Cambridge
Queensland Mathematics B Year 11

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Introduction

This textbook for the new Queensland Mathematics B Year 11 syllabus has been written to develop the knowledge and skills required to handle the requisite computation and algebraic methods and procedures for the course. It emphasises mathematical modelling and problem-solving strategies and skills, the capacity to justify mathematical arguments and make decisions, and the capacity to communicate about mathematics, including the writing of reports.

In respect of the latter two capacities, communication and justification, we encourage teachers to ask the following two questions:

- Did the student communicate **what-they-did** with the information given?
- Did the student justify **why-they-did-it**?

We have provided an example of how this can be done in practice with two problems which are given under the title 'Communication and Justification' on the Teacher CD-ROM accompanying this book, and on the companion websites. The solutions in many of the examples in the textbook and the solutions to some of the worksheets on the teacher disk are intended to model good communication and justification.

The Teacher CD-ROM and companion websites also provide a curriculum grid and teaching program, as well as details of other resources.

About the authors

Joe Ousby has a double major in Pure Mathematics and Mathematical Statistics and a Masters degree in Mathematics Education. Joe has spoken at state and national conferences, written articles in QAMT Teaching Mathematics journal and currently teaches at Mount Alvernia College.

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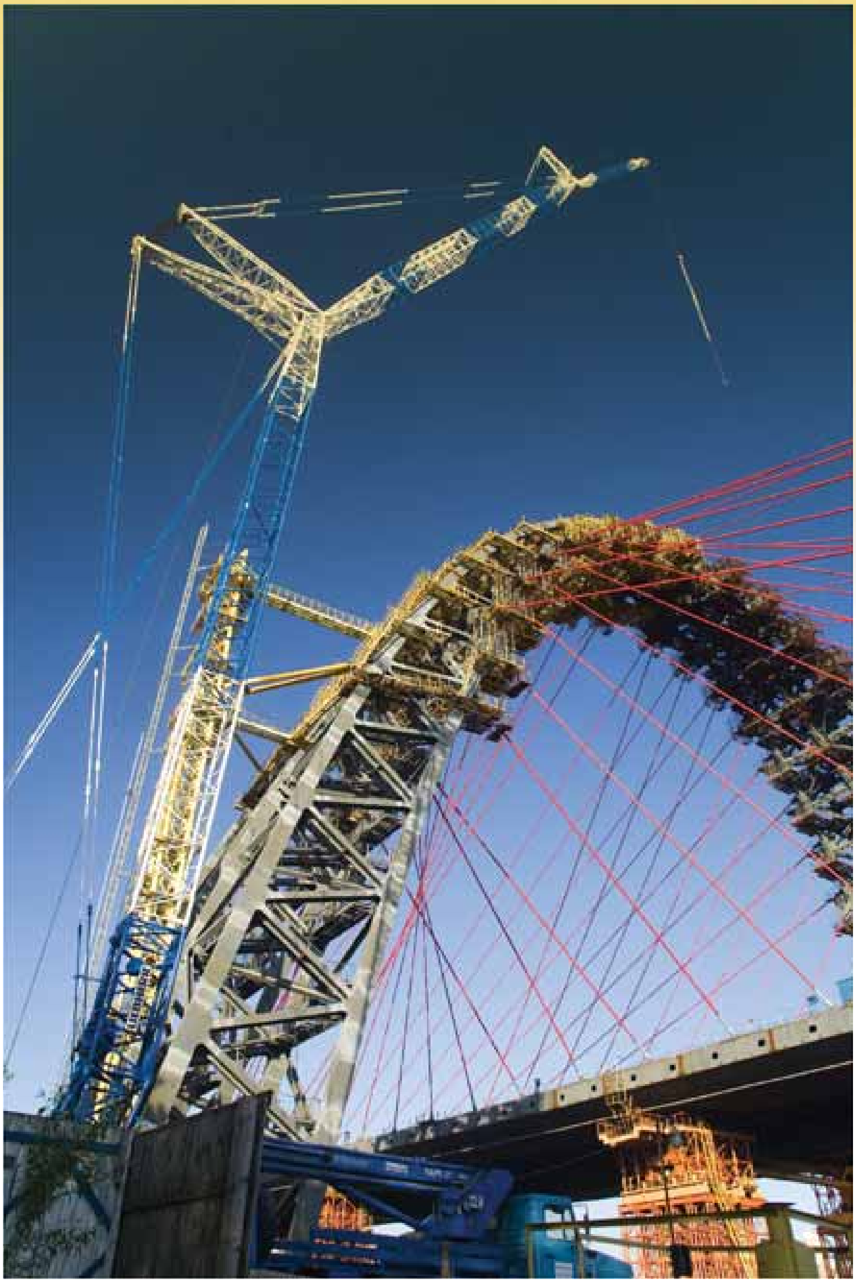
CHAPTER

1

Linear and quadratic equations

Objectives

- To **solve linear and quadratric equations** in one unknown.
- To **factorise quadratic expressions**.
- To sketch the graphs of **linear and quadratic relations**.
- To solve pairs of **simultaneous** linear and quadratic equations.
- To find the equations of **linear and quadratic relations**.
- To investigate quadratic equations using the **discriminant**.
- To develop facility with the **graphics calculator**.



1.1 Linear equations

The solutions to many problems may be found by translating them into mathematical equations that may then be solved using algebraic techniques. An equation is solved by finding the value or values of the unknowns that would make the statement true.

Consider the equation $2x + 5 = 7$.

If $x = 1$, the true statement

$$2(1) + 5 = 7$$

is obtained.

Therefore, the solution to the equation is $x = 1$.

In this case there is no other value of x that would give a true statement.

One way of solving equations is simply to substitute numbers into the equation until the correct answer is found. This method of 'guess and check' the answer is a very inefficient way of solving the problem.

There are a number of standard techniques that can be used for solving linear equations algebraically. The method used in this book involves performing the same calculation to both sides of the equation, producing a simpler equation. This process is repeated until a solution is reached.

Example 1

Solve the equation $3x + 7 = 19$ for x.

Solution

$$\begin{aligned} 3x + 7 &= 19 \qquad (-\,7) \\ 3x &= 12 \qquad (\div\, 3) \\ x &= 4 \end{aligned}$$

Note: Here, 7 is subtracted from both sides of the equation. $3x + 7$ subtract 7 gives $3x$, and 19 subtract 7 gives 12, producing a new simpler equation $3x = 12$. Then both sides of this new equation are divided by 3. Resulting in the final equation $x = 4$.

The solution, 4 in this case, can be checked by substituting the solution back into both sides of the original equation to ensure that the left-hand side (LHS) equals the right-hand side (RHS).

$$\begin{aligned} \text{LHS} &= 3\,(4) + 7 = 19 \\ \text{RHS} &= 19 \end{aligned}$$

$\therefore$ solution is correct.

Example 2

Solve $6x + 3 = 2x - 7$.

Solution

$$\begin{aligned} 6x + 3 &= 2x - 7 && (-\,2x) \\ 4x + 3 &= -7 && (-\,3) \\ 4x &= -10 && (\div\,4) \\ x &= -2.5 \end{aligned}$$

Example 3

Solve $5(2x + 13) = 147$.

Solution

$$\begin{aligned} 5\,(2x + 13) &= 147 \\ 10x + 65 &= 147 && (-\,65) \\ 10x &= 82 && (\div\,10) \\ x &= 8.2 \end{aligned}$$

Example 4

Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

Solution

$$\begin{aligned} \frac{x}{5} - 2 &= \frac{x}{3} && (\times\,15) \\ \frac{{}^{3}\cancel{15}x}{\cancel{5}} - 30 &= \frac{{}^{3}\cancel{15}x}{\cancel{3}} \\ 3x - 30 &= 5x && (-\,3x) \\ -30 &= 2x && (\div\,2) \\ -15 &= x \\ \text{i.e. } x &= -15 \end{aligned}$$

Note: When solving equations with fractions in them, multiply both sides of the equation by a common denominator of the fractions.

When multiplying a fraction by a number, only the top line of the fraction gets multiplied. In the example above, $15 \times \frac{x}{5} = \frac{15x}{5}$.

Example 5

Solve $\dfrac{x-3}{4} - \dfrac{2x-7}{6} = 5$.

Solution

$$\begin{aligned}
\frac{x-3}{4} - \frac{2x-7}{6} &= 5 && (\times 12)\\
\frac{\overset{3}{\cancel{12}}(x-3)}{\cancel{4}} - \frac{\overset{2}{\cancel{12}}(2x-7)}{\cancel{6}} &= 60\\
3(x-3)-2(2x-7) &= 60\\
3x-9-4x+14 &= 60\\
x+5 &= 60 && (-5)\\
-x &= 55 && (\times -1)\\
x &= -55
\end{aligned}$$

Literal equation

An equation for the variable x in which the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.

Example 6

Solve $ax + b = cx + d$ for x.

Solution

$$\begin{aligned}
ax + b &= cx + d && (-b)\\
ax &= cx + d - b && (-cx)\\
ax - cx &= d - b\\
(a-c)x &= d - b && (\div (a-c))\\
x &= \frac{d-b}{a-c}
\end{aligned}$$

Using technology

Linear equations can be solved using the graphics calculator. Students should be able to solve equations both by hand and using forms of this technology.

Example 7

Solve $2.5\,(x + 7) = 8$.

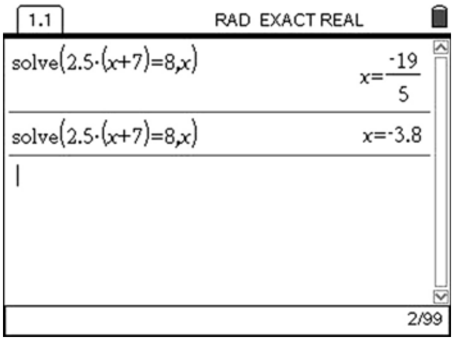
Solution

Using the TI-Nspire:

1 Press menu and select *solve* from the Algebra menu.

2 Type **2.5 × (x + 7) = 8, x)** and then press enter.

3 To obtain a decimal answer while in exact mode, press ctrl enter.

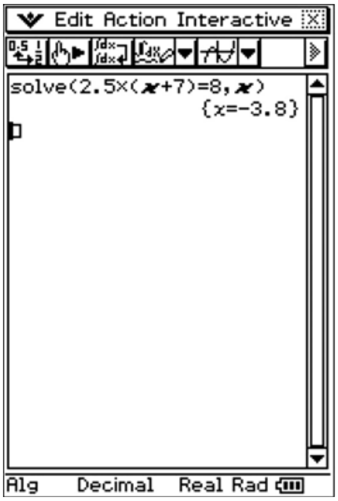


```
solve(2.5·(x+7)=8,x)        x=-19/5
solve(2.5·(x+7)=8,x)        x=-3.8
```

Using the ClassPad:

1 Select the appropriate mode. From the Action menu select *solve* from the Advanced submenu.

2 Type **2.5 × (x + 7) = 8, x)** and then press EXE.



```
solve(2.5×(x+7)=8,x)
                {x=-3.8}
```

Exercise 1A

1 Solve:

Example 1 **a** $2y - 42 = 73$ **b** $5t + 23 = 17$ **c** $16x - 231 = 157$

d $19 - x = 35$ **e** $26 - 3y = 38$ **f** $42 - 4y = 31$

g $\frac{2t}{3} = 4$ **h** $\frac{t}{3} = \frac{1}{2}$ **i** $\frac{3y}{8} = \frac{15}{28}$

j $5y - 3 = 12$ **k** $3x - 7 = 14$ **l** $26 - 3y = 38$

Example 2 **m** $12x + 72 = 7x$ **n** $x - 5 = 4x + 10$ **o** $3x - 27 = 18 - 2x$

Example 3 **p** $2(y + 6) = 10$ **q** $2(x + 4) = 7(x + 2)$ **r** $2y + 6 = 1 - 3(y - 4)$

Example 4 **s** $\frac{y - 3}{8} = 6$ **t** $\frac{y + 41}{3} = 41$ **u** $\frac{2\,(5 - x)}{3} = 11$

Example 5 **v** $\frac{x}{2} + \frac{x}{3} = 10$ **w** $x + 4 = \frac{3}{2}x$ **x** $\frac{7x + 3}{2} = \frac{9x - 8}{4}$

y $\frac{4y - 5}{2} - \frac{2y - 1}{6} = y$ **z** $\frac{2(1 - 2x)}{3} - 2x = -\frac{2}{5} + \frac{4(2 - 3x)}{3}$

Example 6 **2** Solve the following linear literal equations for x:

a $ax + b = 0$ **b** $cx + d = e$ **c** $a(x + b) = c$ **d** $ax - b = cx$

e $\frac{x}{a} + \frac{x}{b} = 1$ **f** $\frac{a}{x} + \frac{b}{x} = 1$ **g** $ax - b = cx - d$ **h** $\frac{ax + c}{b} = d$

Example 7 **3** Solve, using the graphics calculator.

a $0.2x + 6 = 2.4$ **b** $0.6(2.8 - x) = 48.6$ **c** $\frac{2x + 12}{7} = 6.5$ **d** $0.5x - 4 = 10$

e $\frac{1}{4}(x - 10) = 6$ **f** $6.4x + 2 = 3.2 - 4x$

1.2 Factorising

Three different types of factorisation will be considered:

1 Removing the **highest common factor**.
2 Factorising **trinomials**.
3 The **difference of two squares**.

Removing the highest common factor (HCF)

Example 8

Factorise $10x^2 + 15x$.

Solution

$$\begin{aligned} &10x^2 + 15x \\ &= 5x\,(2x + 3) \end{aligned}$$

Factorising trinomials

Step 1 Find the product of the number in front of x^2 and the constant term.
Step 2 Find all the factor pairs that give this product. Work up from 1 × . . . then 2 × . . . etc.
Step 3 Find the pair that adds to the coefficient of the term in x (i.e. the number in front of x).
Step 4 Use this pair of numbers to complete the factorisation.

Example 9

Factorise $x^2 + 14x + 40$.

Solution

$$\begin{array}{rcc} & \multicolumn{2}{c}{40} \\ & 1 & 40 \\ x^2 + 14x + 40 & 2 & 20 \\ = (x + 4)(x + 10) & 4 & 10 \\ & 5 & 8 \end{array}$$

Note: The number in front of x^2 was 1 and this was multiplied by 40 to get 40.

To do step 4 (i.e. complete the factorisation), when the coefficient of x^2 is 1 simply put the pair of numbers in brackets with x, as shown.

Example 10

Factorise $x^2 - 18x + 45$.

Solution

$$\begin{array}{lll} & & 45 \\ x^2 - 18x + 45 & -1 & -45 \\ = (x-3)(x-15) & \boxed{-3} & \boxed{-15} \\ & -5 & -9 \end{array}$$

Note: When the constant term is positive and the coefficient of x is negative, both numbers in the pairs will be negative.

Example 11

Factorise $x^2 - 8x - 20$.

Solution

$$\begin{array}{lll} & & -20 \\ x^2 - 8x - 20 & -1 & 20 \\ = (x+2)(x-10) & \boxed{+2} & \boxed{-10} \\ & -4 & 5 \end{array}$$

Note: When the constant term is negative one of the numbers in each pair will be negative. Start by assuming the negative number is the smaller one, and if that doesn't work switch to the larger one being negative.

When the number in front of x^2 is anything but 1, step 4, the process of completing the factorisation is more complex. The second line is created by simply using the pair of numbers from step 3 to split the term x into two parts.

Example 12

Factorise $6x^2 + 13x - 15$.

Solution

$$\begin{array}{lll} & & -90 \\ 6x^2 + 13x - 15 & -1 & 90 \\ = 6x^2 - 5x + 18x - 15 & -2 & 45 \\ = x(6x-5) + 3(6x-5) & -3 & 30 \\ = (x+3)(6x-5) & \boxed{-5} & \boxed{18} \\ & -6 & 15 \\ & -9 & 10 \end{array}$$

Note: The contents of the two pairs of brackets in line 3 of the solution above must be the same. If they aren't, an error has occurred earlier.

Example 13

Factorise $3x^2 - 17x + 10$.

Solution

	30	
$3x^2 - 17x + 10$	-1	-30
$= 3x^2 - 15x - 2x + 10$	-2	-15
$= 3x(x - 5) - 2(x - 5)$	-3	-10
$= (3x - 2)(x - 5)$	-5	-6

Difference of two squares (DOTS)

Factorising expressions such as $x^2 - 9$ can be done using the method for trinomials discussed above. Example 14 shows this process. However, as shown in Example 15, there is a short-cut using the special result $x^2 - a^2 = (x + a)(x - a)$.

Example 14

Factorise $x^2 - 9$.

Solution

	-9	
$x^2 - 9$	-1	9
$= (x - 3)(x + 3)$	-3	3

Example 15

Factorise $4x^2 - 49$.

Solution

$$
\begin{aligned}
&4x^2 - 49 \\
&= (2x)^2 - 7^2 \\
&= (2x - 7)(2x + 7)
\end{aligned}
$$

The next example shows the need to be mindful of always removing the highest common factor whenever it is possible.

Example 16

Factorise $3x^2 - 75$.

Solution

$$\begin{aligned} & 3x^2 - 75 \\ &= 3\left(x^2 - 25\right) \\ &= 3\left(x^2 - 5^2\right) \\ &= 3\,(x-5)\,(x+5) \end{aligned}$$

Exercise 1B

Example 8

1 Factorise:

a $x^2 + 3x$
b $x^2 - 5x$
c $x^2 + x$
d $3x^2 - 4x$
e $5x^2 - x$
f $5x^2 - 15x$
g $6x^2 - 15x$
h $-x^2 - 5x$
i $-4x^2 + 16x$

Examples 9–13

2 Factorise:

a $x^2 + 10x + 24$
b $x^2 + 9x + 8$
c $x^2 - 11x + 24$
d $x^2 - x - 30$
e $x^2 - 9x + 20$
f $x^2 - 37x - 120$
g $x^2 - 7x - 18$
h $x^2 - 19x + 48$
i $x^2 + 5x - 84$
j $5x^2 + 23x + 12$
k $6x^2 - 7x + 2$
l $5x^2 - 19x + 12$
m $6x^2 + 19x - 20$
n $15x^2 - 11x - 14$
o $15x^2 + x - 14$

Examples 14,15

3 Factorise:

a $x^2 - 49$
b $x^2 - 16$
c $1 - x^2$
d $4x^2 - 81$
e $50x^2 - 98$
f $4x^2 - 36$

Example 16

4 Factorise:

a $5x^2 - 80$
b $6x^2 - 54$
c $8x^2 - 50$
d $4x^2 + 20x + 24$
e $3x^2 + 15x + 18$
f $2x^2 - 18x + 28$
g $5y^2 - 20y - 60$
h $6x^2 + 33x + 45$
i $12x^2 + 26x - 56$
j $x^3 - 5x^2 - 6x$
k $5x^3 - 16x^2 + 12x$
l $48x - 24x^2 + 3x^3$

1.3 Solving quadratic equations

Five methods of solution of quadratic equations are considered here:

1 Isolating x
2 Factorising
3 Completing the square
4 The quadratic formula
5 Using technology

Isolating x

Isolating x is recommended whenever there is a term in x^2 but no term in x. This method is consistent with the way linear equations are solved.

Example 17

Solve $3x^2 + 15 = 90$.

Solution

$$\begin{aligned} 3x^2 + 15 &= 90 && (-\ 15)\\ 3x^2 &= 75 && (\div\ 3)\\ x^2 &= 25 && (\sqrt{\ })\\ x &= \pm 5 \end{aligned}$$

Factorising

Factorising is recommended whenever there are terms in x as well as x^2. This method uses the **null factor theorem**. The null factor theorem states:

If $ab = 0$ then either $a = 0$ or $b = 0$.

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the null factor theorem; i.e. If $ab = 0$ then either $a = 0$ or $b = 0$.

Example 18

Solve $x^2 - x = 12$.

Solution

$$\begin{aligned} x^2 - x &= 12 && (-\ 12)\\ x^2 - x - 12 &= 0\\ (x-4)(x+3) &= 0 \end{aligned}$$

$$\swarrow \qquad \searrow \qquad \text{(null factor theorem)}$$

$$\begin{aligned} \therefore\ x - 4 &= 0 \quad \text{or} \quad x + 3 = 0\\ \therefore\ x &= 4 \quad \text{or} \quad x = -3 \end{aligned}$$

Completing the square

Completing the square will work with all quadratic equations and depends on the result $(x + a)^2 = x^2 + 2ax + a^2$.

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Divide both sides by the value of a.

Step 3 Rewrite the equation with a large gap after x.

Step 4 Halve the coefficient of x, and square it.
Step 5 Add and subtract this square number in the gap after x.
Step 6 Group the first three terms together and factorise.
Step 7 Simplify the last two terms.
Step 8 Solve for x by isolating x.

Example 19

Solve for x by completing the square: $x^2 + 12x + 14 = 0$.

Solution

$$\begin{aligned}
x^2 + 12x + 14 &= 0 \\
\underbrace{x^2 + 12x + 36} - \underbrace{36 + 14} &= 0 \\
(x+6)^2 \qquad -22 &= 0 && (+\,22) \\
(x+6)^2 &= 22 && (\sqrt{\ }) \\
x + 6 &= \pm\sqrt{22} && (-\,6) \\
x &= -6 \pm \sqrt{22} \\
x &= -6 - \sqrt{22} && \text{or} \quad -6 + \sqrt{22} \\
x &\approx -10.69 && \text{or} \quad -1.31
\end{aligned}$$

Example 20

Solve for x by completing the square: $2x^2 - 18x + 10 = 0$.

Solution

$$\begin{aligned}
2x^2 - 18x + 10 &= 0 && (\div\, 2) \\
x^2 - 9x + 5 &= 0 \\
x^2 - 9x + 5 &= 0 \\
\underbrace{x^2 - 9x + 20.25} - \underbrace{20.25 + 5} &= 0 \\
(x-4.5)^2 \qquad -15.25 &= 0 && (+\,15.25) \\
(x-4.5)^2 &= 15.25 && (\sqrt{\ }) \\
x - 4.5 &= \pm\sqrt{15.25} && (+\,4.5) \\
x &= 4.5 \pm \sqrt{15.25} \\
x &= 4.5 - \sqrt{15.25} && \text{or} \;\; 4.5 + \sqrt{15.25} \\
x &\approx 0.59 && \text{or} \;\; 8.41
\end{aligned}$$

Quadratic formula

The **quadratic formula** will work to solve all quadratic equations. It is particularly useful when the factorising is difficult and is usually more efficient than completing the square. The quadratic formula states that:

$$\text{If } ax^2 + bx + c = 0 \quad \text{then} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 21

Solve $5x^2 - x - 13 = 0$ using the quadratic formula.

Solution

$$
\begin{aligned}
&5x^2 - x - 13 = 0\\
&a = 5 \qquad b = -1 \qquad c = -13\\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&\quad = \frac{-(-1) \pm \sqrt{(^-1)^2 - 4 \times 5 \times {}^-13}}{2 \times 5}\\
&\quad = \frac{1 \pm \sqrt{261}}{10}\\
&\quad = \frac{1 - \sqrt{261}}{10} \text{ or } \frac{1 + \sqrt{261}}{10}\\
&\quad \approx -1.516 \quad \text{or} \quad 1.716
\end{aligned}
$$

Using technology

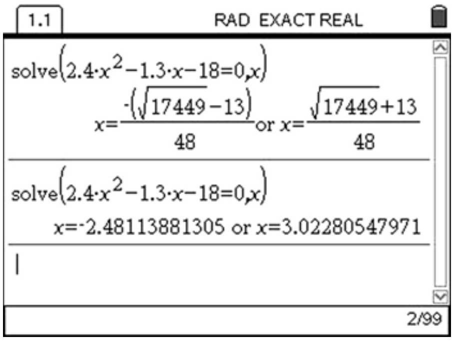
Example 22

Solve $2.4x^2 - 1.3x - 18 = 0$.

