We need to integrate the equation $y = f'(x)$ to find $f(x).$ In the region above the x-axis, the value of $\int_0^x f(x)$ increases if you start from $x = 0$ and move to the right. Beyond $x = 2$, Region $A_2$ however, adds a negative value to the integral, so its value decreases. Thus, it is increasing for $0 \leq x < 2.$
(ii) The value of the integral decreases for, \( x = 2 \) and onwards. We are told that the integral has area \( A_1 = 4 \) units. That is the maximum.

(iii) The value of \( f(6) \) is
\[
\begin{align*}
A_1 - A_2 - 3(4 - 2) \\
= 4 - 4 - 6 \\
= -6.
\end{align*}
\]

(iv) The graph of \( f(x) \) starts at the origin increases to a maximum at \((2, 4)\) and decreases from \((4, 0)\) to \((6, -6)\). Here is a reasonable graphical interpretation:

![Graph of f(x)](image)

**Question 10**

(a) (i) Take a look at \( \Delta ABC \) and \( \Delta ACD \).
1. They are both isosceles triangles with base angles \( \alpha \).
2. \( \angle BAC \) and \( \angle DAC \) are the same angle.
3. The three angles inside \( \Delta ABC \) and \( \Delta ACD \) are the same (reasons 1, 2).
Hence \( \Delta ABC \) and \( \Delta ACD \) are equiangular and thus are similar.

(ii) The base length of \( \Delta ABC \) is \( a + y \).
Using the similarity of \( \Delta ABC \) and \( \Delta ACD \),
\[
\frac{AD}{BC} = \frac{AC}{AB} \quad \frac{a}{x} = \frac{a + y}{x} \quad x = a + y
\]
\[
x^2 = a^2 + ay
\]
As required.

(iii) Note that trigonometry appears despite the fact there is no right angle. Use the cosine rule on \( \triangle ABC \).
\[
(a + y)^2 = x^2 + x^2 - 2(x)(x) \cos \angle ACB
\]
In isosceles triangle \( \triangle ACD \),
\[
\angle CAD + \angle ACD + \theta = \pi
\]
\[
\therefore 2\angle CAD = \pi - \theta
\]
\[
\therefore \angle CAD = \frac{1}{2}(\pi - \theta)
\]

In isosceles triangle \( \triangle ACD \),
\[
\angle CAB = \angle CBA
\]
\[
\therefore \angle CAD = \angle CBA
\]
Now, \( \angle ACB = \pi - 2(\angle CAB) \)
\[
\therefore \angle ACB = \pi - 2(\angle CAD) \quad \therefore \angle ACB = \pi - \frac{1}{2}(\pi - \theta)
\]
\[
\therefore \angle ACB = \pi - (\pi - \theta)
\]
\[
\therefore \theta
\]
So the cosine rule becomes
\[
(a + y)^2 = x^2 + x^2 - 2(x)(x) \cos \theta
\]
\[
(a + y)^2 = 2x^2 - 2x^2 \cos \theta
\]
\[
(a + y)^2 = 2x^2(1 - \cos \theta)
\]
Now refer to the result in part iii.
\[
\therefore (a + y)^2 = 2(a^2 + ay)(1 - \cos \theta)
\]
\[
\therefore (a + y)^2 = 2a(a + y)(1 - \cos \theta)
\]
So either \( a + y = 0 \) (impossible), or
\[
\therefore (a + y) = 2a(1 - \cos \theta)
\]
\[
\therefore y = 2a - 2a \cos \theta - a
\]
\[
\therefore y = a - 2a \cos \theta
\]
\[
\therefore y = a(1 - 2 \cos \theta)
\]
As required.

(iv) Using the result just obtained, the maximum value of \( 1 - 2 \cos \theta \) is \( 1 + 2 = 3 \).
Thus, the maximum value of \( y = a(1 - 2 \cos \theta) \) as required.

(b) The volume of rotation equation required here is
\[
V = \int_a^b \pi y^2 \, dx
\]
The value of \( \theta \) From triangle \( OPA \),
\[
\sin \theta = \frac{OA}{r}
\]
\[
\therefore OA = r \sin \theta
\]
Hence the \( x \)-value at point \( A \) is \( r \sin \theta \).
The $x$-value at point $B$ is $r$.
The function in the first quadrant is $x^2 + y^2 = r^2$
Therefore, $y^2 = r^2 - x^2$

Hence the volume becomes
\[ V = \int_{A}^{B} \pi y^2 \, dx \]
\[ \therefore V = \int_{r \sin \theta}^{r} \pi(r^2 - x^2) \, dx \]
\[ = \pi \left[ r^2 x - \frac{x^3}{3} \right]_{r \sin \theta}^{r} \]
\[ = \pi \left( r^3 - r^3 \right) - \pi \left( r^3 \sin \theta - \frac{r^3 \sin^3 \theta}{3} \right) \]
\[ = \pi \left( \frac{2r^3}{3} \right) - \pi \left( r^3 \sin \theta - \frac{r^3 \sin^3 \theta}{3} \right) \]
\[ = \frac{\pi r^3}{3} (2 - 3 \sin \theta + \sin^3 \theta) \]

As required.

(ii) 1) The dotted line in the surface of the object is the water's original diameter.
The original radius is $r$.

Put the information on a diagram:

The value $\frac{r}{2}$ is the 'half of the original depth' value.
Thus, $\sin \theta = \frac{r}{2} \div r$
\[ \therefore \sin \theta = \frac{1}{2} \]
\[ \therefore \theta = \frac{\pi}{6} \]

(2) The original volume is
\[ \frac{1}{3} \times \frac{4}{2} \pi r^3 \]
\[ = \frac{2}{3} \pi r^3 \]
The new volume is found using the mathematics of part i. Note the diagram form part i rotated through $+ \frac{\pi}{2}$ forms the part ii diagram.

From that result, the volume is
\[ \frac{\pi r^3}{3} (2 - 3 \sin \theta + \sin^3 \theta) \]
with $\theta = \frac{\pi}{6}$
\[ = \frac{\pi r^3}{3} (2 - 3 \sin \frac{\pi}{6} + \sin^3 \frac{\pi}{6}) \]
\[ = \frac{\pi r^3}{3} (2 - 3 \cdot \frac{1}{2} + \frac{1}{8}) \]
\[ = \frac{\pi r^3}{3} \left( \frac{5}{8} \right) \]
\[ = \frac{5 \pi r^3}{24} \]

The required fraction is
\[ \frac{\frac{5 \pi r^3}{24}}{\frac{2}{3} \pi r^3} \]
\[ = \frac{\frac{5 \pi r^3}{24} \times \frac{3}{2}}{2 \pi r^3} \]
\[ = \frac{5}{16} \]