Solution

Using the TI-Nspire:

1. Select *solve* from the Algebra menu.
2. Type $\mathbf{2.4x^2 - 1.3x - 18 = 0, x}$) then press enter or ctrl enter, depending on the type of solution required.



solve$(2.4\cdot x^2-1.3\cdot x-18=0,x)$

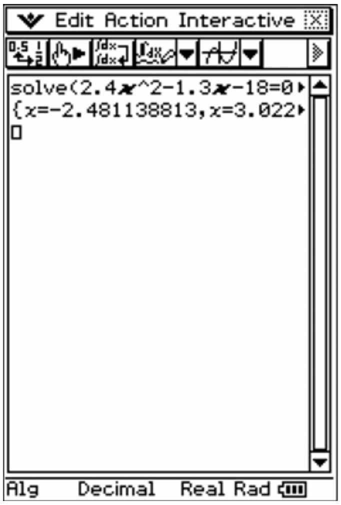
$x=\frac{-(\sqrt{17449}-13)}{48}$ or $x=\frac{\sqrt{17449}+13}{48}$

solve$(2.4\cdot x^2-1.3\cdot x-18=0,x)$

$x={}^-2.48113881305$ or $x=3.02280547971$

Using the ClassPad:

1. From the Action menu select *solve* from the Advanced submenu.
2. Type $\mathbf{2.4x^2 - 1.3x - 18 = 0, x}$) then press EXE.



```
solve(2.4x^2-1.3x-18=0▸
{x=-2.481138813,x=3.022▸
```

Exercise 1C

Example 17

1 Solve the following by isolating x:

a $2x^2 - 18 = 0$ **b** $5x^2 - 180 = 0$ **c** $3x^2 - 180 = 12$ **d** $4x^2 = 1$

e $16 = x^2$ **f** $x^2 = 0.0625$ **g** $1.5x^2 - 37.2 = 0$ **h** $2.3x^2 = 0$

Example 18

2 Solve each of the following by factorising:

a $x^2 - 6x + 8 = 0$ **b** $x^2 - 8x - 33 = 0$ **c** $x^2 + 5x - 14 = 0$

d $x(x + 12) = 64$ **e** $2x^2 + 5x + 3 = 0$ **f** $4x^2 - 8x + 3 = 0$

g $x^2 = 5x + 24$ **h** $6x^2 + 13x + 6 = 0$ **i** $2x^2 - x = 6$

Examples 19, 20

3 Solve by completing the square.

a $x^2 + 8x + 3 = 0$ **b** $x^2 - 12x + 8 = 0$ **c** $x^2 - 6x - 30 = 0$

d $2x^2 + 10x = 13$ **e** $3x^2 + 15x = 14$ **f** $2x(x + 10) = 70$

Example 21

4 Solve for x using the quadratic formula.

a $2x^2 + 5x + 3 = 0$ **b** $4x^2 - 8x + 1 = 0$ **c** $x^2 - x - 33 = 0$

d $1.3x^2 + 5x = 14$ **e** $3x(x + 12) = 64$ **f** $0.2x^2 - 1.4x - 3.5 = 0$

Example 22

5 Use the graphics calculator to solve each of the following equations. Give your answer correct to 2 decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 = 4x + 2$ **c** $3x^2 - 7 = 2x$

6 Solve:

a $x^2 + 9x + 8 = 0$ **b** $x^2 + 8x - 33 = 0$ **c** $x(x - 3) = 28$

d $x^2 - 4x - 14 = 0$ **e** $2x^2 + 5x + 1 = 0$ **f** $4x^2 + 8x + 3 = 0$

g $x^2 = 6x + 16$ **h** $6x^2 - 13x + 5 = 0$ **i** $2x^2 - x = 9$

j $6x^2 + 15 = 23x$ **k** $2x^2 - 3x - 9 = 0$ **l** $10x^2 - 11x + 3 = 0$

m $12x^2 + x = 6$ **n** $4x^2 + 1 = 4x$ **o** $3x(4 - x) = 2x$

p $\frac{1}{7}x^2 = \frac{3}{7}x$ **q** $x^2 + 8x = -15$ **r** $5x^2 = 11x - 2$

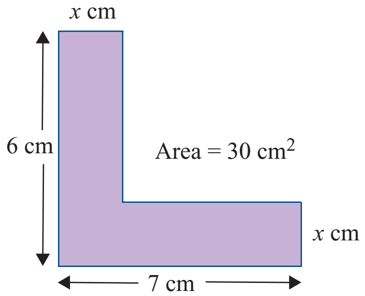
MAPS

7 The bending moment, M, of a simple beam used in bridge construction is given by the formula $M = \frac{wl}{2}x - \frac{w}{2}x^2$.

If $l = 13$, $w = 16$ kg per m and $M = 288$ kg m, calculate the value of x.

MAPS

8 Calculate the value of x.



9 The height, h, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 m.

10 Solve for x if $4x^2 + (m - 4)x - m = 0$.

1.4 Graphing linear and quadratic functions on the number plane

Functions of the form $y = mx + c$ and $ax + by = c$ are said to be **linear** because they form a straight line when graphed on the number plane. Functions of the form $y = ax^2 + bx + c$ are said to be **quadratic**. The fundamental method, which works when graphing all functions, is to make a table of values. The table can be constructed by hand or using the graphics calculator.

Step 1 Create a table of values.

Step 2 Plot the points found in the table on the number plane.

Step 3 Join the dots. Do this freehand when the points are not in line. Use a ruler if they are in line.

Example 23

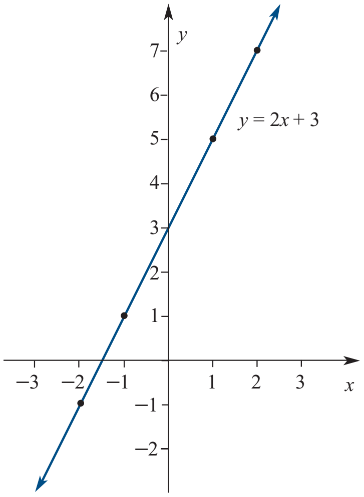
Graph accurately on the number plane:

a $y = 2x + 3$ **b** $y = x^2 - 2x$

Solution

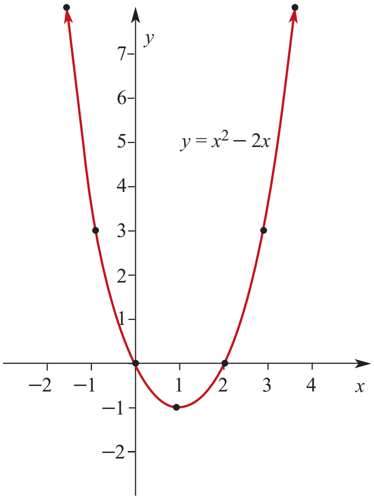
a $y = 2x + 3$

x	−2	−1	0	1	2
y	−1	1	3	5	7



b $y = x^2 - 2x$

x	−2	−1	0	1	2
y	8	3	0	−1	0



Note: Quadratics are symmetrical around the **turning point**. In part **b** it was possible to plot the points (3, 3) and (4, 8), without having calculated them in the table.

Using technology to create tables of values and to graph relations

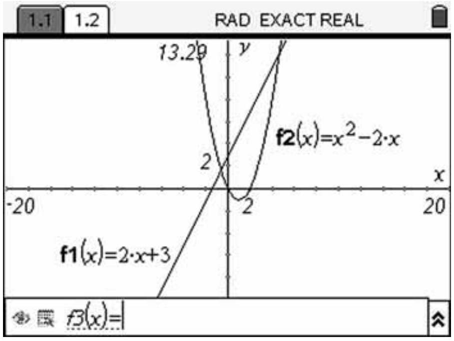
Creating tables of values and sketching relations using a CAS calculator.

Using the TI-Nspire:

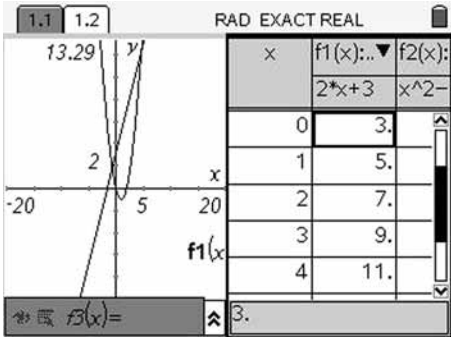
1 Select the Graphs & Geometry application.

2 Type **$2x + 3$** into $f1(x)$ then press (enter).

3 Type **$x^2 - 2x$** into $f2(x)$ then press (enter). (This will graph both functions.)



By pressing (menu) then selecting *Window* you are able to set the boundaries of the graph. To insert a table of values press (ctrl) (T).



x	f1(x):...	f2(x):
	2*x+3	x^2−
0	3.	
1	5.	
2	7.	
3	9.	
4	11.	

To edit the table settings press (menu) and select *Edit Function Table Settings* from the Function Table submenu. Here you are able to change the starting value of the table and the increments.

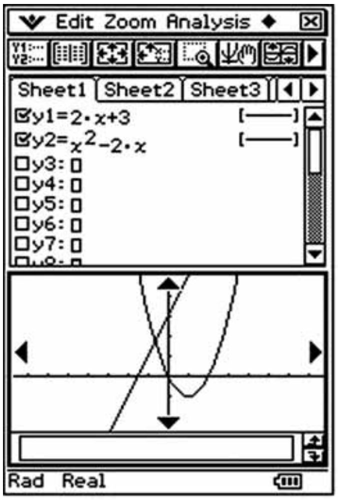
Using the ClassPad:

1 Select the Graphs and Tables application by tapping on Graph&Tab... .

2 Type **$2x + 3$** into $y1$ and press (EXE).

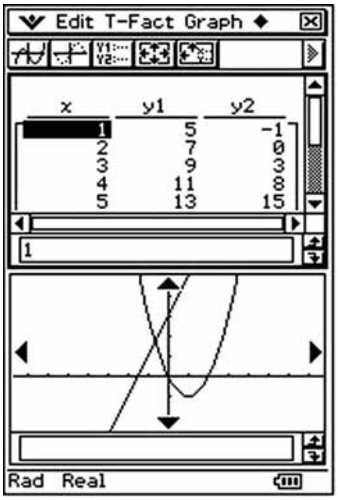
3 Type **$x^2 - 2x$** into $y2$ and press (EXE).

4 Tap ⩘ to sketch the two functions.



☑y1=2·x+3
☑y2=x^2−2·x
☐y3:□
☐y4:□
☐y5:□
☐y6:□
☐y7:□

To set the boundaries of the graph (i.e. the Window settings) tap ⊞. To view a table of values tap ▦.



x	y1	y2
1	5	−1
2	7	0
3	9	3
4	11	8
5	13	15

To edit the table settings tap ⊡.

Mastering these procedures is a worthwhile exercise.

Exercise 1D

Example 23

1 Graph accurately on the number plane:

a $y = 3x + 2$ **b** $y = 4 - 2x$ **c** $3x + 2y = 20$

d $y = x^2 - 3$ **e** $y = (x - 1)(x - 3)$ **f** $y = 3x - x^2$

2 Use the graphics calculator to graph on the number plane:

a $y = \frac{1}{2}x + 1$ **b** $y = -2 - x$ **c** $5x + 2y - 10 = 0$

d $y = 3x(x + 2)$ **e** $y = (x + 2)^2 - 3$ **f** $y = 5 - 2x^2$

1.5 More on graphing linear functions

Although a table of values will always produce accurate graphs, it can be tedious at times. Two more efficient methods are discussed below.

- Using the gradient and y-intercept for linear functions.
- Using the x- and y-intercepts for linear functions.

Using the gradient and *y*-intercept for linear functions $y = mx + c$

When a linear equation is written in the form $y = mx + c$, the gradient is the value m and the y-intercept is the value c. When graphing equations of this form, follow these steps:

Step 1 Identify the gradient and y-intercept of the linear equation; that is, find the value of m and c. This may involve first expressing the equation in the form $y = mx + c$.

Step 2 Express m as a fraction.

Step 3 Plot the y-intercept.

Step 4 Starting at the y-intercept, use the gradient to plot further points.

Step 5 Use a ruler to draw the line.

Example 24

Graph accurately on the number plane:

a $y = \frac{2}{3}x - 1$ **b** $2x + y = 3$

Solution

a

$$y = \frac{2}{3}x - 1$$

$$\therefore\ m = \frac{2}{3} \qquad c = -1$$

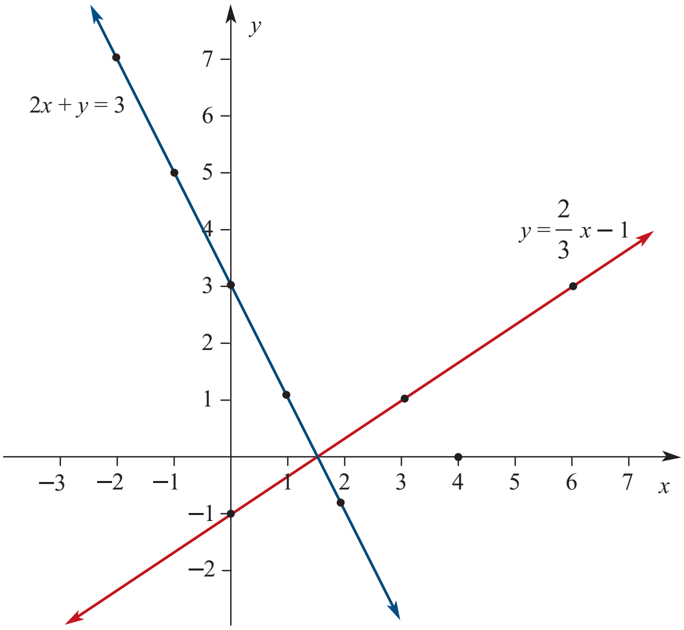
b

$$2x + y = 3 \qquad (-\,2x)$$

$$y = -2x + 3$$

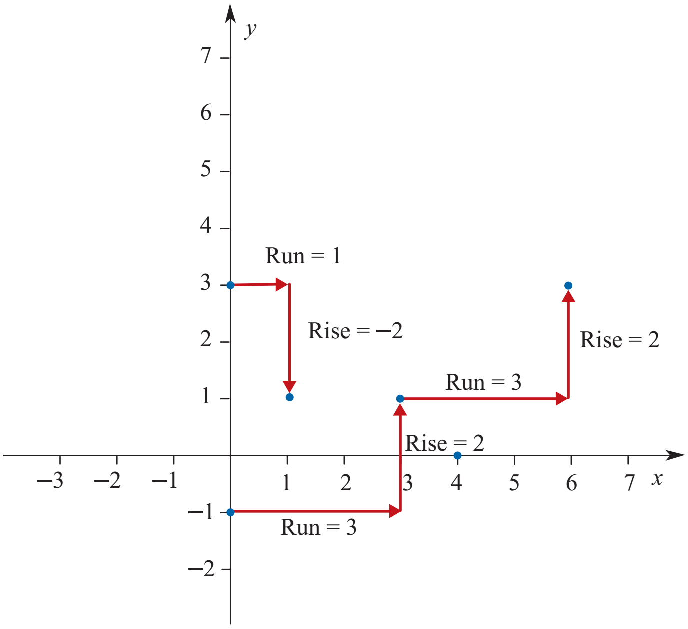
$$m = -2 \qquad c = 3$$

$$= -\frac{2}{1}$$



Note: Gradient $= m = \dfrac{\text{Rise}}{\text{Run}}$.

In part **a** of Example 24, $m = \dfrac{2}{3}$. Hence, run = 3 and rise = 2. The points are then plotted starting at −1 on the *y*-axis, then going 3 to the right and 2 up, then 3 to the right and 2 up again, etc. Similarly, in part **b**, the run = 1 and the rise = –2. Therefore, the points are plotted starting at 3 on the *y*-axis then going 1 to the right and 2 down, then 1 to the right and 2 down again, etc.



Using the *x*- and *y*-intercepts for linear functions

To graph a linear function using the *x*- and *y*-intercepts, follow these steps:

Step 1 Find the *y*-intercept by letting $x = 0$ and solving for *y*.

Step 2 Find the *x*-intercept by letting $y = 0$ and solving for *x*.

Example 25

Graph accurately on the number plane:

a $2x + 3y = 15$ **b** $x - 3y = 6$

Solution

a $2x + 3y = 15$

$x = 0 \therefore 3y = 15 \quad (\div 3)$

$\therefore \quad y = 5$

i.e. *y*-intercept is 5.

$y = 0 \therefore 2x = 15 \quad (\div 2)$

$\therefore \quad x = 7.5$

i.e. *x*-intercept is 7.5

b $x - 3y = 6$

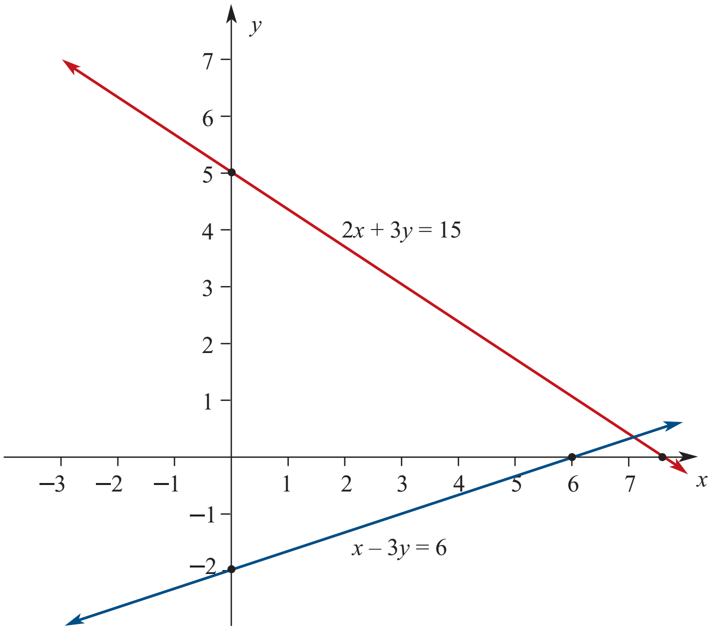
$x = 0 \therefore -3y = 6 \quad (\div -3)$

$\therefore \quad y = -2$

i.e. *y*-intercept is −2.

$y = 0 \quad \therefore x = 6$

i.e. *x*-intercept is 6



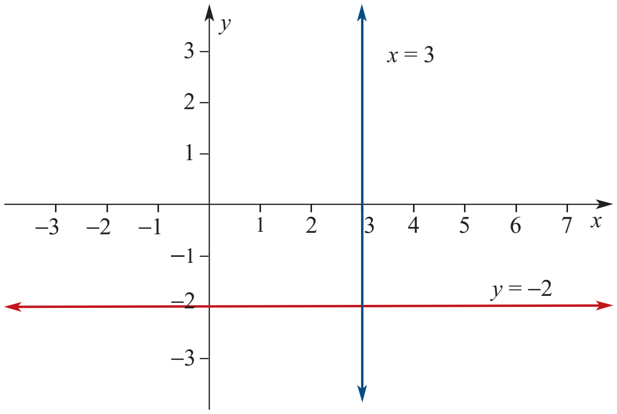
Special cases

Example 26

Graph on the number plane:

a $x = 3$ **b** $y = -2$

Solution



Note: On $x = 3$ all of the points have an x coordinate of 3. On $y = -2$ all of the points have a y coordinate of -2.

Sketching

Sketching is distinct from graphing accurately and will be the most common technique you will use later in the course. When graphing accurately, use graph paper or at least make sure that the axes and points are drawn properly to scale. When sketching, use a ruler to draw any straight lines and estimate where the points should be. There is no need to draw properly to scale.

Example 27

Sketch on the number plane:

a $y = 3x - 1$ **b** $x - 2y = 5$

Solution

a $y = 3x - 1$

$m = 3 \qquad c = -1$

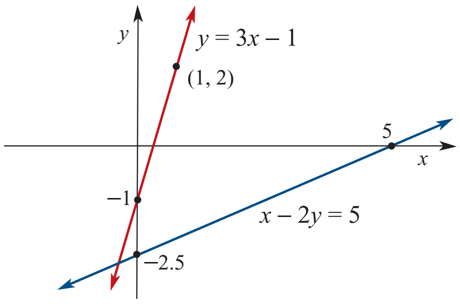
$\therefore \ = \dfrac{3}{1}$

b $x - 2y = 5$

$x = 0 \ \therefore -2y = 5 \qquad (\div -2)$

$\therefore \ y = -2.5$

$y = 0 \ \therefore \ x = 5$



Note: When sketching straight lines always give at least two points.

Exercise 1E

Example 24 **1** Use the gradient and y-intercept method to graph these equations accurately on the number plane:

a $y = \frac{3}{4}x + 2$ **b** $y = \frac{1}{2}x - 2$ **c** $y = 2x + 1$

d $y = 3x - 2$ **e** $y = -2x + 4$ **f** $y = -\frac{1}{2}x$

Example 25 **2** Use the x- and y-intercept method to graph these equations accurately on the number plane:

a $x + 2y = 4$ **b** $3x + y = 6$ **c** $2y - 3x = 9$

d $4y - 2x = 9$ **e** $x = 4y - 4$ **f** $5x - 2y + 12 = 0$

Example 26 **3** Graph accurately on the number plane:

a $y = 6$ **b** $y = 4x - 3$ **c** $x - y = 4$

d $2x - y = 0$ **e** $x = -4$ **f** $2y = 3x$

g $x = 0$ **h** $x + y = 3$ **i** $y = 0$

Example 27 **4** Sketch on the number plane:

a $y = 3x - 4$ **b** $y = -4x - 4$ **c** $3x - 2y = 9$

d $y = 3 - 4x$ **e** $2x + y = -7$ **f** $2x - 5y + 10 = 0$

g $\frac{3}{4}x = 3y + 6$ **h** $y = -3$ **i** $4x = 10$

j $x = \frac{1}{2}y + 4$ **k** $y = 2x - 5$ **l** $y = \frac{3}{4}x + 2$

1.6 More on sketching quadratic functions

Two methods for sketching quadratic functions are treated in this section.

- Using the x- and y-intercepts for sketching quadratic functions.
- Using the turning point form for sketching quadratic functions.

Using the *x*- and *y*-intercepts for sketching quadratic functions

Step 1 Find the y-intercept by substituting $x = 0$ into the equation.

Step 2 Find the x-intercepts by substituting $y = 0$ into the equation, and then solving the resulting quadratic equation.

Step 3 Find a third point if the parabola passes through the origin; that is, through the point (0, 0).

Step 4 Use the points found to sketch the curve.

Example 28

Sketch on the number plane:

a $y = x^2 - 2x - 8$ **b** $y = 3x - x^2$

Solution

a $y = x^2 - 2x - 8$

$x = 0 \quad \therefore y = -8$

i.e. y-intercept is $(0, -8)$.

$y = 0 \quad \therefore 0 = x^2 - 2x - 8$

$x^2 - 2x - 8 = 0$

$(x + 2)(x - 4) = 0$

null factor theorem

$x + 2 = 0$ or $x - 4 = 0$

$\therefore x = -2$ or $x = 4$

i.e. x-intercepts are $(-2, 0)$ and $(4, 0)$.

b $y = 3x - x^2$

$x = 0 \quad \therefore y = 0$

i.e. y-intercept is $(0, 0)$.

$y = 0 \quad \therefore 0 = 3x - x^2$

$3x - x^2 = 0$

$x(3 - x) = 0$

null factor theorem

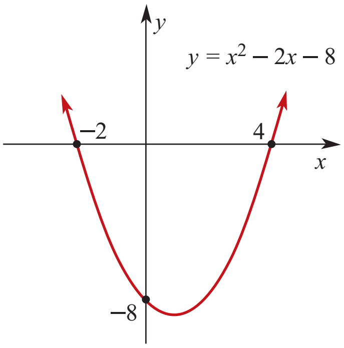
$x = 0$ or $3 - x = 0$

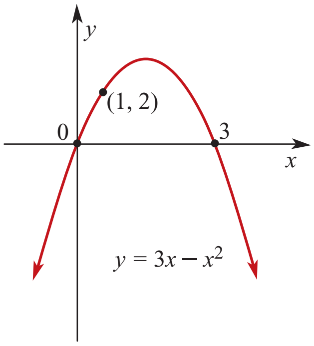
$\therefore x = 0$ or $x = 3$

i.e. x-intercepts are $(0, 0)$ and $(3, 0)$.

$x = 1 \qquad y = 3 \times 1 - 1^2$

$= 2$





Note: A sketch does not have to be drawn properly to scale.

The point $(1, 2)$ was found in part **b** because the graph passed through the origin.

Using the turning point form for quadratic functions $y = a(x - h)^2 + k$

When a quadratic equation is written in the form $y = a(x - h)^2 + k$, the turning point of the parabola is (h, k).

Step 1 Find the turning point by expressing the equation in turning point form by completing the square.

Step 2 Substitute $x = 0$ into the equation to find the y-intercept.

Step 3 Find a second point if the turning point is on the x-axis.

Step 4 Use the points found to sketch the curve.

Example 29

Express the following in the form $y = a(x - h)^2 + k$ and, hence, sketch the curve.

a $y = x^2 - 6x + 11$ **b** $y = 2x^2 + 10x + 7$

Solution

a $y = x^2 - 6x + 11$

$$\begin{aligned} y &= x^2 - 6x + 3^2 - 3^2 + 11 \\ &= (x - 3)^2 \qquad + 2 \end{aligned}$$

$\therefore$ turning point is (3, 2).

When $x = 0$, $y = 11$.

$\therefore$ y-intercept is 11.

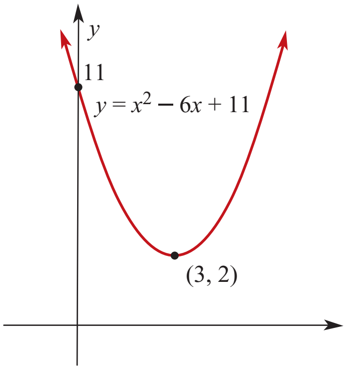
b $y = 2x^2 + 10x + 7$

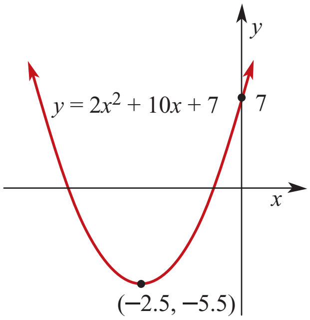
$$\begin{aligned} y &= 2\left(x^2 + 5x\right) + 7 \\ &= 2\left(x^2 + 5x + 2.5^2 - 2.5^2\right) + 7 \\ &= 2\left((x + 2.5)^2 - 6.25\right) + 7 \\ &= 2\,(x + 2.5)^2 - 12.5 + 7 \\ &= 2\,(x + 2.5)^2 - 5.5 \end{aligned}$$

$\therefore$ turning point is (−2.5, −5.5).

When $x = 0$, $y = 7$.

$\therefore$ y-intercept is 7.





Note: It is not necessary to show x-intercepts when using the turning point method to sketch, unless asked to do so.

Exercise 1F

Example 28 **1** Use the x- and y-intercept method to sketch the curves of the following:

a $y = x^2 + 6x + 8$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 - 3x - 10$

d $y = x^2 + 4x$ **e** $y = 2x^2 - x - 3$ **f** $y = 8 - 2x^2$

Example 29 **2** Express each of the following in the form $y = a(x - h)^2 + k$ and, hence, sketch the curve:

a $y = x^2 + 2x + 5$ **b** $y = x^2 - 4x + 6$ **c** $y = x^2 - 3x + 5$

d $y = x^2 + 6x$ **e** $y = 2x^2 - 4x + 9$ **f** $y = 3x^2 + 6x - 1$

g $y = x^2 + 6x + 9$ **h** $y = 3x^2 - 9x + 8$ **i** $y = 9x - 2x^2$

3 Sketch the following parabolas, showing the turning point and all points of intersection with the coordinate axes.

a $y = -2x^2 + 8$ **b** $y = 2(x - 2)(x + 3)$ **c** $y = 2(x - 2)^2 + 3$
d $y = x^2 - 6x + 8$ **e** $y = x^2 - 3x - 10$ **f** $y = x^2 - 5x$
g $y = 2x^2 + 4x$ **h** $y = 2x^2 - 5x - 12$ **i** $y = 3x - 3x^2$

MAPS **4** Sketch the following parabolas. State the exact coordinates of the turning point and all points of intersection with the coordinate axes.

a $y = 4x^2 - 3x$ **b** $y = 3x^2 - 8$ **c** $y = 3x^2 - 5x + 1$

1.7 Simultaneous equations

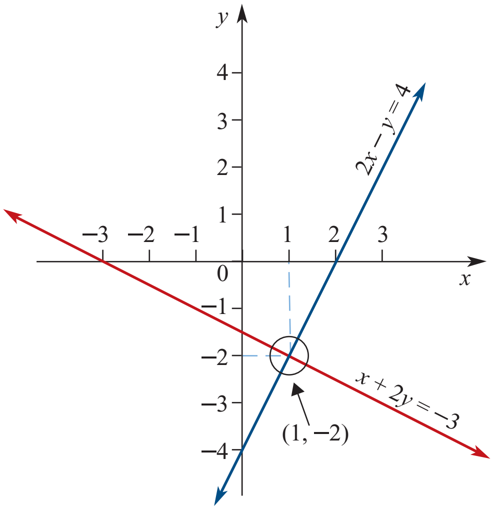
A linear equation that contains two unknowns; for example, $2y + 3x = 10$, does not have a single solution. Such an equation actually expresses a relationship between pairs of numbers, x and y, that satisfy the equation. If all the possible pairs of numbers (x, y) that will satisfy the equation are represented graphically, the result is a straight line. Hence, the name **linear relation**.

If the graphs of two such equations are drawn on the same set of axes, and they are not parallel, the lines will intersect at one point only. Hence, there is one pair of numbers that will satisfy both equations simultaneously.

Finding the intersection of two straight lines can be done graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, this point of intersection may be found algebraically by solving the pair of simultaneous equations.

Three techniques for solving simultaneous equations will be considered: substitution and elimination, both of which are algebraic methods; and graphical, using the graphics calculator.



Substitution

Step 1 Make y the subject of one of the equations.
Step 2 Substitute for y in the other equation.
Step 3 Solve the resulting equation for x.
Step 4 Substitute the resulting value of x back into either equation, to find the value of y.
Step 5 Check by substituting into the other equation.

Example 30

Solve the equations $2x - y = 4$ and $x + 2y = -3$ by substitution.

Solution

$$\begin{aligned} 2x - y &= 4 \,.........\, (1) \qquad \Rightarrow \qquad y = 2x - 4 \\ x + 2y &= -3 \,.......\, (2) \end{aligned}$$

Substituting (1) into (2) gives:

$$\begin{aligned} x + 2(2x - 4) &= -3 \\ x + 4x - 8 &= -3 \\ 5x - 8 &= -3 && (+\,8) \\ 5x &= 5 && (\div\,5) \\ x &= 1 \end{aligned}$$

Substituting $x = 1$ into equation 1:

$$\begin{aligned} y &= 2(1) - 4 \\ &= -2 \end{aligned}$$

Check in equation 2: $\text{LHS} = (1) + 2(-2) = -3$

$\text{RHS} = -3$

$\therefore x = 1$ and $y = -2$

Note: This means that the point (1, –2) is the point of intersection of the graphs of the two linear relations.

Example 31

Solve algebraically the equations $y = 5x - 4$ and $y = 3x + 7$.

Solution

Equating $y = 5x - 4$ and $y = 3x + 7$:

$$\begin{aligned} 5x - 4 &= 3x + 7 && (+\,4) \\ 5x &= 3x + 11 && (-\,3x) \\ 2x &= 11 && (\div\,2) \\ x &= 5.5 \end{aligned}$$

Substituting into equation 1 gives:

$$\begin{aligned} y &= 5(5.5) - 4 \\ &= 23.5 \end{aligned}$$

Check in equation 2: $y = 3(5.5) + 7 = 23.5$

$\therefore x = 5.5$ and $y = 23.5$

Example 32

Solve algebraically the equations $y = x^2 - 3$ and $y = 2x + 5$.

Solution

Equating $y = x^2 - 3$ and $y = 2x + 5$:

$$\begin{aligned} x^2 - 3 &= 2x + 5 \quad (-\,5) \\ x^2 - 8 &= 2x \qquad (-\,2x) \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \end{aligned}$$

$$\swarrow \qquad \downarrow \quad \text{(null factor theorem)}$$

$$\begin{aligned} x - 4 = 0 \quad &\text{or} \quad x + 2 = 0 \\ \therefore x = 4 \quad &\text{or} \quad x = -2 \end{aligned}$$

Substituting $x = 4$ into equation 1 gives $y = 13$, and substituting $x = -2$ into equation 1 gives $y = 1$.

$\therefore$ When $x = 4$, $y = 13$ and when $x = -2$, $y = 1$.

Note: The graphs of $y = x^2 - 3$ and $y = 2x + 5$ intersect at two points, (4, 13) and (−2, 1).

Elimination

Step 1 Compare the coefficients of x and compare the coefficients of y in each of the equations. If either the x coefficients are equal or the y coefficients are equal or they are the opposites of each other, then go to step 3.

Step 2 Create a new pair of equations so that either the x coefficients are equal or the y coefficients are equal. Do this by multiplying at least one of the equations by a number.

Step 3 Add or subtract the pair of equations to eliminate one of the variables.

Step 4 Solve the resulting equation.

Step 5 Substitute the resulting value back into either of the original equations to find the value of the other variable.

Step 6 Check by substituting both the x and y values into the other original equation.

Example 33

Solve by elimination the equations $2x + 5y = 7$ and $2x + 2y = 1$.

Solution

$$\begin{aligned}
& & 2x + 5y &= 7 \,.........\, (1)\\
& & 2x + 2y &= 1 \,.........\, (2)\\
&(1) - (2) & 3y &= 6 \qquad (\div 3)\\
& & y &= 2
\end{aligned}$$

Substituting $y = 2$ into equation 2 gives:

$$\begin{aligned}
2x + 2(2) &= 1\\
2x + 4 &= 1 \qquad (-\,6)\\
2x &= -3 \qquad (\div 2)\\
x &= -1.5
\end{aligned}$$

Check in equation 1: $2(-1.5) + 5(2) = 7$ ✓

$\therefore x = -1.5$ and $y = 2$

Note: Step 2 was skipped because the coefficients of x were equal. In step 3 subtracting the equations resulted in x being eliminated because the coefficients of x were equal.

Example 34

Solve by elimination the equations $2x + y = 11$ and $3x + 2y = 14$.

Solution

$$\begin{aligned}
& & 2x + y &= 11 \,.........\, (1)\\
& & 3x + 2y &= 14 \,.........\, (2)\\
&(1) \times 2 & 4x + 2y &= 22 \,.........\, (3)
\end{aligned}$$

(3) − (2) gives $x = 8$.

Substituting $x = 8$ into equation 1:

$$\begin{aligned}
2\,(8) + y &= 11\\
16 + y &= 11 \qquad (-\,16)\\
y &= -5
\end{aligned}$$

Check in equation 2: $3(8) + 2(-5) = 14$ ✓

$\therefore x = 8$ and $y = -5$

Note: In step 2, doubling equation 1 made the coefficients of y equal. This resulted in y being eliminated in step 3 when the equations were subtracted.

Example 35

Solve by elimination the equations $2x - 3y = -22$ and $5x + 2y = 2$.

Solution

$$\begin{array}{rlrl} & & 2x - 3y = -22 & \ldots\ldots(1) \\ & & 5x + 2y = 2 & \ldots\ldots\ldots(2) \\ (1) \times 2 & & 4x - 6y = -44 & \ldots\ldots(3) \\ (2) \times 3 & & 15x + 6y = 6 & \ldots\ldots\ldots(4) \\ (3) + (4) & \text{gives} & 19x = -38 & (\div 19) \\ & & x = -2 & \end{array}$$

Substituting $x = -2$ into equation 2 gives:

$$\begin{aligned} 5(-2) + 2y &= 2 \\ -10 + 2y &= 2 \qquad (+10) \\ 2y &= 12 \qquad (\div 2) \\ y &= 6 \end{aligned}$$

Check in equation 1: $2(-2) - 3(6) = -22$ ✓

$\therefore x = -2$ and $y = 6$

Note: In step 2, doubling equation 1 and trebling equation 2 made the coefficients of y the opposites of each other. This resulted in y being eliminated in step 3 when the equations were added.

Using technology

Note: Write each of the equations with y as the subject before using the graphics calculator.

Example 36

Use the graphics calculator to solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

$$\begin{array}{rcl} 2x - y = 4 \ldots\ldots(1) & \Rightarrow & y = 2x - 4 \\ x + 2y = -3 \ldots\ldots(2) & \Rightarrow & y = -\frac{1}{2}x - \frac{3}{2} \end{array}$$

Using the TI-Nspire:

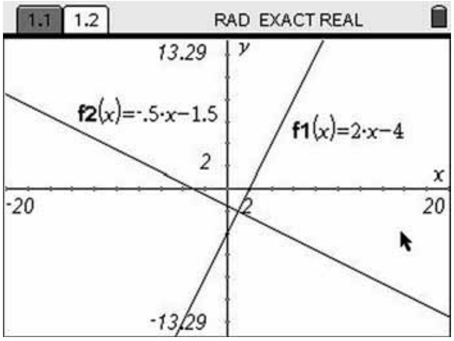
1 Select the Graphs & Geometry application.

Using the ClassPad:

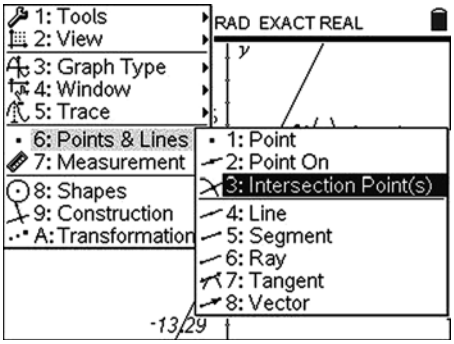
1 Select the Graphs and Tables application by tapping on Graph&Tab.. .

2 Type **2*x* – 4** into $f1(x)$ then press enter.

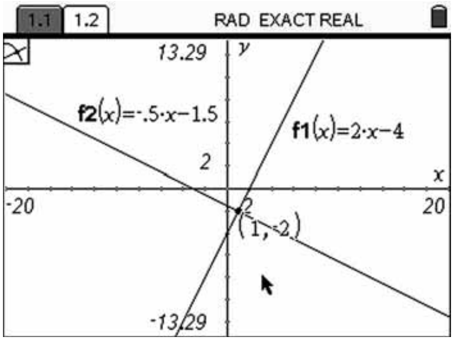
3 Type **–0.5*x* – 1.5** into $f2(x)$ then press enter.



To calculate the point of intersection press menu and select *Intersection Point(s)* from the Points & Lines submenu.



Move the cursor to the point of intersection to display its coordinates.

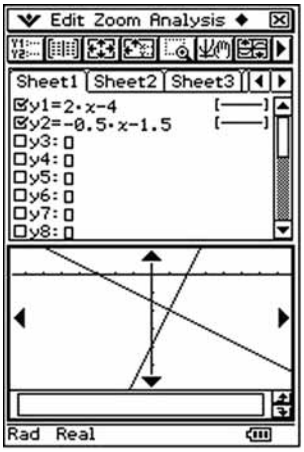


Point of intersection is (1, –2).

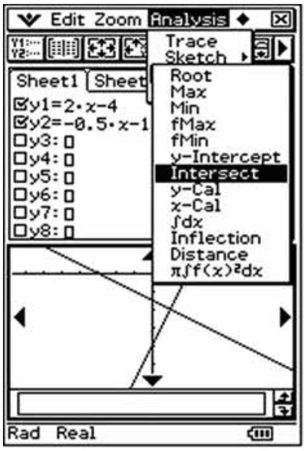
2 Type **2*x* – 4** into *y*1 and press EXE.

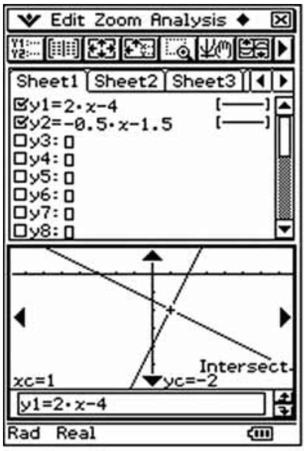
3 Type **– 0.5*x* – 1.5** into *y*2 and press EXE.

4 Tap ▦ to sketch the two functions.



To calculate the point of intersection tap Analysis and select *Intersect* from the G-Solve submenu.





Point of intersection is (1, –2).

Exercise 1G

Example 30

Example 31

1 Solve the following by substitution:

a $y = 2x + 1$
$y = 3x + 2$

b $y = 5x - 4$
$y = 3x + 6$

c $y = 2 - 3x$
$y = 5x + 10$

d $x + y = 6$
$x = y + 10$

e $y = x + 1$
$x + y = 7$

f $x - 2y = 6$
$x + 6y = 10$

Example 32

2 Solve algebraically:

a $y = 3x + 2$
$3x + y = 8$

b $2x - 5y = 10$
$4x + 3y = 7$

c $7x - 6y = 20$
$3x + 4y = 2$

d $t = 3s + 7$
$s = 2t - 9$

e $2p + 5q + 3 = 0$
$7p - 2q - 9 = 0$

f $2x - 4y + 12 = 0$
$2y + 3x - 2 = 0$

g $y = 2x + 4$
$y = x^2 + 1$

h $y = 2 - 2x$
$y = x^2 - 1$

i $y = 2x + 2$
$y = 1 - x^2$

j $y = 3x + 2$
$y = x^2 - 8$

k $y = x^2 + 2x - 4$
$y = x^2 - 2x$

l $y = 2x^2$
$y = 3x - x^2$

Examples 33–35

3 Solve the following by elimination:

a $6x + 2y = 14$
$2x + 2y = 2$

b $2x + 3y = 20$
$x - 3y = 1$

c $5x + 2y = 14$
$3x - 2y = 18$

d $2x + 4y = 26$
$2x + y = 17$

e $3x + 2y = 8$
$x - y = 6$

f $2x + 3y = 9$
$2y - x = -1$

g $5x + 3y = 19$
$x + y = 3$

h $2x + 3y = 0$
$5x - 2y = 19$

i $5x + 2y + 16 = 0$
$4x - 3y - 1 = 0$

Example 36

4 Solve the following using technology:

a $y = 2x$
$y = 3x - 2$

b $3x + y = 4$
$y = 6 - x$

c $3x + y = 8$
$x + 2y = 16$

d $y = 2x + 5$
$y = x^2 + 2x$

e $x + 2y = 5$
$y = x^2 - 3$

f $y = 5 - x^2$
$y = 3x^2 - 2x + 1$

1.8 The discriminant

When graphing quadratics it is apparent that the number of x-axis intercepts a parabola may have is:

i zero – graph is either all above or all below the x-axis.

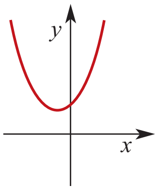
ii one – graph touches the x-axis; turning point is the x-axis intercept.

or **iii** two – graph crosses the x-axis.

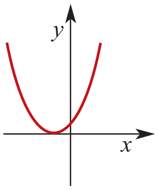
By considering the formula for the general solution to a quadratic equation, $ax^2 + bx + c = 0$, it can be established whether a parabola will have zero, one or two x-axis intercepts.

The expression $b^2 - 4ac$, which is part of the quadratic formula, is called the **discriminant**.

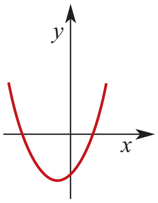
i If the discriminant, $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has zero solutions and the corresponding parabola will have no x-axis intercepts.



ii If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



iii If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



Note: Discriminant is denoted by the symbol Δ.

Example 37

Write the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

a

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-6)^2 - (4 \times 1 \times 8)\\ &= 4\end{aligned}$$

As $\Delta > 0$, the graph intersects the x-axis at two distinct points.

b

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-8)^2 - (4 \times 1 \times 16)\\ &= 0\end{aligned}$$

As $\Delta = 0$, the graph touches the x-axis at one point.

c

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - (4 \times 2 \times 4)\\ &= -23\end{aligned}$$

As $\Delta < 0$, the graph does not intersect the x-axis.

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation. Assuming a, b, c are rational and $\Delta \geq 0$:

- If $\Delta = b^2 - 4ac$ is a perfect square, which is not zero, then the quadratic equation has two rational solutions.
- If $\Delta = b^2 - 4ac = 0$, the quadratic equation has one rational solution.
- If $\Delta = b^2 - 4ac$ is not a perfect square, then the quadratic equation has two irrational solutions.

Extension exercise 1H

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 - 4$
d $2x^2 + 3x$ **e** $-2x^2 + 3x + 4$

Example 37 **2** Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis.

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$
d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - 2x - 1$

3 By examining the discriminant, find the number of roots of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $x^2 - 4x + 4 = 0$
d $2x^2 + 8x - 7 = 0$ **e** $4x^2 - 12x + 9 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of roots for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$
d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 10 = 0$

MAPS **5** Find the discriminant for the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number and, hence, show that the equation has rational solution(s).

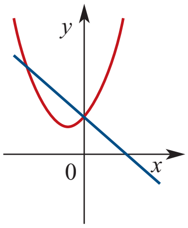
MAPS **6** Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions

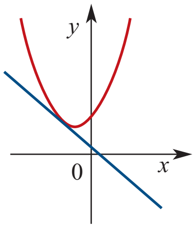
1.9 More on points of intersection

To find the point or points of intersection between a straight line and a parabola, solve the equations simultaneously, as discussed earlier in this chapter.

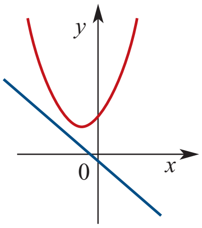
It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola, two, one or zero points of intersection occur, respectively.



Two points of intersection



One point of intersection



No point of intersection

If there is one point of intersection between the parabola and the straight line then the line is a **tangent** to the parabola.

Example 38

Show that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

Solution

At the point of intersection:

$$\begin{aligned} x^2 - 3x + 2 &= 1 - x \\ x^2 - 2x + 1 &= 0 \\ (x - 1)^2 &= 0 \\ \therefore x &= 1 \end{aligned}$$

There is only one solution to the pair of equations.
$\therefore$ The straight line just touches the parabola.

Alternatively:

$$\begin{aligned} x^2 - 3x + 2 &= 1 - x \\ x^2 - 2x + 1 &= 0 \\ \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4 \times 1 \times 1 \\ &= 0 \end{aligned}$$

$\therefore$ There is only one solution to the pair of equations.
$\therefore$ The straight line just touches the parabola.

Exercise 1I

Example 38

1 Show that, for the pairs of equations given, the straight line is a tangent to the parabola.

- **a** $y = x^2 - 6x + 8$
  $y = -2x + 4$
- **b** $y = x^2 - 2x + 6$
  $y = 4x - 3$
- **c** $y = 2x^2 + 11x + 10$
  $y = 3x + 2$
- **d** $y = x^2 + 7x + 4$
  $y = -x - 12$

2 **a** Find the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$.

b Find the values of m for which the straight line $y = mx - \frac{1}{2}$ is tangent to the parabola $y = 2x^2 + 5x$.

c Find the value of c such that the line with equation $y = 2x + c$ is a tangent to the parabola with equation $y = x^2 + 3x$.

3 **a** Solve for a such that the line with equation $y = x$ is tangent to the parabola $y = x^2 + ax + 1$.

b Solve for b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

c Solve for c such that the line with equation $y = 2x + c$ intersects twice with the parabola $y = x^2 + 3x$.

1.10 Finding the equation of a straight line

The equation of a straight line can be found if the gradient, m, and y-axis intercept are known.

Example 39

Find the equation of the straight line with $m = -3$ and $c = 10$.

Solution

$$y = -3x + 10 \qquad (y = mx + c)$$

Example 40

Find the equation of the straight line with gradient -3 that passes through the point $(-5, 10)$.

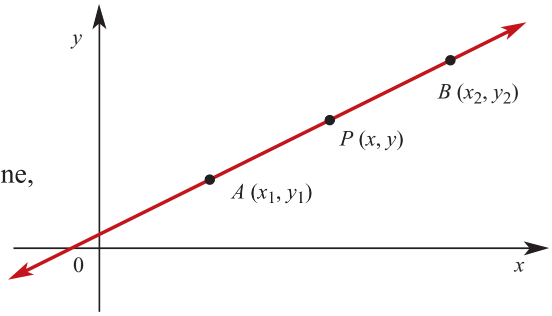
Solution

$$\begin{aligned} y &= mx + c \\ y &= -3x + c && \text{(gradient given as } -3) \\ \text{When } x = -5,\ y &= 10 && \text{(given that coordinate } (-5, 10) \text{ lies on the line)} \\ \therefore \quad 10 &= -3 \times -5 + c \\ \therefore \quad 10 &= 15 + c && (-\,15) \\ \therefore \quad c &= -5 \end{aligned}$$

The equation of the line is $y = -3x - 5$.

In general, the equation of a straight line can be determined through two ‘independent pieces of information’. Two cases will now be considered.

Case 1: Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$



Using these two points, the gradient of the line *AB* can be determined:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Using the general point $P(x, y)$, also on the line,

$$m = \frac{y - y_1}{x - x_1}$$

Therefore, the equation of the line is

$$y - y_1 = m\,(x - x_1)\,,\ \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 41

Find the equation of the straight line passing through the points (1, –2) and (3, 2).

Solution

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-(-2)}{3-1} \\ &= \frac{4}{2} \\ &= 2 \\ y - y_1 &= m\,(x - x_1) \\ m &= 2 \text{ and the line passes through } (1, -2)\,. \\ \therefore y - (-2) &= 2\,(x - 1) \\ \therefore y + 2 &= 2x - 2 \qquad (-\,2) \\ \therefore y &= 2x - 4 \end{aligned}$$

Case 2: Given the gradient *m* and one other point, $A(x_1, y_1)$

As the gradient, *m*, is already known, the rule can be found using:

$$y - y_1 = m(x - x_1)$$

Example 42

Find the equation of the line that passes through the point (3, 2) and has a gradient of −4. Express the answer in general form.

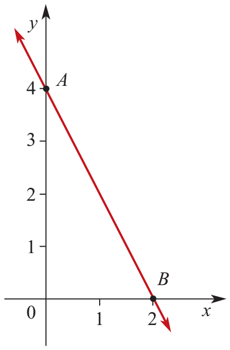
Solution

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ m &= -4 \text{ and the line passes through } (3, 2). \\ y - 2 &= -4(x - 3) \\ y - 2 &= -4x + 12 \qquad (+\,2) \\ y &= -4x + 14 \\ y &= -4x + 14 \text{ is the equation.} \end{aligned}$$

In general form, this is $4x + y - 14 = 0$.

Example 43

Find the equation of the line shown in the graph.



Solution

y-intercept is (0, 4); that is, $c = 4$.
A is (0, 4) and B is (2, 0).

$$\begin{aligned} \text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - 4}{2 - 0} \\ &= -\frac{4}{2} \\ &= -2 \end{aligned}$$

∴ The equation of the line is $y = -2x + 4$. $\qquad (y = mx + c)$

Vertical and horizontal lines

If $m = 0$, then the line is **horizontal** and the equation is simply $y = c$, where c is the y-axis intercept. Note that the equation of a horizontal line is in the form $y = mx + c$, however, $m = 0$. If the line is **vertical**, the gradient is undefined and its rule is given as $x = a$, where a is the x-axis intercept. Note that the equation of a vertical line is not in the form $y = mx + c$.

Example 44

Find the equation of the line passing through:

a (2, 5) and (4, 5) **b** (3, 1) and (3, –2)

Solution

a

$$\begin{aligned}\text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{5-5}{4-2}\\ &= \frac{0}{2}\\ &= 0\end{aligned}$$

$\therefore$ The equation of the line is $y = 5$. $(y = c)$

b

$$\begin{aligned}\text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2-1}{3-3}\\ &= \frac{-3}{0}\\ &= \text{Undefined}\end{aligned}$$

$\therefore$ The equation of the line is $x = 3$. $(x = a)$

Exercise 1J

Example 39 **1** **a** Find the equation of the straight line with gradient 3 and y-axis intercept 5.

b Find the equation of the straight line with gradient -4 and y-axis intercept 6.

c Find the equation of the straight line with gradient 3 and y-axis intercept -4.

Example 40 **2** Find the equation of the straight line with:

a gradient 3 and which passes through the point (6, 7).

b gradient -2 and which passes through the point (1, 7).

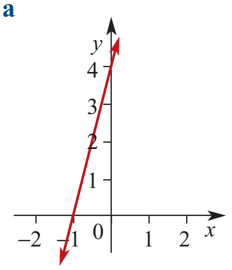
3 Find the equation of the straight line with:

a y-intercept 6 and passing through the point (1, 8).

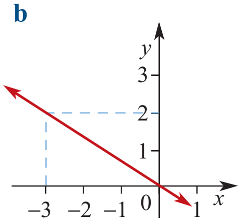
b y-intercept -2 and passing through the point (4, 8).

Examples 41–43 **4** Find the equation of each of the following lines. Express the answer in general form.

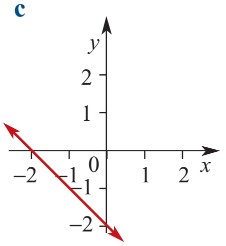
a



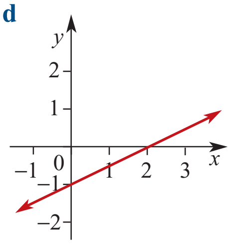
b



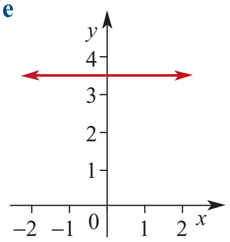
c



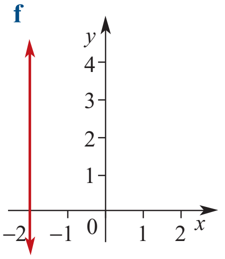
d



e



f



Examples 40–44

5 Find the equation of the line that passes through the pair of points:

a (0, 4), (6, 0)	**b** (−3, 0), (0, −6)	**c** (2, 6), (5, 3)
d (−2, 3), (4, 0)	**e** (1, 0), (4, 2)	**f** (3, 0), (3, 5)
g (−1.5, 2), (4.5, 8)	**h** (2, 5), (3, 5)	**i** (−3, 1.75), (4.5, −2)

MAPS

6 Do the points $P(1, -3)$, $Q(2, 1)$ and $R(2\frac{1}{2}, 3)$ lie on the same straight line?

7 Find the equations defining each of the three sides of the triangle ABC, where coordinates of the vertices are $A(-2, -1)$, $B(4, 3)$ and $C(6, 0)$.

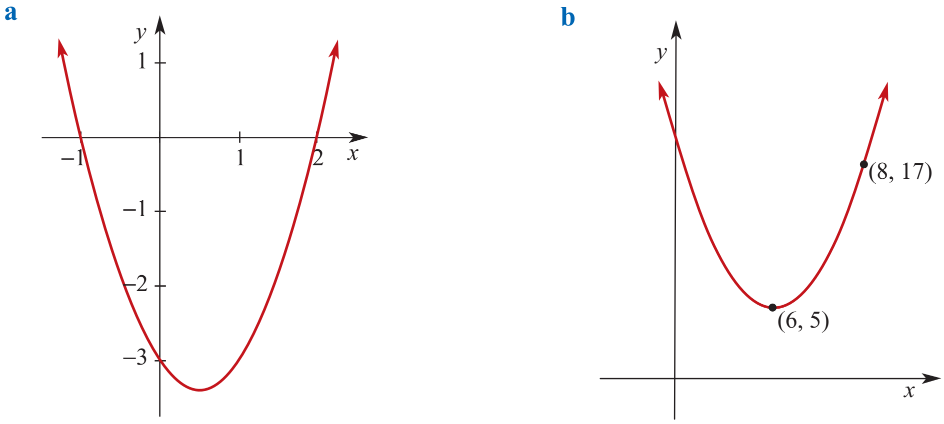
1.11 Determining quadratic equations

It is possible to find the quadratic rule to fit given points. Although the general form of a quadratic equation is $y = ax^2 + bx + c$, there are two other forms that can be useful to find the equation of a quadratic. These are:

- The x-intercept form $y = a(x - e)(x - f)$, where e and f are the x-intercepts.
- The turning point form $y = a(x - h)^2 + k$, where (h, k) is the turning point.

Example 45

Determine the quadratic rule for each of the graphs, assuming each is a parabola.



Solution

a

$$y = a(x - e)(x - f)$$
$$e = -1,\ f = 2$$
$$\therefore y = a(x - -1)(x - 2)$$
$$\text{i.e. } y = a(x + 1)(x - 2)$$

The curve also passes through (0, −3).

$$\therefore -3 = a(0 + 1)(0 - 2)$$
$$\text{i.e. } -3 = -2a \qquad (\div -2)$$
$$\therefore a = 1.5$$
$$\therefore y = 1.5(x + 1)(x - 2)$$

b

$$y = a(x - h)^2 + k$$

Turning point is (6, 5).

$$\therefore y = a(x - 6)^2 + 5$$

The graph also passes through (8, 17).

$$\therefore 17 = a(8 - 6)^2 + 5$$
$$\therefore 17 = 4a + 5 \qquad (-\ 5)$$
$$\therefore 12 = 4a \qquad (\div 4)$$
$$\therefore a = 3$$
$$\therefore y = 3(x - 6)^2 + 5$$

Using technology

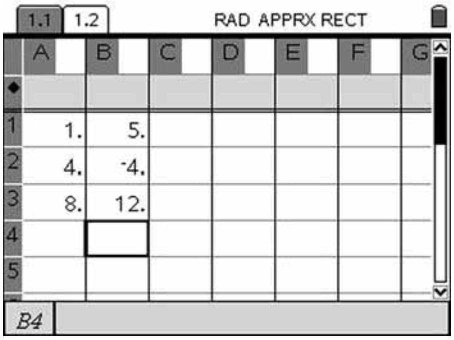
Example 46

Find the equation of the parabola passing through the points (1, 5), (4, –4) and (8, 12).

Solution

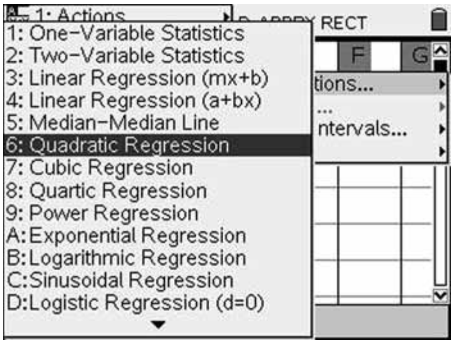
Using the TI-Nspire:

1. Press ⌂ and enter into the Lists & Spreadsheet application.
2. Type the *x* coordinates **1**, **4** and **8** into the first column.
3. Type the *y* coordinates **5**, **– 4** and **12** into the second column.



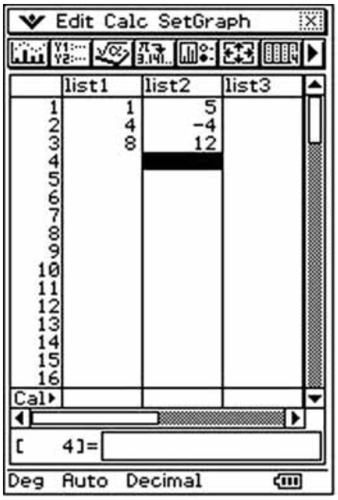
	A	B	C	D	E	F	G
1	1.	5.					
2	4.	⁻4.					
3	8.	12.					
4							
5							

4. To highlight both columns move the cursor to the extreme top of column A until it is highlighted. Press and hold the ⇧ key, then press the right arrow key.
5. To perform a Quadratic regression on the highlighted data, press menu and navigate as follows: 4:Statistics, 1:Stat Calculations, 6:Quadratic Regression.



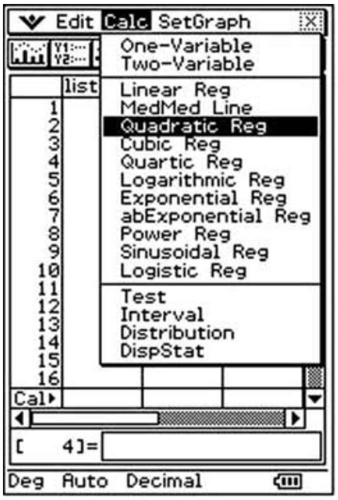
Using the ClassPad:

1. Enter into the Statistics application.
2. Type the *x* coordinates **1**, **4** and **8** into list1.
3. Type the *y* coordinates **5**, **– 4** and **12** into list2.

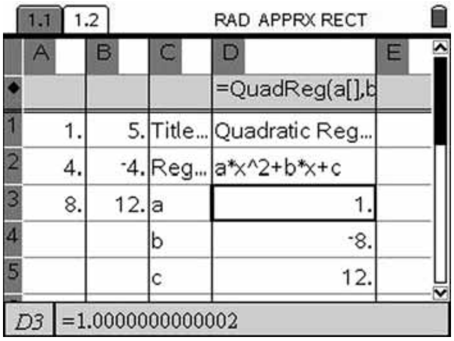


	list1	list2	list3
1	1	5	
2	4	−4	
3	8	12	

4. To perform a Quadratic regression on the data, enter into the Calc menu and then tap Quadratic Reg.



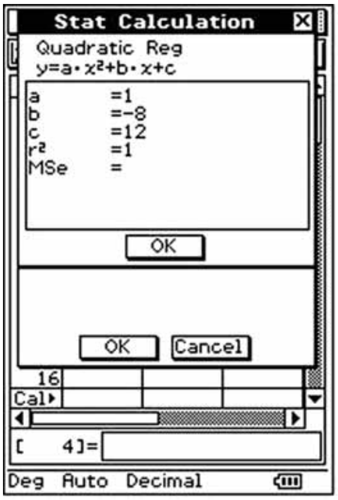
6 Press (enter) twice.



	A	B	C	D	E
◆				=QuadReg(a[],b	
1	1.	5.	Title...	Quadratic Reg...	
2	4.	-4.	Reg...	a*x^2+b*x+c	
3	8.	12.	a	1.	
4			b	-8.	
5			c	12.	

D3 =1.0000000000002

7 Ensure the following is set: XList: list1 and YList: list2. Tap OK.



Quadratic Reg
y=a·x²+b·x+c

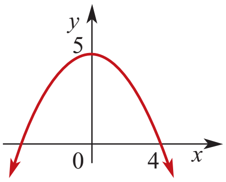
a =1
b =-8
c =12
r² =1
MSe =

Thus, the equation of the parabola is $y = x^2 - 8x + 12$.

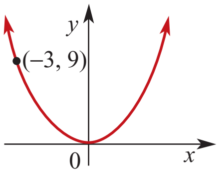
Exercise 1K

Example 45 **1** Determine the equation of each of the following parabolas:

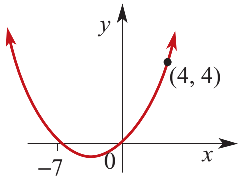
a



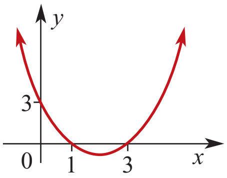
b



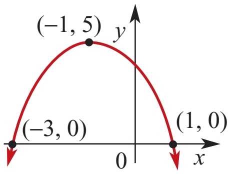
c



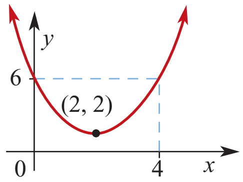
d



e



f



Example 46 **2** Find the equation of the parabola with:

a x-intercepts 2 and −4 and passes through the point (3, 10).

b x-intercepts 0 and 4 and passes through the point (1, 12).

c x-intercepts 2 and 5 and passes through the point (0, 8).

Example 46 **3** Find the equation of the parabola with:

a vertex at (−1, 2) and passes through the point (3, 10).

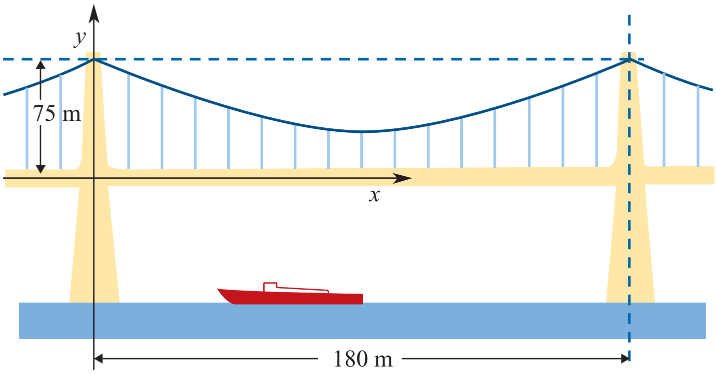
b vertex at (2, 3) and passes through the point (4, −3).

c vertex at (0, 0) and passes through the point (3, 12).

Example 46 **4** Find the equation of the parabola passing through the points:

- **a** (−1, 2), (1, 4) and (2, −1)
- **b** (−2, 5), (0, 3) and (2, 3)
- **c** (−2, 30), (1, −3) and (4, 36)

5 Assuming that the suspension cable shown in the diagram forms a parabola, find the rule that describes its shape. The minimum height of the cable above the roadway is 30 m.

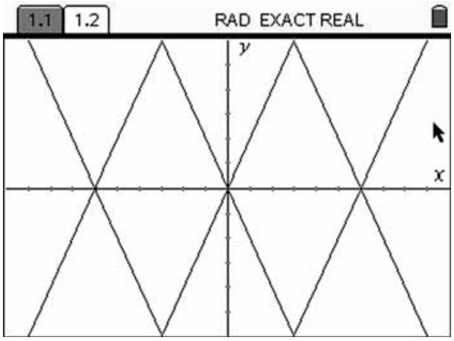


1.12 Modelling and problem solving

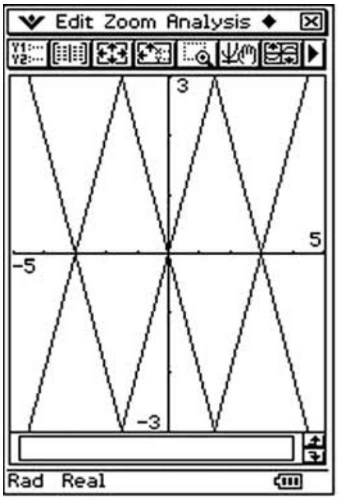
Exercise 1L

1 Produce these screens using the graphics calculator.

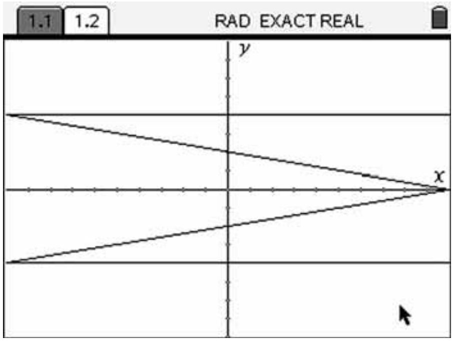
a Using the TI-Nspire

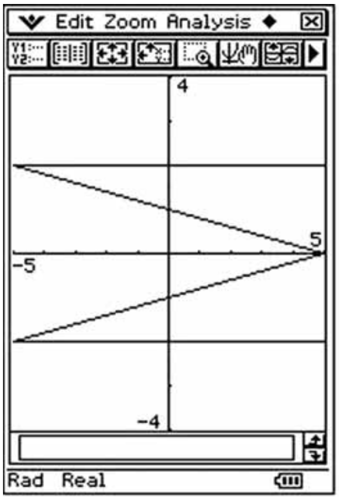


Using the ClassPad



b





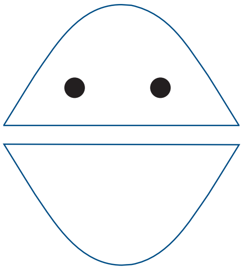
2 The correct conversion formula for degrees Celsius to degrees Fahrenheit is $F = \frac{9}{5}C + 32$. The Celsius scale for measuring temperature was adopted in Australia in the 1970s. At that time the public were advised to use the method 'Double it and add thirty' to make conversions of degrees Celsius to degrees Fahrenheit. Although the method was only approximate, it worked reasonably well. Actually, it works correctly for one particular value of C.

- **a** Draw accurate graphs of the conversion formula and the method the public were advised to use on the same set of coordinate axes. Hence, find the values of C and F for which both methods give the same answer.
- **b** At what temperature do both the Celsius and Fahrenheit scales give the same value?

3 In the television program *South Park*, Timmy and Phillip, look like this picture. Using the relations

$$y = 5,\ y = -5,\ y = \frac{x^2}{25} - 30 \quad \text{and} \quad y = 30 - \frac{x^2}{25}$$

- **a** Draw a neat, accurate picture of a 'Canadian' on grid paper.
- **b** Produce the image of a 'Canadian' on your graphics calculator screen.



4 A polygon with n sides has $\frac{n(n-3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

5 Solve for t:

a $\frac{a - t}{b - t} = c$ **b** $\frac{at + b}{ct - b} = 1$

6 Solve $\dfrac{a}{x+a} + \dfrac{b}{x-b} = \dfrac{a+b}{x+c}$ for x.

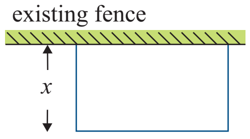
7 Tom leaves town A and rides at a constant speed towards town B. Julie leaves town B at the same time that Tom leaves town A, and travels towards town A. Julie rides at a speed that is 5 km/h faster than Tom. Town B is 100 km from town A. They meet after 4 hours. How far has Julie travelled at the time that they meet?

8 For a particular electric train the tractive 'resistance', R, at speed, V km/h is given by $R = 1.6 + 0.03V + 0.003V^2$. Find V when the tractive 'resistance' is 10.6.

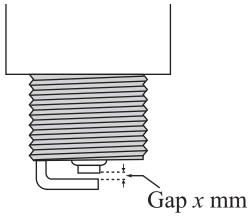
9 The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate algebraically the length and width of the rectangle.

10 The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the length of the base algebraically.

11 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence. If the width of paddock is x m and the area inside the yard is A m^2, sketch the graph of A against x.

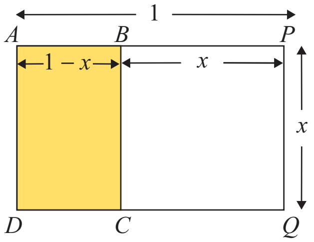


12 The efficiency rating, E, of the particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$. Determine the values of x for which the efficiency rating is 70 or better.



13 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other to form a rectangle shape of which the length is twice its width. Find the length of the side of the square if the combined area of the two shapes is 4.25 cm^2.

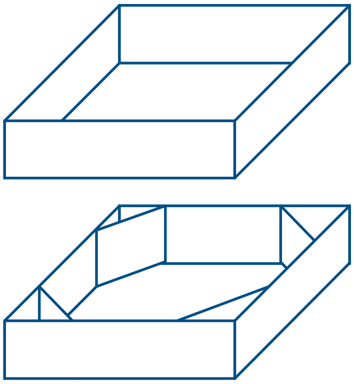
14 A shape that has been of interest to architects and artists over the centuries is the 'golden rectangle'. Many have thought that it gave the perfect proportions for buildings. The rectangle is such that if a square is drawn on one of the longer sides then the new rectangle is similar to the original; that is, rectangle $ABCD$ is similar to rectangle $APQD$. Find the value of x. (This is known as the 'golden ratio'.)



Extension **15** **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution?

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions?

c Show that there are real solutions of the equation $4mx^2 + 4(m-1)x + m - 2 = 0$ for all real values of x.



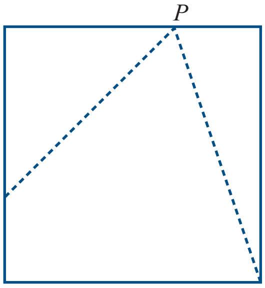
16 The pastry cook has decided to cook an octagonal cake, however, he only has a square cake tin, which is 24 cm × 24 cm × 7 cm. He has decided to cut four rectangular pieces of cardboard, one for each corner, in order to make the interior of the tin octagonal. He wants the cake to be a regular octagon, and so the pieces of card must be just the right length.

Find the size that each piece of cardboard must be to have the cake turn out as a regular octagon.

17 Using a square sheet of paper:

- Choose a point, P, on the top edge.
- Join P to the bottom right-hand corner.
- Create a 45° triangle in the top left-hand corner, using P.

What position, of the original point P, will make the area of the quadrilateral a maximum?



Chapter summary

- An equation is solved by finding the value or values of the unknowns that will make the statement true.
- When solving equations, perform the same calculation to both sides of the equation.
- An equation for the variable x in which the coefficients of x, including the constants, are pronumerals is known as a literal equation.
- When solving equations with fractions in them, multiply both sides of the equation by a common denominator of the fractions.
- When multiplying a fraction by a number, only the top line of the fraction gets multiplied.
- When factorising, remove the highest common factor whenever it is possible.
- The null factor theorem states: If $ab = 0$ then either $a = 0$ or $b = 0$.
- When solving quadratic equations:
  If x appears once, then use the isolating method of solution.

  If x appears twice, then write the equation in the form $ax^2 + bx + c = 0$ and use either factorising or the quadratic formula.
- The quadratic formula states: If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The fundamental method that works when graphing all functions is to make a table of values.
- Functions of the form $y = mx + c$ and $ax + by = c$ are said to be linear.
- Functions of the form $y = ax^2 + bx + c$ are said to be quadratic.
- When graphing linear equations in the form $y = mx + c$, the gradient is m and the y-intercept is c.
- Gradient $= m = \dfrac{\text{Rise}}{\text{Run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- To find the y-intercept, let $x = 0$ and solve for y.
  To find the x-intercept, let $y = 0$ and solve for x.
- Lines of the form $x = c$ are parallel to the y-axis, and lines of the form $y = c$ are parallel to the x-axis.
- The line $x = 0$ is the y-axis, and the line $y = 0$ is the x-axis.
- To find the equation of a straight line use $y - y_1 = m(x - x_1)$.
- When a quadratic equation is written in the form $y = a(x - e)(x - f)$, the x-intercepts are e and f.
- When a quadratic equation is written in the form $y = a(x - h)^2 + k$, the turning point is (h, k).
- Discriminant $\Delta = b^2 - 4ac$.
- If $\Delta > 0$, then the quadratic equation has two solutions and the parabola crosses the x-axis in two places.
- If $\Delta = 0$, then the quadratic equation has one solution and the turning point of the parabola is on the x-axis.

- If $\Delta < 0$, then the quadratic equation has no solutions and the parabola does not cross the x-axis.
- If there is one point of intersection between a parabola and a straight line then the line is a tangent to the parabola.

Multiple-choice questions

1 The solution of the equation $x - 8 = 3x - 16$ is:

A $x = -\frac{8}{3}$ **B** $x = \frac{11}{3}$ **C** $x = 4$ **D** $x = 2$ **E** $x = -2$

2 The solution of the simultaneous equations $2x - y = 10$ and $x + 2y = 0$ is:

A $x = -2$ and $y = 3$ **B** $x = 2$ and $y = -3$ **C** $x = 4$ and $y = -2$

D $x = 6$ and $y = 2$ **E** $x = 1$ and $y = -8$

3 If $A = \frac{hw + k}{w}$ then:

A $w = \frac{k}{A - h}$ **B** $w = \frac{ht + k}{A}$ **C** $w = \frac{A - 2k}{2h}$

D $w = \frac{3Ah}{2} - k$ **E** $w = \frac{2}{3}h(A + k)$

4 The gradient of the line passing through the points $(5, -8)$ and $(6, -10)$ is:

A -2 **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $-\frac{1}{18}$ **E** $\frac{3}{2}$

5 The equation of a straight line with gradient 3 and passing through the point $(1, 9)$ is:

A $y = x + 9$ **B** $y = 3x + 9$ **C** $y = 3x + 6$

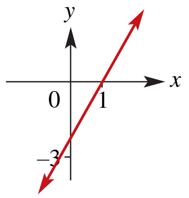
D $y = -\frac{1}{3}x + 1$ **E** $y = -\frac{1}{3}x + 6$

6 The relation with graph as shown has rule:

A $y = -3x - 3$ **B** $y = -\frac{1}{3}x - 3$

C $y = \frac{1}{3}x - 3$ **D** $y = 3x + 3$

E $y = 3x - 3$



7 The linear factors of $12x^2 + 7x - 12$ are:

A $4x - 3$ and $3x + 4$ **B** $3x - 4$ and $4x + 3$ **C** $3x - 2$ and $4x + 6$

D $3x + 2$ and $4x - 6$ **E** $6x + 4$ and $2x - 3$

8 The solutions of the equation $x^2 - 56 = x$ are:

A $x = -8$ or 7 **B** $x = -7$ or 8 **C** $x = 7$ or 8 **D** $x = -9$ or 6

E $x = 9$ or -6

9 The equation $y = 5x^2 - 10x - 2$ in turning point form $y = a(x - h)^2 + k$, by completing the square, is:

A $y = (5x + 1)^2 + 5$ **B** $y = (5x - 1)^2 - 5$ **C** $y = 5(x - 1)^2 - 5$
D $y = 5(x + 1)^2 - 2$ **E** $y = 5(x - 1)^2 - 7$

10 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two real roots is (are):

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

Short-response questions

1 Solve:

a $2x + 6 = 8$ **b** $3 - 2x = 6$ **c** $2x + 5 = 3 - x$ **d** $\frac{3 - x}{5} = 6$
e $\frac{x}{3} = 4$ **f** $\frac{13x}{4} - 1 = 10$ **g** $\frac{3x + 2}{5} - \frac{4 - x}{2} = 5$

2 Solve each of the following for t:

a $a - t = b$ **b** $\frac{at + b}{c} = d$ **c** $a(t - c) = d$

3 Factorise:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$
d $36 - x^2$ **e** $x^2 + x - 12$ **f** $x^2 + x - 2$
g $2x^2 + 3x - 2$ **h** $6x^2 + 7x + 2$ **i** $6x^2 - 13x - 15$

4 Solve:

a $2x^2 = 50$ **b** $x^2 - 33 = 0$ **c** $2(x^2 + 12) = 122$

5 Solve for x by factorising:

a $x^2 + x - 12 = 0$ **b** $x^2 + 11x + 24 = 0$ **c** $x(x - 5) = 66$
d $2x^2 - 13x + 15 = 0$ **e** $6x^2 + 7x = 20$ **f** $2x^2 = x + 6$

6 Use the quadratic formula to solve:

a $x^2 - 6x + 3 = 0$ **b** $2x^2 + 7x + 4 = 0$ **c** $3x^2 + 9x - 1 = 0$

7 Create a table of values and, hence, graph accurately on the number plane:

a $y = 3x + 4$ **b** $y = x^2 - 5x + 2$ **c** $2x - 3y = 7$

8 Use the gradient and y-intercept method to graph accurately on the number plane:

a $y = \frac{1}{2}x + 3$ **b** $y = 4x - 2$ **c** $y = \frac{3}{4}x$

9 Use the x- and y-intercept method to graph accurately on the number plane:

a $x - 2y = 6$ **b** $3x + y = 9$ **c** $5x + 2y - 14 = 0$

10 Sketch on the number plane:

a $y = 3x - 6$ **b** $y = -2x + 3$ **c** $3x - 5y + 30 = 0$

11 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = 2(x - 4)^2 - 3$ **c** $y = (3 - x)(x + 2)$

12 Express in the form $y = a(x - h)^2 + k$ and, hence, sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = 3x - x^2$ **c** $y = 2x^2 - 8x + 3$

13 Sketch the graphs of each of the following, showing all points of intersection with the coordinate axes and the turning point:

a $y = x^2 - 7x + 6$ **b** $y = 4x^2 - 25$ **c** $y = 6x^2 - 13x - 5$

14 Solve each of the following, using substitution:

a $y = 2x + 4$, $y = 3x + 5$
b $y - x = 5$, $x = 3 - y$
c $x - 3y = 1$, $x + 6y = 10$

15 Solve each of the following, using elimination:

a $6x + 3y = 12$, $2x + 3y = 2$
b $5x + 2y = 16$, $x - y = 6$
c $3x + 6y = 0$, $5x - 4y - 7 = 0$

16 Solve algebraically:

a $y = 2x + 5$, $4x + y = 8$
b $2x - 5y = 10$, $2y - 4x = 4$
c $y = x^2 + 2x - 9$, $y = 2x - x^2$

17 Solve the following, using technology:

a $y = 2x$, $y = 3x - 2$
b $y = x^2 - 2x + 5$, $y = 2x - x^2$
c $4x + y = 8$, $x - 3y = 16$

18 Find the coordinates of the points of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$ **b** $y = 5x + 12$ and $y = 2x^2$

19 By examining the discriminant, state the nature and number of roots for each of the following:

Extension

a $x^2 - 4x + 2 = 0$ **b** $-x^2 + 6x - 9 = 0$ **c** $3x^2 - x + 4 = 0$

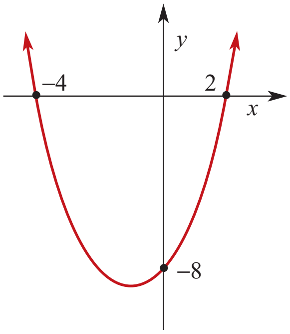
20 Show that, for the pairs of equations given, the straight line is a tangent to the parabola.

Extension

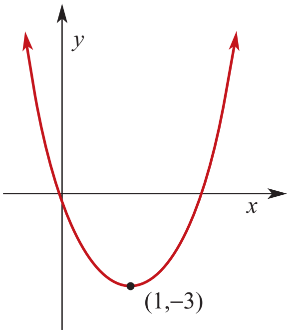
a $y = x^2 + 2$, $y = -2x + 1$
b $y = 4x^2 - 8x$, $y = 12x - 25$

21 Find the equation of each of the following parabolas:

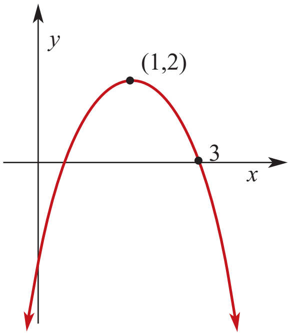
a



b



c



22 Find the equation of the parabola:

a with x-intercepts -1 and 3 and passes through the point $(0, -4)$.
b with vertex at $(3, 2)$ and passes through the point $(4, 0)$.
c passing through $(0, -7)$, $(1, 0)$ and $(3, 32)$.

CHAPTER

2

Trigonometric ratios and applications

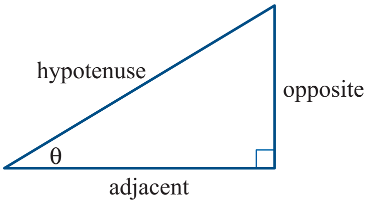
Objectives

- To solve practical problems using the **trigonometric ratios sine, cosine and tangent**.
- To use the **sine rule** and the **cosine rule** to solve problems.
- To solve problems involving **angles of depression, angles of elevation and bearings**.
- To understand the connection between **degrees and radians**.
- To determine exact trigonometric ratios for **special angles 30°, 45° and 60°**.
- To use the **unit circle** to determine trigonometric ratios of angles of any magnitude.
- To **apply trigonometry** to a range of modelling and problem solving contexts.

2.1 Defining sine, cosine and tangent

In Year 10 mathematics, the basic definitions of the sine, cosine and tangent ratios of an angle within the context of a right-angled triangle were introduced:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}}\left(\frac{\text{opposite}}{\text{adjacent}}\right)$$



Students will have also studied Pythagoras' theorem in previous years and are reminded of it here: 'In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.'

$$a^2 = b^2 + c^2$$

In the context of the triangle above:

$$\textit{Hypotenuse}^2 = \textit{Opposite}^2 + \textit{Adjacent}^2$$

Degrees, minutes and seconds

Students will be aware that hours are broken into minutes and seconds when smaller divisions and more accuracy is needed. For example, $\frac{1}{4}$ hour = 15 minutes, $\frac{1}{2}$ minute = 30 seconds. The same convention is used when measuring angles. Fractions of a degree can be written in minutes and seconds in the same way that fractions of an hour can be written in minutes and seconds.

$2\frac{1}{4}$ h = 2 : 15 h (Read as 2 hours 15 minutes.)

$2\frac{1}{4}^\circ = 2^\circ\ 15'$ (Read as 2 degrees 15 minutes.)

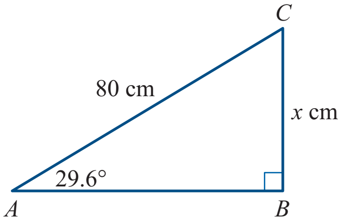
1 : 46 : 23 h (Read as 1 hour 46 minutes 23 seconds.)

$1^\circ\ 46'\ 23''$ (Read as 1 degree 46 minutes 23 seconds.)

It is worth noting that 2.41 cm rounds off to 2 cm because $0.41 < 0.5$; however, 2:41 hours rounds off to 3 hours because $0{:}41 > 0.5$ h. $2^\circ\ 41'$ also rounds off to 3° for the same reason.

Example 1

Find the value of x, correct to 2 decimal places.

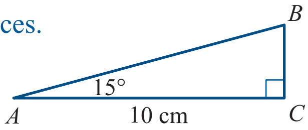


Solution

$$\begin{aligned}\sin\theta &= \frac{opp}{hyp}\\ \sin 29.6^\circ &= \frac{x}{80}\\ \therefore\quad x &= 80\sin 29.6^\circ\\ &= 39.5153\ldots\\ \therefore\quad x &\approx 39.52\text{ cm}\end{aligned}$$

Example 2

Find the length of the hypotenuse, correct to 2 decimal places.



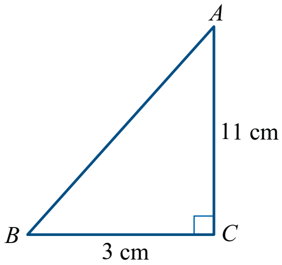
Solution

$$\begin{aligned}\cos\theta &= \frac{adj}{hyp}\\ \cos 15^\circ &= \frac{10}{AB}\\ \therefore\quad 10 &= AB\cos 15^\circ\\ \therefore\quad AB &= \frac{10}{\cos 15^\circ}\\ &= 10.3527\ldots\end{aligned}$$

The length of the hypotenuse = 10.35 cm, correct to 2 decimal places.

Example 3

Find the magnitude (i.e. size) of $\angle ABC$.
Give your answer correct to the nearest minute.



Solution

$$\begin{aligned}\text{Let } x &= \angle ABC\\ \tan\theta &= \frac{opp}{adj}\\ \tan x &= \frac{11}{3}\\ x &= \tan^{-1}\left(\frac{11}{3}\right)\\ \therefore\quad x &= 74.74488\ldots\\ \therefore\quad x &\approx 74^\circ 45\\ \angle ABC &= 74^\circ\ 45',\ \text{correct to the nearest minute.}\end{aligned}$$

Using technology

Note: On CAS calculators decimal degrees can be readily converted to degrees, minutes and seconds.

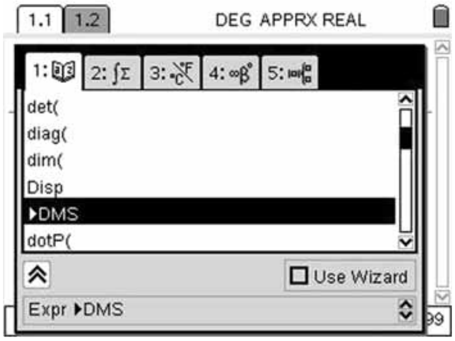
Using the TI-Nspire:

1 Set to Degree and Approximate mode.

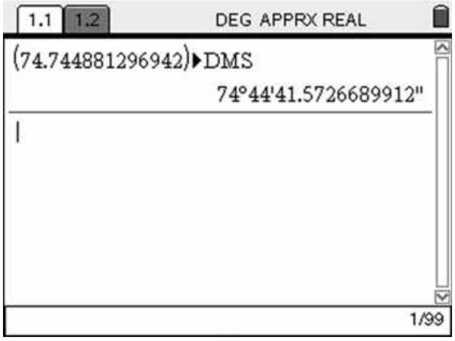
2 With 74.7448813 . . . on the screen enter into the catalog by pressing ∞β°.

3 Ensure you are in submenu 1.

4 Press D then scroll down and press enter on ▶DMS.



5 Press enter.

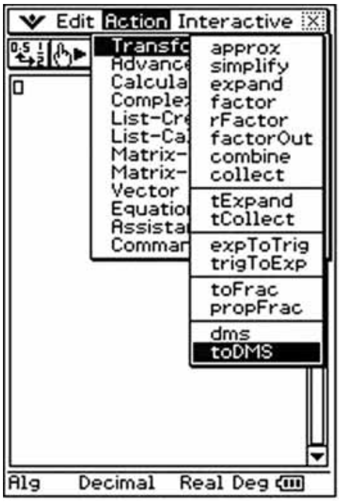


```
(74.744881296942)▶DMS
                  74°44'41.5726689912"
```

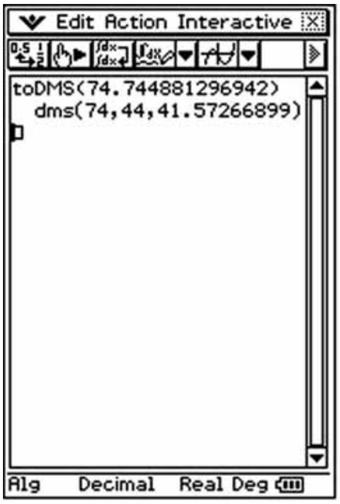
Using the ClassPad:

1 Set to Decimal and Degree mode.

2 In the Main screen go to the Action menu and select the *toDMS* command from the Transformation submenu.



3 Now type **74.744881296942** followed by) then press EXE.



```
toDMS(74.744881296942)
  dms(74,44,41.57266899)
```

Note: When rounding to the nearest minute recall that there are 60 seconds in a minute.

Using the TI-Nspire:

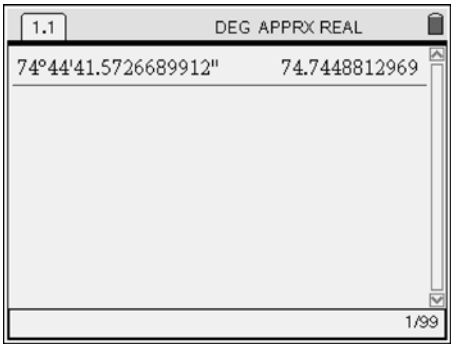
1 Set to Degree and Approximate mode.

2 Type **74** then press ctrl ′.

3 Type **44** then press ′.

4 Type **41.5726689912** then press ′ *twice*.

5 Press enter.



DEG APPRX REAL

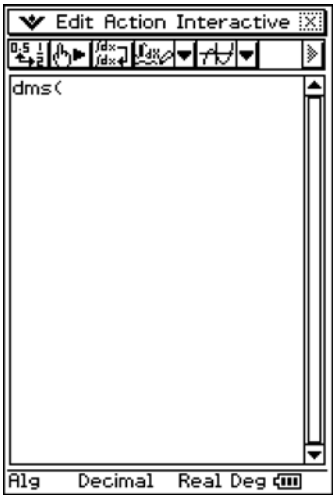
```
74°44'41.5726689912"        74.7448812969
```

1/99

Using the ClassPad:

1 Set to Degree and Decimal mode.

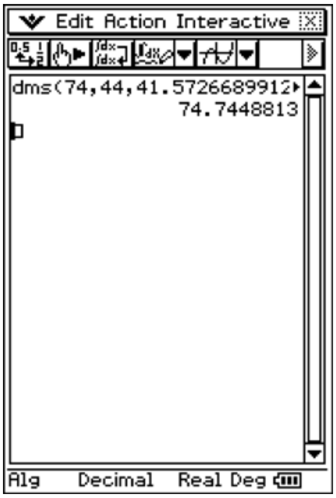
2 Tap Action and select *dms* from the Transformation submenu.



```
dms(
```

Alg Decimal Real Deg

3 Now type **74,44,41.5726689912**) and then press EXE.



```
dms(74,44,41.5726689912▸
                74.7448813
```

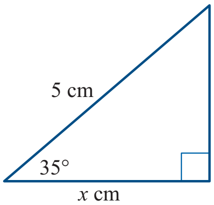
Alg Decimal Real Deg

Exercise 2A

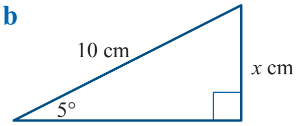
Examples 1–3

1 Find the value of x in each of the following:

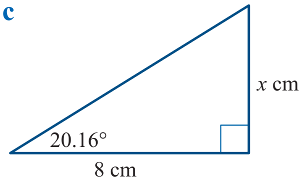
a



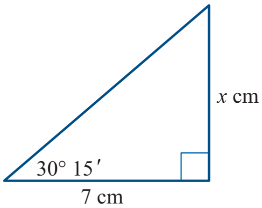
b



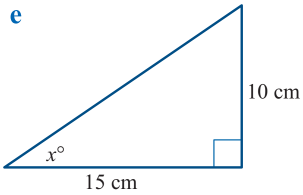
c



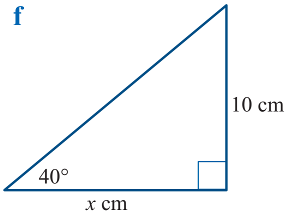
d



e



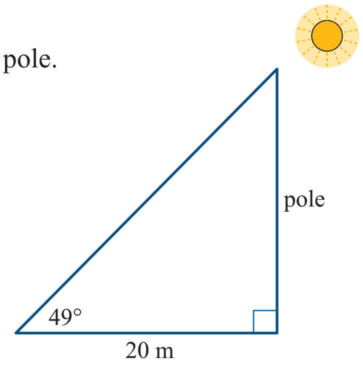
f



2 An equilateral triangle has perpendicular height of length 20 cm. Find the length of one side.

3 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

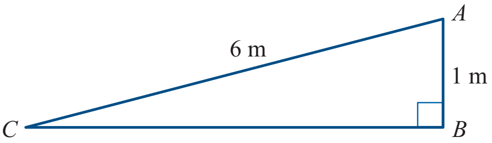
4 A pole casts a shadow 20 m long when the angle of elevation of the sun is 49°. Calculate the height of the pole.



5 This figure represents a ramp. Find the:

a magnitude of $\angle ACB$

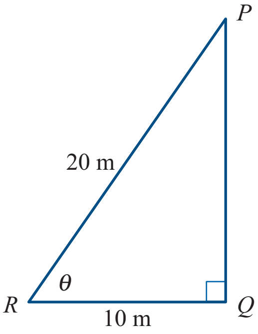
b distance BC



6 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.
Calculate the:

a angle of elevation of the wire to the ground, θ

b height of the mast



7 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate the:

a length of the ladder **b** height it reaches up the wall

8 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park 13 m above street level. Calculate the angle of the ramp to the horizontal.

9 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

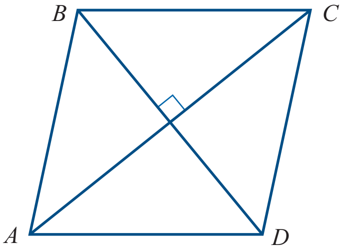
10 A mountain railway with a straight track rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

11 The diagonals of a rhombus bisect each other at right angles.

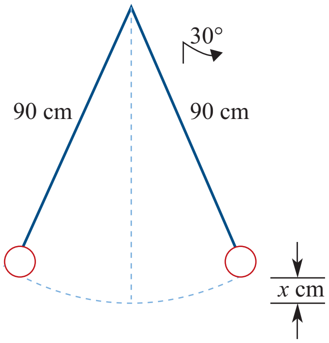
If $BD = AC = 10$ cm, find the:

a length of the sides of the rhombus

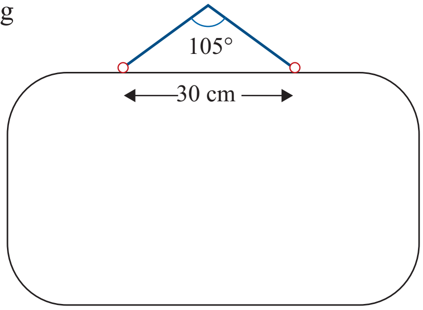
b magnitude of $\angle ACB$



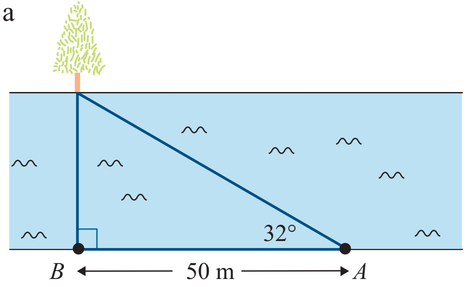
12 A pendulum swings through an angle of 30°. If the pendulum is 90 cm long, what is the distance x cm between its highest and lowest point?



13 A picture is hung symmetrically by means of a string passing over a nail with its ends attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm and the angle between the two portions is 105°. Find the length of the string.

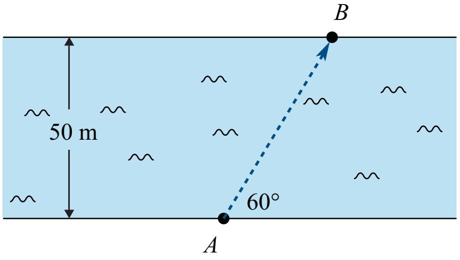


14 The distance $AB = 50$ m. If the line of sight of a person standing at A to the tree makes an angle of 32° with the bank, how wide is the river?



15 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flowerbed, which extends a distance of 1.7 m from the foot of the wall. How high up the wall can the ladder reach?

16 A river is known to be 50 m wide. A swimmer sets off from *A* to cross the river and the path *AB* of the swimmer is as shown. How far does the person swim? Assume there is no current.



2.2 The sine rule

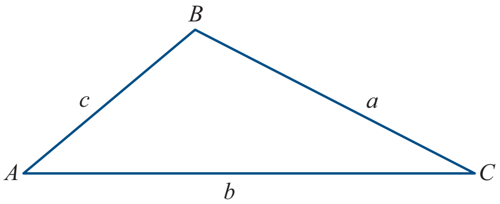
In Section 2.1, methods for finding unknown lengths and angles for right-angled triangles were discussed. This section and the next will deal with methods for finding unknown quantities in triangles that are *not* right angled.

The sine rule is used to find unknown quantities in a triangle when one of the following situations arises:

- one side and two angles are given
- two sides and a non-included angle are given.

Labelling convention

The following convention is followed in the remainder of this section. Interior angles are denoted by uppercase letters and the length of the side opposite an angle is denoted by the corresponding lowercase letter. For example:

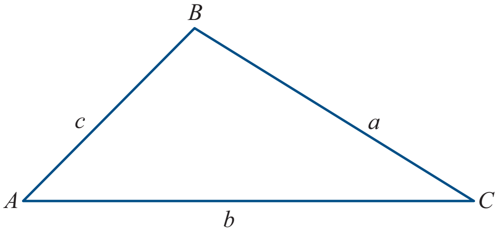


The magnitude of angle *BAC* ($\angle BAC$) is denoted by *A*.

The length of side *BC* is denoted by *a*, as it is the side opposite $\angle A$ (or ($\angle BAC$).

The **sine rule** states that for triangle *ABC*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



A proof will be given only for the acute-angled triangle case. The proof for obtuse-angled triangles is similar.

Proof

In triangle ACD, $\sin A = \dfrac{h}{b}$.

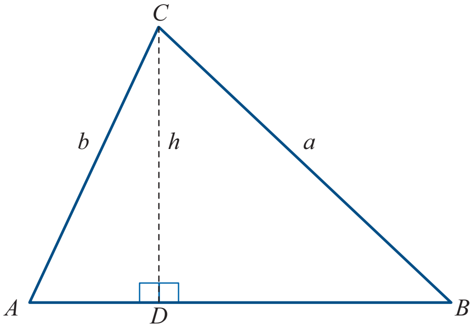
$$\therefore \quad h = b \sin A$$

In triangle BCD, $\sin B = \dfrac{h}{a}$.

$$\therefore \quad h = a \sin B$$

$$\therefore \quad b \sin A = a \sin B$$

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

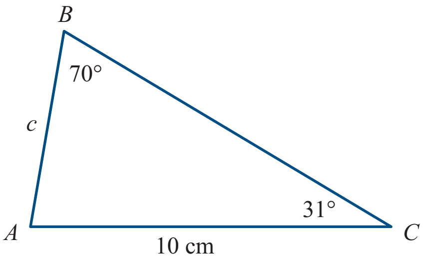


Similarly, starting with a perpendicular from A to BC would give:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

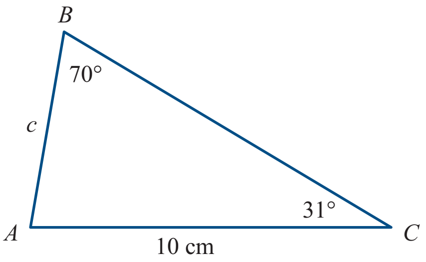
Example 4

Use the sine rule to find the length of AB, to 2 decimal places.



Solution

$$\begin{aligned}
\frac{b}{\sin B} &= \frac{c}{\sin C}\\
\frac{10}{\sin 70^\circ} &= \frac{c}{\sin 31^\circ}\\
\therefore \quad 10 \sin 31^\circ &= c \sin 70^\circ\\
\therefore \quad c &= \frac{10 \sin 31^\circ}{\sin 70^\circ}\\
\therefore \quad c &= 5.4809\ldots
\end{aligned}$$

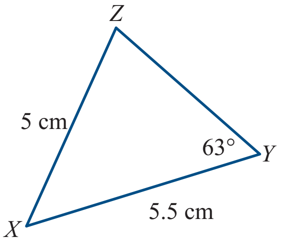


The length of AB is 5.48 cm, correct to 2 decimal places.

Note: In practice, only two angle-sided pairs are required to form the equation. In this case, the third pair; that is, $\angle A$ and side a, are not required.

Example 5

Use the sine rule to find the magnitude of $\angle XZY$ in the triangle, given that $\angle Y = 63^\circ$, $y = 5$ cm and $z = 5.5$ cm.



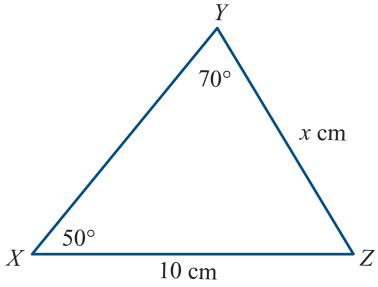
Solution

$$
\begin{aligned}
\frac{a}{\sin A} &= \frac{b}{\sin B}\\
\frac{5}{\sin 63^\circ} &= \frac{5.5}{\sin Z}\\
5\sin Z &= 5.5\sin 63^\circ\\
\sin Z &= \frac{5.5\sin 63^\circ}{5}\\
&= 0.98\ldots.\\
Z &= \sin^{-1}(0.98\ldots)\\
Z &\approx 78.55^\circ \text{ or } 78^\circ\ 33'\ 9''
\end{aligned}
$$

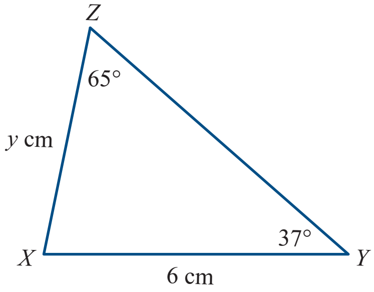
Exercise 2B

Example 4 **1** Find the value of the pronumeral for each of the following triangles:

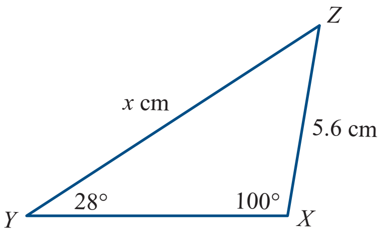
a



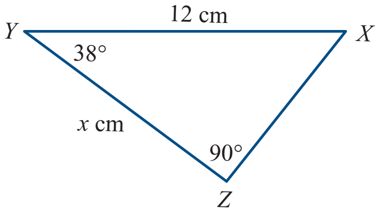
b



c

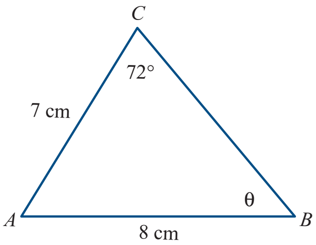


d

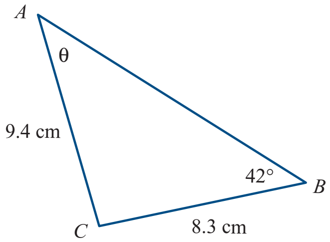


Example 5 **2** Find the value of θ for each of the following triangles:

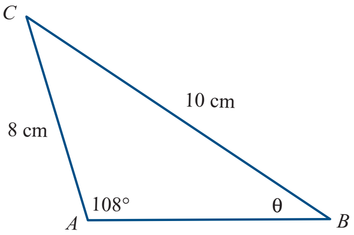
a



b



c



3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12$, $B = 59°$, $C = 73°$

b $A = 75.3°$, $b = 5.6$, $B = 48.25°$

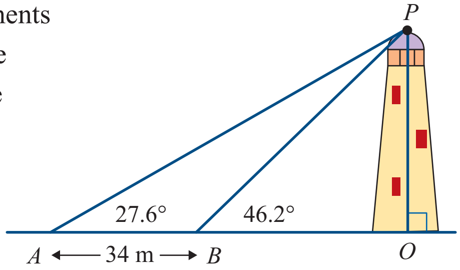
c $A = 123.2°$, $a = 11.5$, $C = 37°$

d $B = 140°$, $b = 20$, $A = 10°$

4 Find all sides and angles in a triangle *ABC* having $B = 129°$, $b = 7.89$ cm and $c = 4.56$ cm.

5 A landmark, *A*, is observed from two points, *B* and *C*, which are 400 m apart. The magnitude of $\angle ABC$ is found to be 68° and the magnitude of $\angle ACB$ is 70°. Find the distance of *A* from *C*.

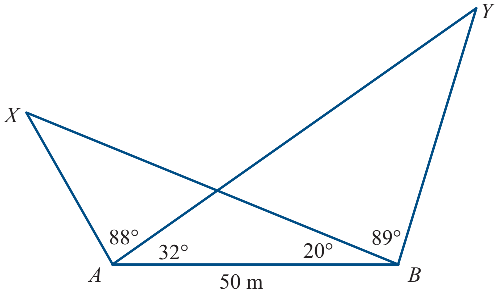
6 *P* is a point at the top of a lighthouse. Measurements of the length of *AB* and angles *PBO* and *PAO* are taken, and are as shown in the diagram. Find the height of the lighthouse.



7 *A* and *B* are two points on a coastline. They are 1070 m apart. *C* is a point at sea. The angles *CAB* and *CBA* have magnitudes of 74° and 69°, respectively. Find the distance of *C* from *A*.

8 Find the distance:

a *AX* **b** *AY*



2.3 The cosine rule

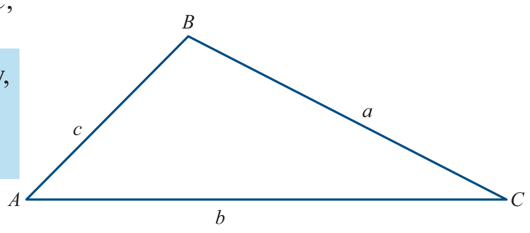
The **cosine rule** is used to find unknown quantities in a triangle when one of the following situations arises:

- two sides and an included angle are given
- three sides are given.

The **cosine rule** states that for triangle *ABC*,

$$a^2 = b^2 + c^2 - 2bc\ \cos A \text{ or, equivalently,}$$

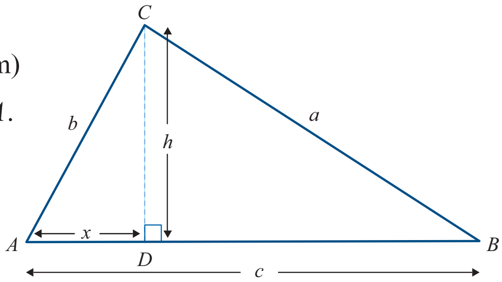
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$



The result will be proved for an acute-angled triangle. The proof for obtuse-angled triangles is similar.

Proof

In triangle ACD:



$$b^2 = x^2 + h^2 \text{ (Pythagoras' theorem)}$$

$$\cos A = \frac{x}{b} \text{ and, therefore, } x = b\cos A.$$

In triangle BCD:

$$a^2 = (c - x)^2 + h^2 \text{ (Pythagoras' theorem)}$$

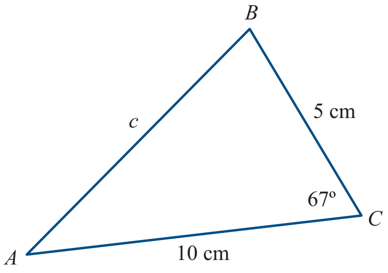
Expanding gives:

$$\begin{aligned} a^2 &= c^2 - 2cx + x^2 + h^2 \\ &= c^2 - 2cx + b^2 && (\text{as } x^2 + h^2 = b^2) \\ \therefore \quad a^2 &= b^2 + c^2 - 2bc\cos A && (\text{as } x = b\cos A) \end{aligned}$$

The student should note that a is opposite A in the triangle and that a^2 and $\cos A$ appear at opposite ends of the formulae. The first formula reads '$a^2 = \ldots\ldots \cos A$' and the second formula reads '$\cos A = \ldots\ldots - a^2$'. Noting this may aid in memorising and using the formulae.

Example 6

For triangle ABC, find the length of AB in centimetres, correct to 2 decimal places.



Solution

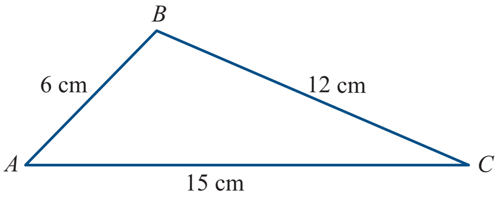
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 5^2 + 10^2 - 2 \times 5 \times 10 \times \cos 67^\circ \\ &\approx 85.9268\ldots && (\sqrt{\ }) \\ \therefore \quad c &\approx 9.2696\ldots \\ &\approx 9.27 \end{aligned}$$

$AB = 9.27$ cm, correct to 2 decimal places.

Note: $C = 67^\circ$. Also AB is the side c. Therefore, the formula reads $c^2 = \ldots\ldots \cos C$.

Example 7

Find the magnitude of ∠*ABC* for triangle *ABC*.



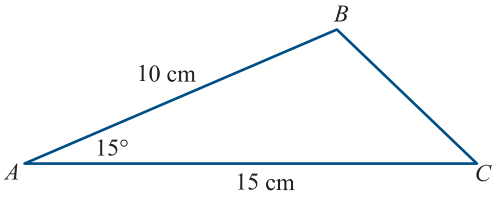
Solution

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6} \\ &\approx -0.3125 \qquad (\cos^{-1}) \\ \therefore \quad B &\approx 108.2099\ldots \\ &\approx 108^\circ \\ \angle ABC &\approx 108^\circ \end{aligned}$$

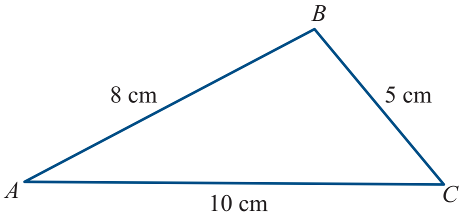
Note: B is the angle being sought. Therefore, the formula reads $\cos B = \ldots\ldots - b^2$.

Exercise 2C

Example 6 **1** Find the length of *BC*.



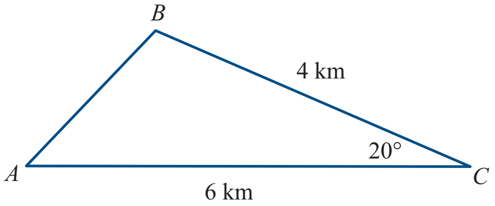
Example 7 **2** Find the magnitude of angles ∠*ABC* and ∠*ACB*.



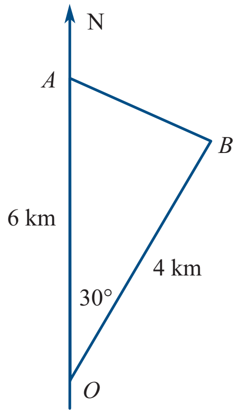
3 For triangle *ABC* with:

- **a** $A = 60^\circ$, $b = 16$, $c = 30$, find a.
- **b** $a = 14$, $B = 53^\circ$, $c = 12$, find b.
- **c** $a = 27$, $b = 35$, $c = 46$, find the magnitude of angle $\angle ABC$.
- **d** $a = 17$, $B = 120^\circ$, $c = 63$, find b.
- **e** $a = 31$, $b = 42$, $C = 140^\circ$, find c.
- **f** $a = 10$, $b = 12$, $c = 9$, find the magnitude of angle $\angle BCA$.
- **g** $a = 11$, $b = 9$, $c = 43.2^\circ$, find c.
- **h** $a = 8$, $b = 10$, $c = 15$, find the magnitude of angle $\angle CBA$.

4 A section of an orienteering course is as shown. Find the length of leg AB.



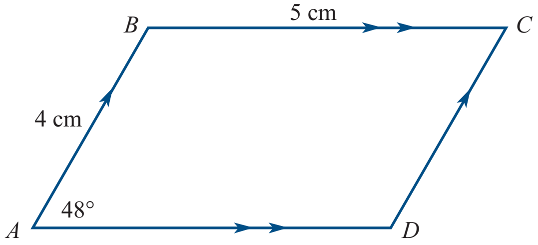
5 Two ships sail from point O. At a particular time their positions A and B are as shown. Find the distance between the ships at this time.



6 $ABCD$ is a parallelogram. Find the length of the diagonals:

a AC

b BD

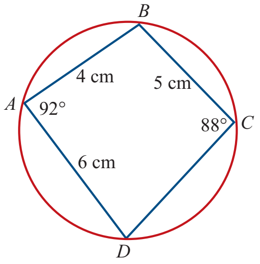


7 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.

8 For the diagram opposite:

a Find the length of chord BD.

b Use the sine rule to find the length of CD.



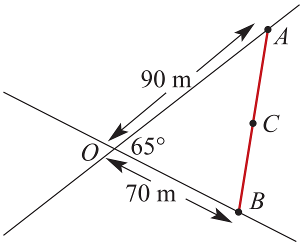
9 Two circles of centre O and O', and radii 7.5 cm and 6 cm, respectively, have a common chord AB of length 8 cm. Find the magnitude of each of the following angles:

a $AO'B$ **b** AOB

10 Two straight roads OA and OB intersect at O at an angle of 65°. A point A on one road is 90 m from the intersection and a point B on the other road is 70 m from the intersection, as shown on the diagram.

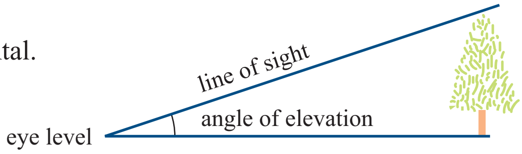
a Find the distance of A from B.

b C is the midpoint of AB. Find the distance of C from the intersection O.

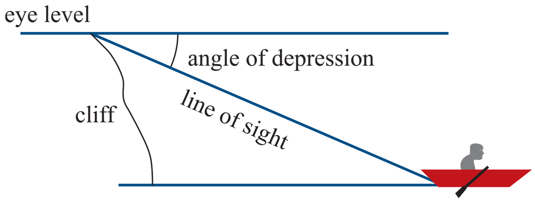


2.4 Angles of elevation and depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.

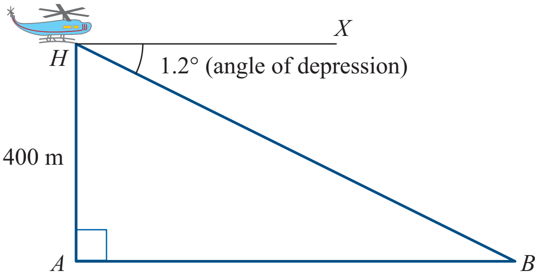


The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 8

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.



(diagram not to scale)

Solution

As $\angle XHB = \angle HBA$ (alternate angles: Z rule)

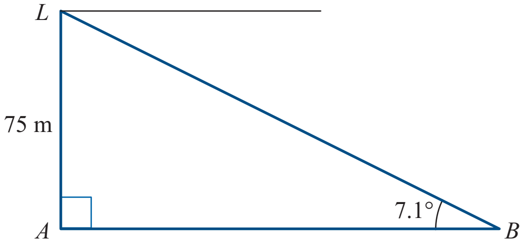
$$\begin{aligned}
\tan\theta &= \frac{opp}{adj}\\
\tan 1.2^\circ &= \frac{AH}{AB}\\
\therefore \quad \tan 1.2^\circ &= \frac{400}{AB}\\
AB &= \frac{400}{\tan 1.2^\circ}\\
AB &\approx 19\,095.8\ldots \qquad (\div 1000)\\
&= 19.0958\ldots\\
&\approx 19
\end{aligned}$$

The horizontal distance is approximately 19 km.

Note: The distance is rounded to the nearest kilometre because of the context of the question.

Example 9

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the horizontal distance of the boat from the lighthouse.



Solution

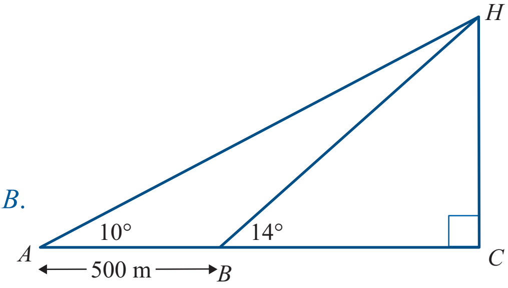
$$\begin{aligned} \tan\theta &= \frac{opp}{adj} \\ \tan 7.1^\circ &= \frac{75}{AB} \\ \therefore \quad AB &= \frac{75}{\tan 71^\circ} \\ &\approx 602.135\ldots \\ &\approx 600 \end{aligned}$$

The distance of the boat from the lighthouse is approximately 600 m.

Example 10

From point *A*, a man observes that the angle of elevation of the summit (*H*) of a hill is 10°. He then walks towards the hill for 500 m along flat ground to point *B*. The summit of the hill is now at an angle of elevation of 14°.

Find the height of the hill above the level of *AB*.

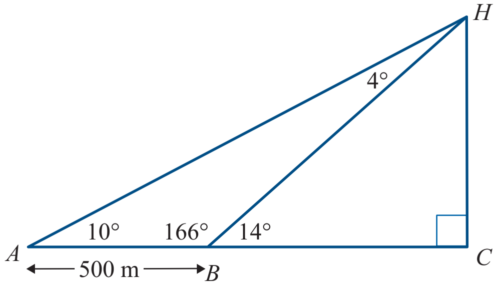


Solution

$\angle HBA = (180 - 14)^\circ = 166^\circ$ (straight angle)
$\angle AHB = 180^\circ - (166 + 10)^\circ = 4^\circ$ (angle sum of triangle)

Using the sine rule in triangle *ABH*:

$$\begin{aligned} \frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{500}{\sin 4^\circ} &= \frac{HB}{\sin 10^\circ} \\ \therefore \quad HB &= \frac{500 \times \sin 10^\circ}{\sin 4^\circ} \\ &\approx 1244.67\ldots \end{aligned}$$



In triangle *BCH*:

$$\begin{aligned}\sin\theta &= \frac{opp}{hyp}\\ \sin 14^\circ &= \frac{HC}{HB}\\ \therefore\quad HC &= HB\sin 14^\circ\\ &\approx 301.11\ldots\\ &\approx 300\end{aligned}$$

The height of the hill is approximately 300 m.

Bearings

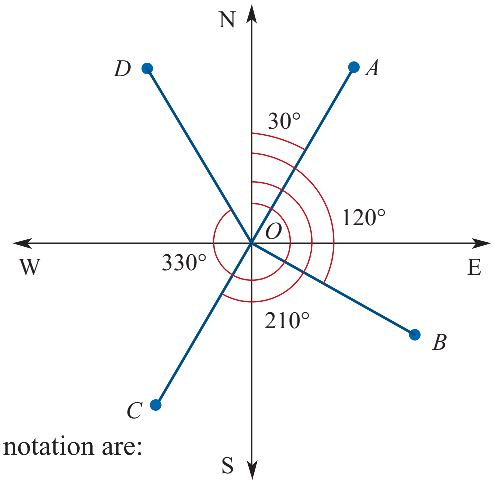
The **true bearing** is the direction measured from north (N) in a clockwise sense, where north is taken as 0°.

The bearing of *A* from *O* is 030°.

The bearing of *B* from *O* is 120°.

The bearing of *C* from *O* is 210°.

The bearing of *D* from *O* is 330°.



The **compass (or conventional) bearing** is measured with reference to the axes: N or S first, then E or W.

For example, the above four bearings using this notation are:

The bearing of *A* from *O* is N30°E.

The bearing of *B* from *O* is S60°E.

The bearing of *C* from *O* is S30°W.

The bearing of *D* from *O* is N30°W.

During this chapter, unless otherwise stated, assume that when a bearing is required, it is understood to be a **true bearing**.

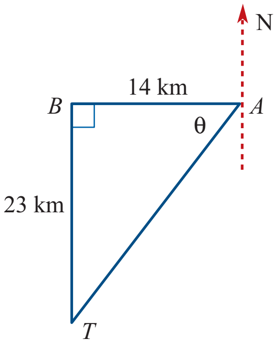
Magnetic bearings vary from true bearings, depending on time and location. They will not be considered in this chapter.

Example 11

The road from town *A* runs due west for 14 km to town *B*. A television mast is located due south of *B* at a distance of 23 km. Calculate the:

a distance *AT*

b true bearing of the mast from the centre of town *A*



Solution

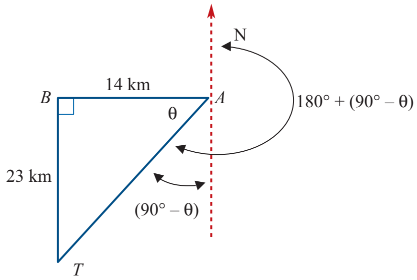
a By Pythagoras' theorem

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \\ AT &= \sqrt{725} \quad (-\sqrt{725} \text{ is excluded since } AT \text{ is a distance.}) \\ \therefore \quad AT &\approx 26.925\ldots \\ &\approx 27 \end{aligned}$$

AT is approximately 27 km.

b $\tan\theta = \dfrac{23}{14} \left(\text{using } \tan\theta = \dfrac{opp}{adj}\right)$

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{23}{14}\right) \\ \therefore \quad \theta &\approx 58.67^\circ \\ \therefore \quad \text{True bearing} &= 180 + (90 - \theta) \\ &\approx 211.33 \\ &\approx 211 \end{aligned}$$



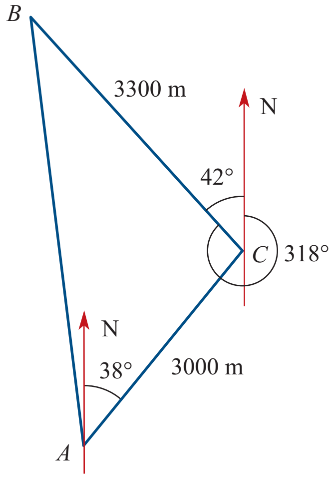
$\therefore$ The bearing of the mast from the centre of town is approximately 211° T.

Example 12

A yacht starts from a point *A* and sails on a bearing of 038°T for 3000 m. It then alters its course to one in a direction with a bearing of 318°, and after sailing for 3300 m it reaches a point *B*. Find the:

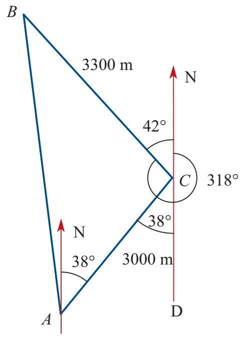
a distance *AB*

b bearing of *B* from *A*



Solution

a
$$\begin{aligned} \angle NAD &= \angle ACD && \text{(alternate angles)} \\ \angle ACB &= 180^\circ - (38 + 42)^\circ && \text{(straight angle)} \\ &= 100^\circ \\ c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \times \cos 100^\circ \\ &\approx 23\,328\,233.92\ldots (\sqrt{\ }) \\ \therefore \quad c &\approx 4829.93\ldots && (\div 1000) \\ &\approx 4.82993\ldots \\ &\approx 4.8 \end{aligned}$$



The distance *AB* is approximately 4.8 km.

b

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{3300}{\sin A} = \frac{4829.9...}{\sin 100}$$

$$\sin A = \frac{3300 \times \sin 100}{4829.9...}$$

$$\sin A \approx 0.6728\ldots \qquad (\sin^{-1})$$

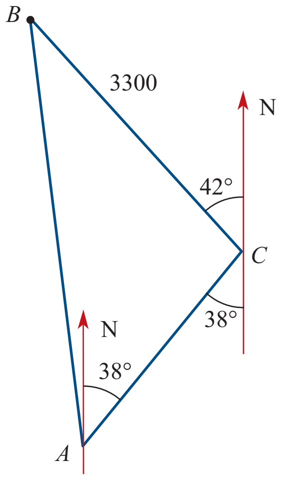
$$\therefore \quad A \approx 42.288\ldots$$

$$\text{True bearing} = 360° - (A - 38)°$$

$$\approx 355.71\ldots$$

$$\approx 356°$$

The True bearing of *B* from *A* is approximately 356°T.



Exercise 2D

Example 8 **1** From the top of a vertical cliff 130 m high the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

Example 9 **2** The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

3 A man standing on top of a mountain observes that the angle of depression to the foot of a building is 41°. If the height of the man above the foot of the building is 500 m, find the horizontal distance from the man to the building.

4 A woman lying down on the top edge of a vertical cliff 40 m high observes the angle of depression to a buoy in the sea to be 20°. Calculate the distance between the buoy and the foot of the cliff directly below the woman.

Example 10 **5** A person standing on top of a cliff 50 m high is in line with two buoys, whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

6 The bearing of a point, *A*, from another point, *B*, is 207°. What is the bearing of *B* from *A*?

Example 11 **7** A ship sails 10 km north and then 15 km east. What is its bearing from the starting point?

8 A ship leaves port *A* and steams 15 km due east. It then turns and steams for 22 km due north.

- **a** What is the bearing of the ship from port *A*?
- **b** What is the bearing of port *A* from the ship?

Example 12 **9** A yacht starts from point *A* and sails on a bearing of N35°E for 2000 m. It then alters its course to one in a direction with a bearing of N40°W and, after sailing for 2500 m, it reaches point *B*. Find the:

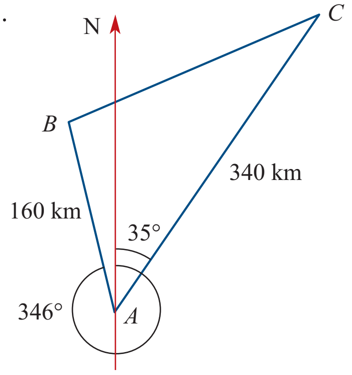
a distance *AB* **b** bearing of *B* from *A*, to the nearest degree

10 The bearing of a ship, *S*, from a lighthouse, *A*, is N55°E. A second lighthouse, *B*, is due east of *A*. The bearing of *S* from *B* is N58°W. Find the magnitude of $\angle ASB$.

11 A yacht starts from *L* and sails 12 km due east to *M*. It then sails 9 km on a bearing of S38°E to *K*. Find the magnitude of $\angle MLK$.

12 As shown in the diagram, the bearing of *C* from *A* is 035°. The bearing of *B* from *A* is 346°. The distance of *C* from *A* is 340 km. The distance of *B* from *A* is 160 km.

- **a** Find the magnitude of $\angle BAC$.
- **b** Use the cosine rule to find the distance from *B* to *C*.



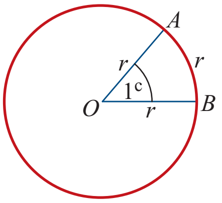
13 From a ship, *S*, two other ships, *P* and *Q*, are on bearings 320° and 075°, respectively. The distance $PS = 7.5$ km and the distance $QS = 5$ km. Find the distance *PQ*.

2.5 Definition of a radian

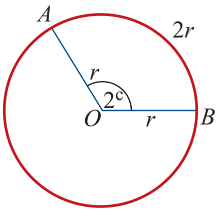
Consider the circle centre *O* and radius *r* shown at right.

Suppose radii *OA*, *OB* are drawn so that the arc *AB* is equal in length to the radius, *r*.

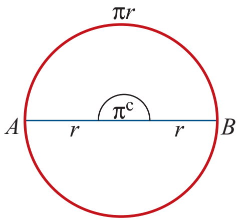
The angle $\angle AOB$ is defined as 1 radian (which is written as 1^c).



Similarly, if *OA*, *OB* are drawn so that arc *AB* is equal to 2*r*, then the angle $\angle AOB$ is equal to 2 radians (2^c).



Continuing with this logic, if *OA*, *OB* are now drawn so that arc *AB* is a semicircle, then its length will be half the circumference; that is, πr, and so the angle at the centre ($\angle AOB$) will equal π radians and, in fact, will equal 180°.



Thus, we may say that $\pi^c = 180°$.

Applying multiples and fractions of this identity, we may construct this table.

Angles in degrees	0	30	45	60	90	180	360
Angles in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

As

$$1^c = \left(\frac{180}{\pi}\right)^{\circ} \quad \text{and} \quad 1^{\circ} = \left(\frac{\pi}{180}\right)^{c}$$

We may also perform direct conversions using the rules below.

Degrees to radians	Multiply by $\frac{\pi}{180}$
Radians to degrees	Multiply by $\frac{180}{\pi}$

Note: It is usual to write θ^c as θ, i.e. *unitless* as it is a **ratio**.

i.e. The angle of the centre of the sector is the **ratio** of the arc length to radius $\left(\frac{AB}{r}\right)$.

As such, it is the **number** of radii in the arc length AB, therefore being unitless.

For example, $\left(\frac{\pi}{3}\right)^c = \frac{\pi}{3}$, $\quad 1.04^c = 1.04 \quad \sin\pi = 0 \quad \cos 1 \approx 0.54$ etc.

Example 13

Convert 30° to radians.

Solution

$$\begin{aligned} 30^\circ &= \left(30 \times \frac{\pi}{180}\right)^c \\ &= \left(\frac{\pi}{6}\right)^c \\ &= \frac{\pi}{6} \end{aligned}$$

Example 14

Convert $\left(\frac{\pi}{4}\right)^c$ to degrees.

Solution

$$\left(\frac{\pi}{4}\right)^c = \frac{\pi}{4} = \left(\frac{\pi}{4} \times \frac{180}{\pi}\right)^\circ = 45^\circ$$

Using technology

Example 15

Convert an angle of 2.7 to degrees.

Solution

$$\begin{aligned} 2.7 &= \left(2.7 \times \frac{180}{\pi}\right)^\circ \\ &= 154.7^\circ \text{ (by calculator)} \end{aligned}$$

Example 16

Convert 73° to radians.

Solution

$$\begin{aligned}73^\circ &= \left(73 \times \frac{\pi}{180}\right)^c \\ &= 1.27^c \text{ (by calculator)} \\ &= 1.27\end{aligned}$$

To convert from degrees to radians

Using the TI-Nspire

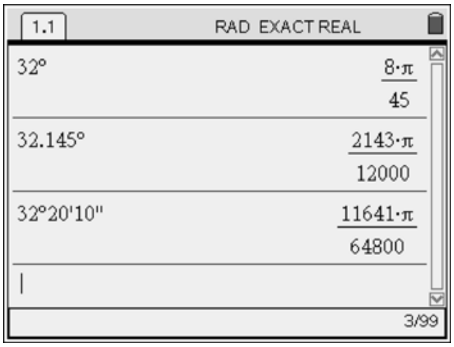
Exact mode

1 Set the calculator to Exact and Radian mode.

2 Type **32**, press ctrl ′, then press enter.

3 Type **32.145**, press ctrl ′ and then press enter.

4 Type **32**, then press ctrl ′. Type **20** and then press ′. Type **10** then press ′ *twice*. Now press enter.



1.1 RAD EXACT REAL

32° $\frac{8\cdot\pi}{45}$

32.145° $\frac{2143\cdot\pi}{12000}$

32°20'10" $\frac{11641\cdot\pi}{64800}$

3/99

Approximate mode

1 Set the calculator to Approximate and Radian mode.

2 Type **82**, press ctrl ′, then press enter.

3 Type **(–)123.45**, press ctrl ′ and then press enter.

Using the ClassPad

Standard mode

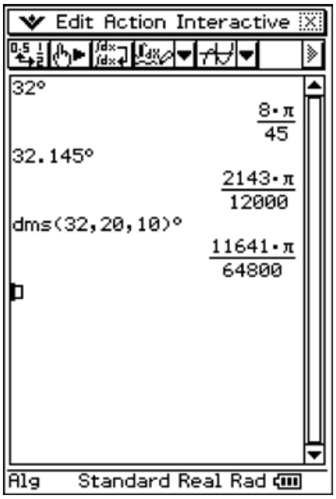
1 Set the calculator to Standard and Radian mode.

2 Press Keyboard, then tap TRIG. (This gives access to the degree symbol.)

3 Type **32**, tap ° and then press EXE.

4 Type **32.145**, tap ° and then press EXE.

5 Type **dms(32,20,10)**, tap ° and then press EXE.



32° $\frac{8\cdot\pi}{45}$

32.145° $\frac{2143\cdot\pi}{12000}$

dms(32,20,10)° $\frac{11641\cdot\pi}{64800}$

Alg Standard Real Rad

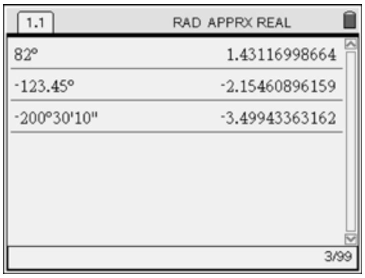
Decimal mode

1 Set the calculator to Decimal and Radian mode.

2 Type **82**, tap ° and then press EXE.

3 Type **(–)123.45**, tap °, then press EXE.

4 Type **(–)200**, then press ctrl ′. Type **30**, then press ′. Type **10**, then press ′ *twice*. Now press enter.



82°	1.43116998664
-123.45°	-2.15460896159
-200°30'10"	-3.49943363162

Note: To approximate an exact answer press ctrl clear.

4 Type **dms((–)200,30,10)**, tap ° and then press EXE.

```
82°
          1.431169987
-123.45°
         -2.154608962
dms(-200,30,10)°
         -3.481883376
```

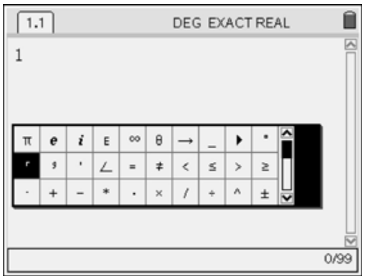
To convert from radians to degrees

Using the TI-Nspire

Exact mode

1 Set the calculator to Exact and Degree mode.

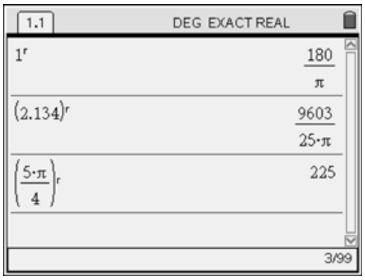
2 Type **1**, press ctrl ∞β°, then press enter on the radian symbol *r*.



3 Press enter to evaluate.

4 Type **2.134** (access the radian symbol, as described above) then press enter.

5 Type $\frac{5\pi}{4}$ using the fraction template, then press the right arrow key. Access the radian symbol, as described above, and then press enter.



1^r	$\frac{180}{\pi}$
$(2.134)^r$	$\frac{9603}{25\cdot\pi}$
$\left(\frac{5\cdot\pi}{4}\right)^r$	225

Using the ClassPad

Standard mode

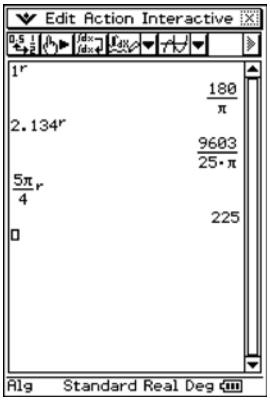
1 Set the calculator to Standard and Degree mode.

2 Press Keyboard, then tap TRIG. (This gives access to the radian symbol *r*.)

3 Type **1**, tap r and then press EXE.

4 Type **2.134**, tap r and then press EXE.

5 Type $\frac{5\pi}{4}$, tap r and then press EXE.



1^r	$\frac{180}{\pi}$
2.134^r	$\frac{9603}{25\cdot\pi}$
$\frac{5\pi}{4}^r$	225

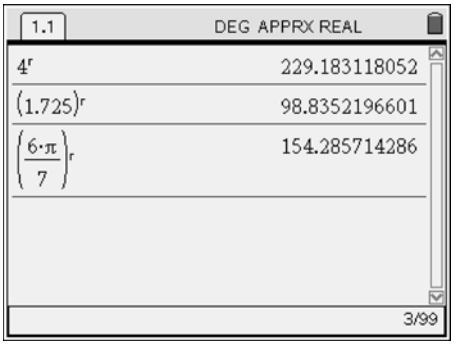
Approximate mode

1 Set the calculator to Approximate and Degree mode.

2 Type **4** (access the radian symbol, as described above) and then press enter.

3 Type **1.725** (access the radian symbol, as described above) and then press enter.

4 Type $\frac{6\pi}{7}$ using the fraction template, then press the right arrow key. Access the radian symbol, as described above, and then press enter.



4^r 229.183118052

$(1.725)^r$ 98.8352196601

$\left(\frac{6\cdot\pi}{7}\right)^r$ 154.285714286

Note: To approximate an exact answer press ctrl enter .

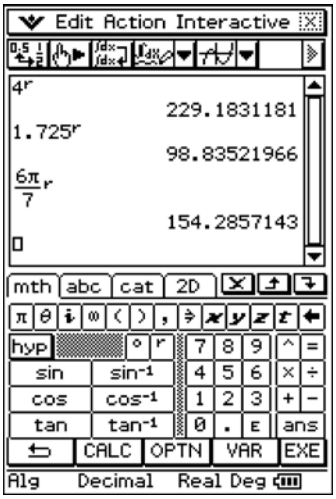
Decimal mode

1 Set the calculator to Decimal and Degree mode.

2 Type **4**, tap r and then press EXE.

3 Type **1.725**, tap r and then press EXE.

4 Type $\frac{6\pi}{7}$, tap r and then press EXE.



4^r 229.1831181

1.725^r 98.83521966

$\frac{6\pi}{7}{}^r$ 154.2857143

Exercise 2E

Example 13 **1** Express the following angles in radian measure in terms of π:

a 60° **b** 144° **c** 240°
d 330° **e** 420° **f** 480°

Example 14 **2** Express, in degrees, the angles with the following radian measures:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{6}$ **d** 0.9π
e $\frac{5\pi}{9}$ **f** $\frac{9\pi}{5}$ **g** $\frac{11\pi}{9}$ **h** 1.8π

Example 15 **3** Use a calculator to convert the following angles from radians to degrees:

a 0.6 **b** 1.89 **c** 2.9 **d** 4.31
e 3.72 **f** 5.18 **g** 4.73 **h** 6.00

Example 16 **4** Use a calculator to express the following in radian measure:

a 38° **b** 73° **c** 107° **d** 161°
e 84.1° **f** 228° **g** 136.4° **h** 329°

5 Evaluate the following, using your calculator:

a $\sin 1.9$ **b** $\sin 2.3$ **c** $\sin 4.1$ **d** $\cos 0.3$
e $\cos 2.1$ **f** $\cos(-1.6)$ **g** $\sin(-2.1)$ **h** $\sin(-3.8)$
i $\tan 1.6$ **j** $\tan(-2.8)$

6 For each of the following angles θ, determine the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$:

a $\theta = 27\pi$ **b** $\theta = -\frac{5\pi}{2}$ **c** $\theta = \frac{29\pi}{2}$ **d** $\theta = -\frac{9\pi}{2}$
e $\theta = \frac{11\pi}{2}$ **f** $\theta = 56\pi$ **g** $\theta = 211\pi$ **h** $\theta = -53\pi$

2.6 Exact trigonometric ratios and angles of any magnitude

Exact trigonometric ratios

A calculator can be used to find the values of the trigonometric functions for different values of θ. For many values of θ the calculator gives an approximation. Here, special values of θ are considered for which the sine, cosine and tangent can be calculated exactly.

Exact values for $\frac{\pi}{6}$ (i.e. 30°) and $\frac{\pi}{3}$ (i.e. 60°)

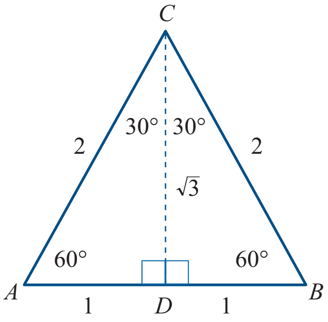
Consider an equilateral triangle ABC of side length 2 units.
In $\triangle ACD$, by the Theorem of Pythagoras:

$$\begin{aligned} DC &= \sqrt{AC^2 - AD^2} \\ &= \sqrt{3} \end{aligned}$$

$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}}$$



$$\sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

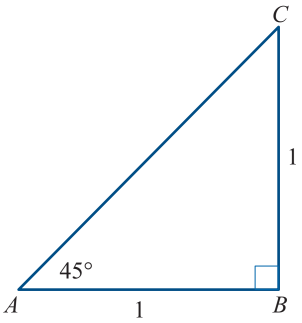
Exact values for $\frac{\pi}{4}$ (i.e. 45°)

$$AC = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$



As an aid to memory, the exact values for circular functions can be tabulated.

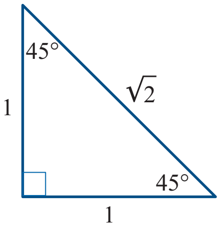
Summary

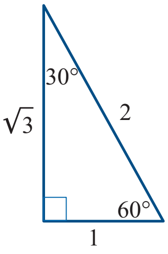
Learn

θ ($\theta°$)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

OR

Learn





Angles of any magnitude

It is worth noting that the definitions you have been using for sine, cosine and tangent only work if the angles in use are between 0° and 90°.

i.e. $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ have meaning only if the angle θ is an acute angle in a right-angled triangle.

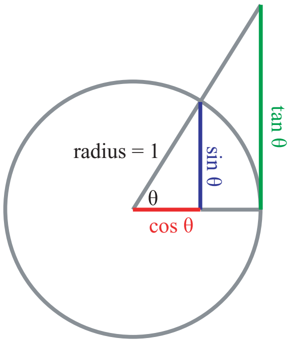
Despite this limitation, your calculator had no problem finding the sine and cosine of obtuse angles when using the Sine and Cosine Rules in the previous sections. The basis behind this contradiction is that your calculator was built with a broader definition in mind.

Although you will still use the old definitions on many occasions, the definitions discussed below will allow trigonometry to be applied to situations other than right-angled triangles.

Historically, the sine, cosine and tangent were defined by the Greeks, using the diagram shown.

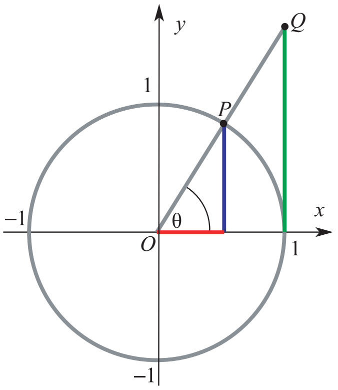
Note: The radius of the circle is one unit. This circle is often referred to as a **unit circle**.

$\sin\theta$ = length of blue line
$\cos\theta$ = length of red line
$\tan\theta$ = length of green line (or tangent)



Drawing the Cartesian plane over the diagram above makes it possible to define the trigonometric ratios as follows:

$\sin\theta = y$ coordinate of P
$\cos\theta = x$ coordinate of P
$\tan\theta = y$ coordinate of Q



Notes:

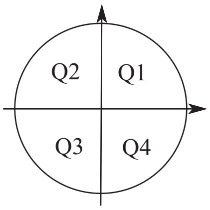
The angle θ is measured anticlockwise from the positive direction of the *x*-axis.
P is the point where the angle being drawn crosses the circle.
Q is the point where the line *OP* crosses the line $x = 1$.

On tan θ

Some books will define tan θ as $\frac{\sin\theta}{\cos\theta}$. This result follows directly from recognising that the two triangles with hypotenuses *OP* and *OQ* are similar and recalling that 'Corresponding sides in similar triangles are in the same ratio'. This leads to $\frac{\sin\theta}{\cos\theta} = \frac{\tan\theta}{1}$ and, therefore, that:

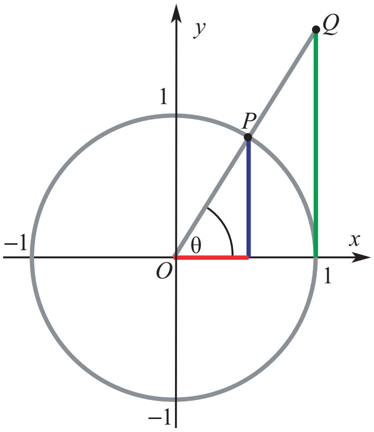
$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

A Convention: The *x*- and *y*-axes, as drawn, divide the circle into four parts, which are referred to as the first, second, third and fourth quadrants, Q1, Q2, Q3 and Q4, respectively, as shown.



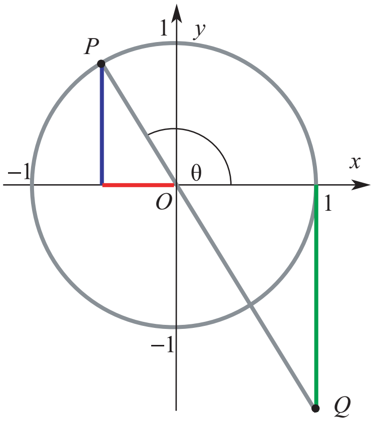
Further discussion of the new definitions

If θ is between 0° and 90° then it lies in Q1 and the diagram, as shown, gives **All** of sin θ, cos θ and tan θ to be positive, because the *x* and *y* coordinates of *P* and *Q* are all positive.

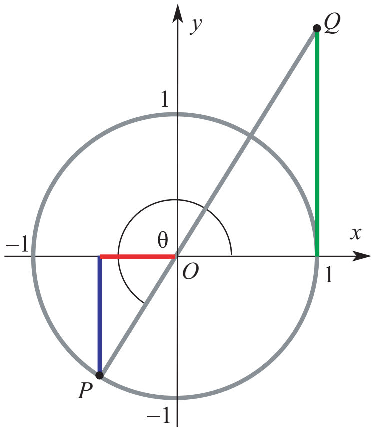


If θ is between 90° and 180° then it lies in Q2 and only **sin** θ is positive (cos θ and tan θ are both negative). This is because the *y* coordinate of *P* is positive, whereas the *x* coordinate of *P* and the *y* coordinate of *Q* are both negative.

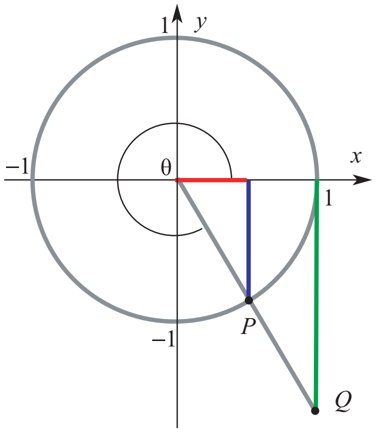
Note: The lines $x = 1$ and *OP* must be extended downwards to meet at *Q* and, hence, to find tan θ.



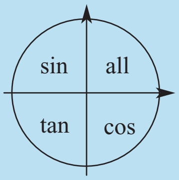
If θ is between 180° and 270° then it lies in Q3 and only **tan** θ is positive (sin θ and cos θ are both negative). This is because the *y* coordinate of *Q* is positive, whereas the *x* and *y* coordinates of *P* are both negative.



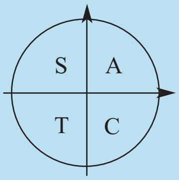
If θ is between 270° and 360° then it lies in Q4 and only **cos** θ is positive (sin θ and tan θ are both negative). This is because the *x* coordinate of *P* is positive, whereas the *y* coordinate of *P* and the *y* coordinate of *Q* are both negative.



These results are often summarised as:



or



Also referred to as ASTC or CAST.

Simplifying the work load

Considering angles in Q2, the student should note that:

$$\sin 150° = \sin 30°, \sin 170° = \sin 10°, \text{ and } \sin 163° = \sin 27°.$$

These illustrate the general result:

$$\mathbf{\sin \theta = \sin(180 - \theta)}$$

Similarly:

$$\cos 150° = -\cos 30°, \cos 170° = -\cos 10°, \text{ and } \cos 163° = -\cos 27°.$$

and

$$\tan 150° = -\tan 30°, \tan 170° = -\tan 10°, \text{ and } \tan 163° = -\tan 27°,$$

illustrating the general results:

$$\mathbf{\cos\theta = -\cos(180 - \theta)} \text{ and } \mathbf{\tan\theta = -\tan(180 - \theta)}$$

Considering angles in Q3, the student also should note that:

$$\sin 210° = -\sin 30°, \cos 200° = -\cos 20°, \text{ and } \tan 226° = \tan 46°.$$

These illustrate the general results:

$$\mathbf{\sin\theta = -\sin(\theta - 180)}, \mathbf{\cos\theta = -\cos(\theta - 180)} \text{ and } \mathbf{\tan\theta = \tan(\theta - 180)}$$

Similarly, for angles in Q4:

$$\mathbf{\sin\theta = -\sin(360 - \theta)}, \mathbf{\cos\theta = \cos(360 - \theta)} \text{ and } \mathbf{\tan\theta = -\tan(360 - \theta)}$$

Summary

Q2	Q3	Q4
$\sin\theta = \sin(180 - \theta)$	$\cos\theta = -\cos(180 - \theta)$	$\tan\theta = -\tan(180 - \theta)$
$\sin\theta = -\sin(\theta - 180)$	$\cos\theta = -\cos(\theta - 180)$	$\tan\theta = \tan(\theta - 180)$
$\sin\theta = -\sin(360 - \theta)$	$\cos\theta = \cos(360 - \theta)$	$\tan\theta = -\tan(360 - \theta)$

Note: Although the results above are general results and true for all values of θ, it is not necessary for students to extend their understanding beyond the discussion above.

Students may want to learn the results above off by heart; however, it is worth noting the following underlying principles:

1. The results 180 – θ, θ – 180 and 360 – θ simply calculate the acute angle between *OP* and the *x*-axis.
2. The *x* and *y* coordinates of all points on the circle, and therefore cos θ and sin θ, can be related back to points in Q1 using ASTC and the acute angle with the *x*-axis.

Use the following procedures when finding the exact value of angles of any magnitude:

1. Draw the angle required in a unit circle.
2. Calculate the acute angle that *OP* makes with the *x*-axis.
3. Use ASTC to establish whether the answer is positive or negative.
4. Use the exact angle results discussed in Section 2.6 if required.

Using technology

Using TI-Nspire:

1 Set the calculator to Radian mode.

2 Open the Graphs & Geometry application.

3 Press menu and select *Parametric* from the Graph Type submenu.

4 Input the following equations:

$x1(t) = \mathbf{cos(}\boldsymbol{t}\mathbf{)}$

$y1(t) = \mathbf{sin(}\boldsymbol{t}\mathbf{)}$

and change tstep to $\frac{\pi}{12}$.

5 Press enter.

6 Press menu, select *Window Settings* from the Window submenu and set the following:

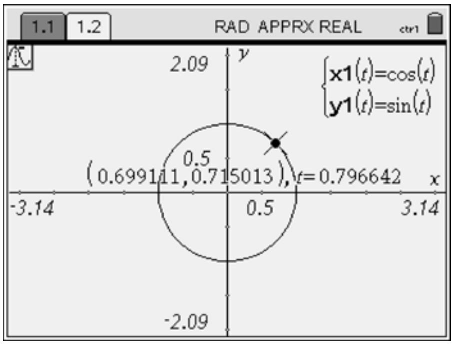
XMin: $-\pi$

XMax: π

YMin: –2.09

YMax: 2.09

7 Press menu and select *Graph Trace* from the Trace submenu.



```
x1(t)=cos(t)
y1(t)=sin(t)
(0.699111,0.715013), t=0.796642
```

Using ClassPad:

1 Set the calculator to Radian and Standard mode.

2 Tap Graph&Tab.

3 Tap ▼ and select xt=.

4 Input the following equations:

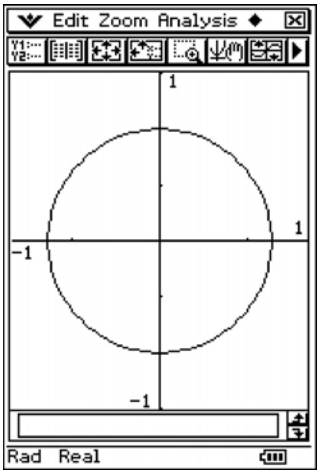
$xt1$: $\mathbf{cos(}\boldsymbol{t}\mathbf{)}$

$yt1$: $\mathbf{sin(}\boldsymbol{t}\mathbf{)}$

and then press EXE.

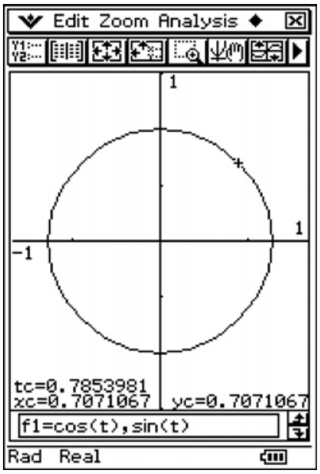
5 Tap ⌗ to view the graph and then tap Resize for a full-screen view.

6 Tap Zoom, then select Square.



7 Tap ⊞ and change tstep to $\frac{\pi}{12}$.

8 Tap Analysis and select Trace.



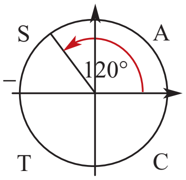
```
tc=0.7853981
xc=0.7071067   yc=0.7071067
f1=cos(t),sin(t)
```

Example 17

Find the exact value of cos 120°, without using a calculator.

Solution

$$\begin{aligned} \cos 120^\circ &= -\cos(180-120)^\circ \\ &= -\cos 60^\circ \\ &= -\frac{1}{2} \end{aligned}$$

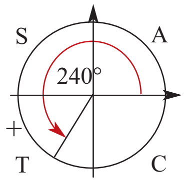


Example 18

Find the exact value of tan 240°, without using a calculator.

Solution

$$\begin{aligned} \tan 240^\circ &= +\tan(240-180)^\circ \\ &= \tan 60^\circ \\ &= \sqrt{3} \end{aligned}$$



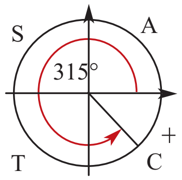
Example 19

Find the exact value of $\cos\left(\frac{7\pi}{4}\right)$, without using a calculator.

Solution

$$\frac{7\pi}{4} \times \frac{180^\circ}{\pi} = 315^\circ$$

$$\begin{aligned} \cos 315^\circ &= +\cos(360-315)^\circ \\ &= \cos 45^\circ \\ &= \frac{1}{\sqrt{2}} \\ \therefore \cos\frac{7\pi}{4} &= \frac{1}{\sqrt{2}} \end{aligned}$$



Note: Converting to degrees is not essential.

Boundary angles: 0° and 360°, 90°, 180° and 270°

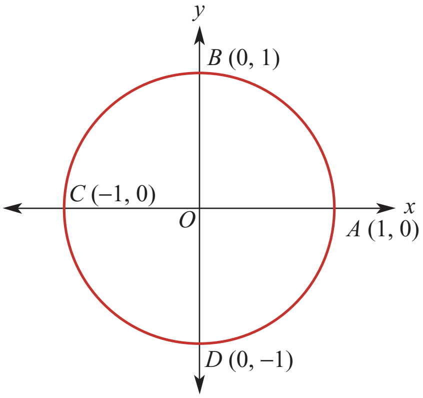
Remembering that, for all angles represented on the unit circle,

$$\begin{aligned} \boldsymbol{x} &= \mathbf{cos}\ \boldsymbol{\theta} \\ \boldsymbol{y} &= \mathbf{sin}\ \boldsymbol{\theta} \end{aligned}$$

we are able to specify values for the trigonometric functions of angles not belonging to any particular quadrant; that is, **boundary angles**.

On the diagram, the four points A, B, C and D are represented, corresponding to the angles 0° (360°), 90°, 180° and 270°, respectively.

Their (x, y) coordinates are also labelled.



Noting that $x = \cos\theta$, $y = \sin\theta$ and $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$,

we can set up the following table of values:

Angle	Sine	Cosine	Tangent
$0°, 360°$ (i.e. $0, 2\pi$)	0	1	0
$90°$ $\left(\text{i.e. } \dfrac{\pi}{2}\right)$	1	0	Undefined
$180°$ (i.e. π)	0	-1	0
$270°$ $\left(\text{i.e. } \dfrac{3\pi}{2}\right)$	-1	0	Undefined

Example 20

Find the exact value of $\cos\left(-\dfrac{\pi}{3}\right)$, without a calculator.

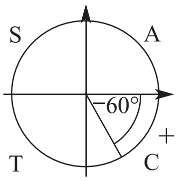
Solution

$$-\frac{\pi}{3} \times \frac{180°}{\pi} = -60°$$

$$\cos(-60°) = +\cos 60°$$

$$= \frac{1}{2}$$

$$\therefore \quad \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$



Note: In Example 20, negative angles are measured clockwise and the angle between OP and the x-axis is 60°, leading to cos (–60°) = +cos 60°.

Example 21

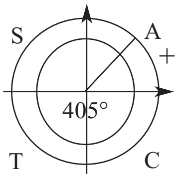
Find the exact value of sin 405°.

Solution

$$\sin 405° = \sin(405 - 360)°$$

$$= +\sin 45°$$

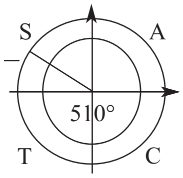
$$= \frac{1}{\sqrt{2}}$$



Example 22

Find the exact value of $\tan\left(\frac{17\pi}{6}\right)$.

Solution



$$\frac{17\pi}{6} \times \frac{180^\circ}{\pi} = 510^\circ$$

$$\begin{aligned}\tan(510^\circ) &= -\tan(540 - 510)^\circ \\ &= -\tan 30^\circ \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$

$$\therefore \quad \tan\left(\frac{17\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

Using technology

Using the TI-Nspire:

Degrees

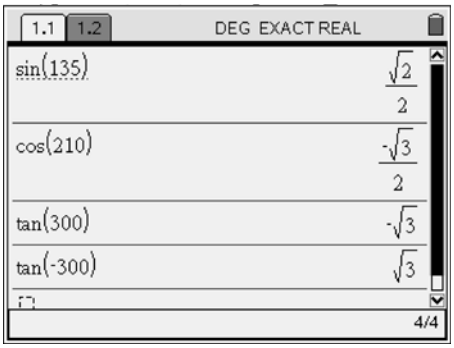
1 Set the calculator to Degree and Exact mode.

2 Type **sin(135)**, then press enter.

3 Type **cos(210)**, then press enter.

4 Type **tan(300)**, then press enter.

5 Type **tan(–300)**, then press enter.



sin(135) $\frac{\sqrt{2}}{2}$

cos(210) $\frac{-\sqrt{3}}{2}$

tan(300) $-\sqrt{3}$

tan(-300) $\sqrt{3}$

Using the ClassPad:

Degrees

1 Set the calculator to Degree and Standard mode.

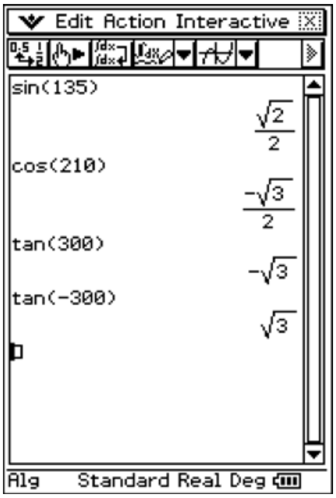
2 Press Keyboard, then tap TRIG to access the sin, cos and tan commands.

3 Type **sin(135)** and then press EXE.

4 Type **cos(210)** and then press EXE.

5 Type **tan(300)** and then press EXE.

6 Type **tan(–300)** and then press EXE.



sin(135) $\frac{\sqrt{2}}{2}$

cos(210) $\frac{-\sqrt{3}}{2}$

tan(300) $-\sqrt{3}$

tan(-300) $\sqrt{3}$

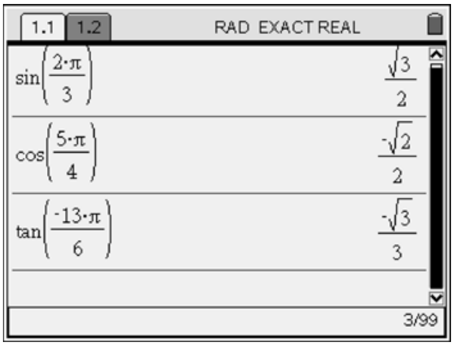
Radians

1 Set the calculator to Radian and Exact mode.

2 Type $\sin\left(\frac{2\pi}{3}\right)$, then press enter.

3 Type $\cos\left(\frac{5\pi}{3}\right)$, then press enter.

4 Type $\tan\left(-\frac{13\pi}{6}\right)$, then press enter.



$\sin\left(\frac{2\cdot\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$

$\cos\left(\frac{5\cdot\pi}{4}\right)$ $\frac{-\sqrt{2}}{2}$

$\tan\left(\frac{-13\cdot\pi}{6}\right)$ $\frac{-\sqrt{3}}{3}$

3/99

Radians

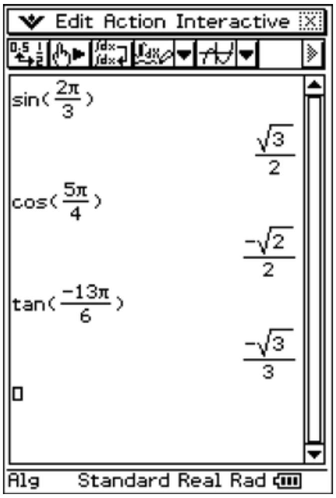
1 Set the calculator to Radian and Exact mode.

2 Press Keyboard, then tap TRIG to access the sin, cos and tan commands.

3 Type $\sin\left(\frac{2\pi}{3}\right)$ and then press EXE.

4 Type $\cos\left(\frac{5\pi}{3}\right)$ and then press EXE.

5 Type $\tan\left(-\frac{13\pi}{6}\right)$ and then press EXE.



$\sin(\frac{2\pi}{3})$

$\frac{\sqrt{3}}{2}$

$\cos(\frac{5\pi}{4})$

$\frac{-\sqrt{2}}{2}$

$\tan(\frac{-13\pi}{6})$

$\frac{-\sqrt{3}}{3}$

Exercise 2F

Examples 17,18

1 Without using a calculator, evaluate the sin, cos and tan of each of the following, expressing your answer in exact form. Validate using a calculator.

a 120° **b** 135° **c** 180° **d** 240°

e 315° **f** 390° **g** 420° **h** −135°

i −300° **j** −60° **k** 270° **l** 0°

Examples 19–22

2 Write down the exact values of each of the following. Validate using a calculator.

a $\sin\frac{2\pi}{3}$ **b** $\cos\frac{3\pi}{4}$ **c** $\tan 2\pi$ **d** $\sin\frac{7\pi}{6}$

e $\sin\pi$ **f** $\tan\frac{4\pi}{3}$ **g** $\sin\frac{5\pi}{3}$ **h** $\cos\frac{7\pi}{4}$

i $\tan\frac{11\pi}{6}$ **j** $\cos\pi$ **k** $\sin\frac{3\pi}{2}$ **l** $\tan\frac{\pi}{2}$

3 Write down the exact values of each of the following. Validate using a calculator.

a $\sin\left(-\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{11\pi}{4}\right)$ **c** $\tan\left(\frac{13\pi}{6}\right)$ **d** $\tan\left(\frac{15\pi}{6}\right)$

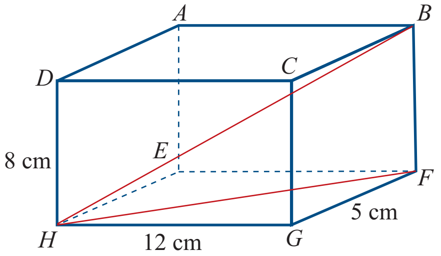
e $\cos\left(\frac{14\pi}{4}\right)$ **f** $\cos\left(-\frac{3\pi}{4}\right)$ **g** $\sin\left(\frac{11\pi}{4}\right)$ **h** $\cos\left(-\frac{21\pi}{3}\right)$

2.7 Modelling and problem solving

Exercise 2G

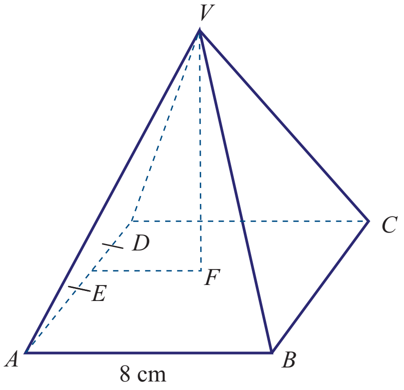
1 *ABCDEFGH* is a rectangular prism with dimensions as shown. Find the:

a length of *FH*
b length of *BH*
c magnitude of angle *BHF*
d magnitude of angle *BHG*

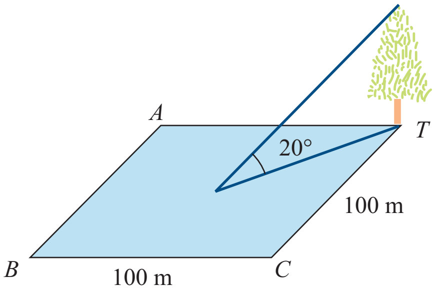


2 *VABCD* is a right pyramid with a square base. The sides of the base are 8 cm in length. The height, *VF*, of the pyramid is 12 cm. Find the:

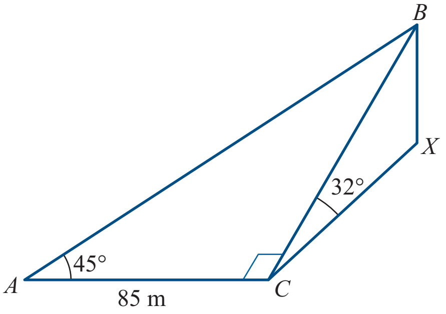
a length of *EF*
b magnitude of angle *VEF*
c length of *VE*
d length of a sloping edge
e magnitude of angle *VAD*
f surface area of the pyramid



3 A tree stands at the corner of a square playing field. Each side of the square is 100 m long. At the centre of the field the tree subtends an angle of 20°. What angle does it subtend at each of the other three corners of the field?



4 Suppose that *A*, *C* and *X* are three points in a horizontal plane and *B* is a point vertically above *X*. If the length of *AC* = 85 m, and the magnitudes of angles *BAC*, *ACB* and *BCX* are 45°, 90° and 32°, respectively, find the height *XB*.



5 Standing due south of a tower 50 m high, the angle of elevation to the top is 26°. What is the angle of elevation after walking a distance 120 m due east?

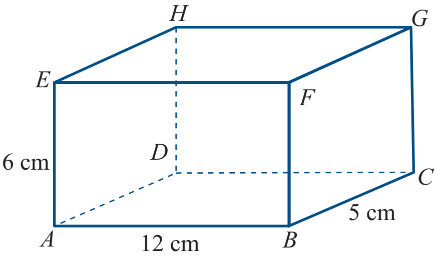
6 From the top of a cliff 160 m high two buoys are observed. Their bearings are 337° and 308°. Their respective angles of depression are 3° and 5°. Calculate the distance between the buoys.

7 Find the magnitude of each of the following angles for the rectangular prism shown:

a $\angle ACE$

b $\angle HDF$

c $\angle ECH$



8 From a point A due north of a tower, the angle of elevation to the top of the tower is 45°. From point B, 100 m on a bearing of 120° from A, the angle of elevation is 26°. Find the height of the tower.

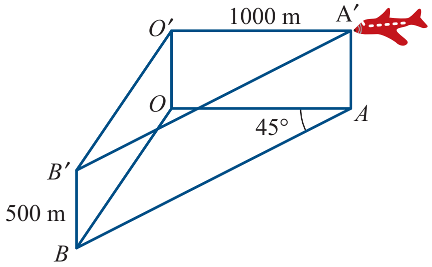
9 A and B are two positions on level ground. From an advertising balloon at a vertical height of 750 m, A is observed in an easterly direction and B at a bearing of 160°. The angles of depression of A and B, as viewed from the balloon, are 40° and 20°, respectively. Find the distance between A and B.

10 A right pyramid, height 6 cm, stands on a square base of side 5 cm. Find the:

a length of a sloping edge

b area of a triangular face

11 A light aircraft flying at a height of 500 m above the ground is sighted at a point A' due east of an observer stationed at a point O on the ground, who is measured horizontally to be 1 km from the plane. The aircraft is flying south-west (along $A'B'$) at 300 km/h.



a How far will it travel in 1 minute?

b Find its bearing from O (O') at this time.

c What will be its angle of elevation from O at this time?

12 Three points, A, B and C, are on a horizontal line such that $AB = 70$ m and $BC = 35$ m. The angles of elevation of the top of a tower are α, β and γ, where $\tan\alpha = \frac{1}{13}$, $\tan\beta = \frac{1}{15}$ and $\tan\gamma = \frac{1}{20}$. The foot of the tower is at the same level as A, B and C. Find the height of the tower.

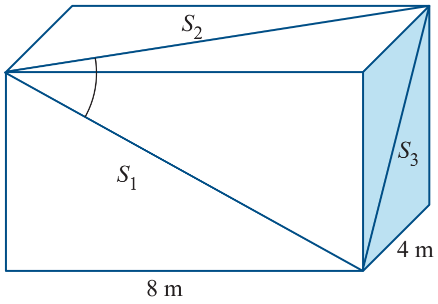
13 A cat sitting on the edge of a straight river bank spots a bird sitting in a tree directly across the river and on the river's edge. The angle of elevation from the cat to the bird is 15°. The cat then moves 25 m along the river bank, and now spots the same bird at an angle of elevation of 13°. How high is the tree?

14 Andrew and Rob have parked their cars directly under the take-off flight path of a passenger aircraft. Andrew's car is parked 2 km directly to the west of Rob's car, and both cars are at the same height.

Not long after the plane has taken off, Andrew observes that the plane is to the east at an angle of elevation of 28°. From Rob's position, the plane appears to the east at an angle of elevation of 43°. Thirty seconds later, Andrew sees the plane to the east at an angle of elevation of 40°, whereas from Rob's position the plane now appears to the west at an angle of elevation of 70°.

Determine the speed of the plane over this 30-second period. Give your answer to the nearest kilometre per hour. (You may assume that the plane's speed was constant. Do not assume that it travels at a constant altitude.)

15 A shipping company has decided to start the manufacture of its own shipping containers. Supporting beams (S_1, S_2 and S_3) line three of the sides of the container, as shown in the diagram. For the greatest structural stability under the expected transport conditions, engineers have calculated that the angle between the two supports S_1 and S_2 (as indicated in the diagram) must be equal to 30°.



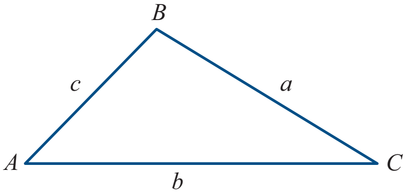
Determine the height of a container with floor dimensions of 8 m × 4 m if it is to be built to these specifications.

Chapter summary

- The **sine rule** is used to find unknown quantities in a triangle when one of the following situations arises:
  - one side and two angles are given
  - two sides and the non-included angle are given.

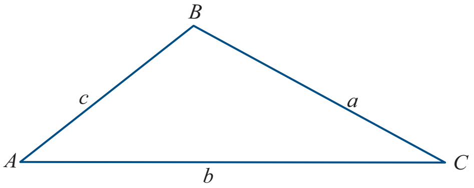
  The **sine rule** states that for a triangle ABC,

$$\frac{a}{\sin\ A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- The **cosine rule** is used to find unknown quantities in a triangle when one of the following situations arises:
  - two sides and an included angle are given
  - three sides are given.

  The **cosine rule** states that for a triangle ABC,



$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or, equivalently, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- A radian is defined as the size of the angle formed when two radii in a circle cut off an arc whose length is equal to that of the radii.

  To convert degrees to radians, multiply by $\frac{\pi}{180}$.

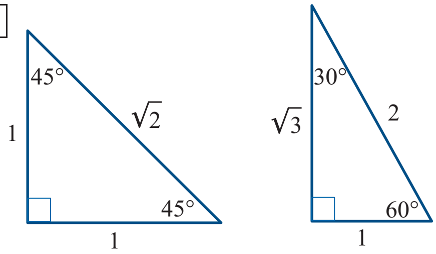
  To convert radians to degrees, multiply by $\frac{180}{\pi}$.

- Sine, cosine and tangent ratios of special angles and boundary angles are summarised thus:

Special angles

θ (θ°)	**sin θ**	**cos θ**	**tan θ**
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

OR



Boundary angles

θ	**sin θ**	**cos θ**	**tan θ**
0, 2π (0°, 360°)	0	1	0
$\frac{\pi}{2}$ (90°)	1	0	Undefined
π (180°)	0	−1	0
$\frac{3\pi}{2}$ (270°)	−1	0	Undefined

For angles larger than 90°, trigonometric ratios may be found as follows:

θ in quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$
1	$+\sin\theta$	$+\cos\theta$	$+\tan\theta$
2	$+\sin(180^\circ-\theta)$	$-\cos(180^\circ-\theta)$	$-\tan(180^\circ-\theta)$
3	$-\sin(180^\circ-\theta)$	$-\cos(180^\circ-\theta)$	$+\tan(180^\circ-\theta)$
4	$-\sin(360^\circ-\theta)$	$+\cos(360^\circ-\theta)$	$-\tan(360^\circ-\theta)$

Multiple-choice questions

1 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62^\circ$. The magnitude of $\angle XYZ$, correct to 1 decimal place, is:

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°

2 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c, to the nearest whole number, is:

A 9 **B** 10 **C** 11 **D** 81 **E** 129

3 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is:

A 43° **B** 63° **C** 74° **D** 82° **E** 98°

4 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is:

A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m

5 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is:

A 1° **B** 4° **C** 53° **D** 86° **E** 89°

6 A bushwalker walks 5 km due east and then 7 km due south. The bearing the bushwalker must take to return to the start is:

A 036° **B** 306° **C** 324° **D** 332° **E** 348°

7 A boat sails at a bearing of 215° from A to B. The bearing it must take from B to return to A is:

A 035° **B** 055° **C** 090° **D** 215° **E** 250°

8 The exact value of $\sin 60^\circ$ is:

A $\frac{1}{2}$ **B** $\frac{1}{\sqrt{2}}$ **C** $\frac{\sqrt{3}}{2}$ **D** 0 **E** 1

9 The exact value of $\cos\frac{2\pi}{3}$ is:

A $\frac{1}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $-\frac{\sqrt{3}}{2}$ **E** 1

10 The exact value of $\tan\left(-\frac{3\pi}{2}\right)$ is:

A 0 **B** 1 **C** -1 **D** $\sqrt{3}$ **E** Not defined

Short-response questions

1 Convert to radians:

a 210° (Give your answer in terms of π.)

b 29° (Give your answer to 2 decimal places.)

c $117°63'$ (Give your answer to 2 decimal places.)

2 Convert to degrees:

a $\frac{3\pi}{4}^{c}$

b 2.7^c (Give your answer to the nearest minute.)

3 Write each of the following in exact (surd or fraction) form (without using a calculator):

a $\sin 210°$ **b** $\cos 315°$ **c** $\tan \frac{2\pi}{3}$ **d** $\cos 270°$ **e** $\sin\left(-\frac{\pi}{6}\right)$

f $\tan \frac{\pi}{2}$ **g** $\cos\left(-\frac{2\pi}{3}\right)$ **h** $\sin(570°)$ **i** $\tan\left(-\frac{19\pi}{6}\right)$

4 From a port P, a ship Q is 20 km away on a bearing of 125°, and a ship R is 35 km away on a bearing of 050°. Find the distance between the two ships.

5 In a quadrilateral $ABCD$, $AB = 5$ cm, $BC = 6$ cm, $CD = 7$ cm, $B = 120°$ and $C = 90°$. Find the length of the diagonal AC.

6 If $\sin x = \sin 37°$ and x is obtuse, find x.

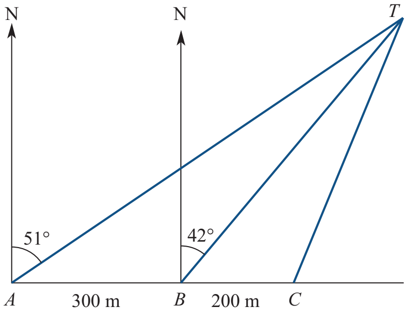
7 A point T is 10 km due north of a point S, and a point R, which is east of a straight line joining T and S, is 8 km from T and 7 km from S.

a Calculate the bearing of R from S.

b A fourth point, Q, is on a bearing of 319° from S and is 12 km from T. Calculate the magnitude of $\angle TQS$ and, hence, the bearing of T from Q.

8 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 80°$ and $\angle ABC = 70°$. Calculate the length of AC.

9 The diagram shows three survey points, A, B and C, which are on an east–west line on level ground. From point A, the bearing of the foot of a tower is 051°, whereas from B the bearing of the tower is 042°. Find:



a **i** $\angle TAB$ **ii** $\angle ATB$

b AT **c** CT

10 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour?

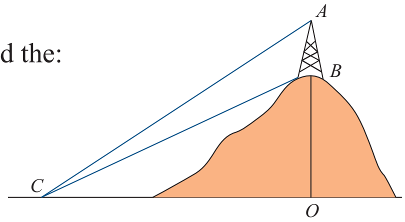
11 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C.

a If the bearing of C from A is 120°, find the distances AC and BC.

b The helicopter flies at a constant speed throughout and takes 5 minutes to fly from A to C. Find its speed.

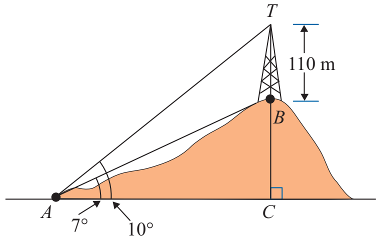
12 From a cliff top 150 m above sea level, two boats are observed. One has an angle of depression of 18° and is due east, and the other has an angle of depression of 9° on a bearing of 148°. Calculate, correct to the nearest metre, the distance between the boats.

13 *AB* is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°. Find the:



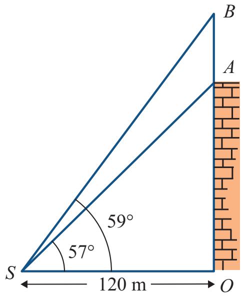
- **a** magnitude of angles *ACB*, *CBO* and *CBA*
- **b** length of *BC*
- **c** height of the hill; that is, the length of *OB*

14 A tower 110 m high stands on the top of a hill. From a point *A* at the foot of the hill the angle of elevation of the bottom of the tower is 7°, and that of the top of the tower is 10°.



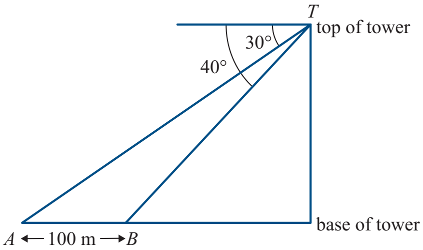
- **a** Find the magnitude of angles *TAB*, *ABT* and *ATB*.
- **b** Use the sine rule to find the length of *AB*.
- **c** Find *CB*, the height of the hill.

15 Point *S* is a distance of 120 m from the base of a building. On the building is an aerial, *AB*. The angle of elevation from *S* to *A* is 57°. The angle of elevation from *S* to *B* is 59°. Find the following distances:



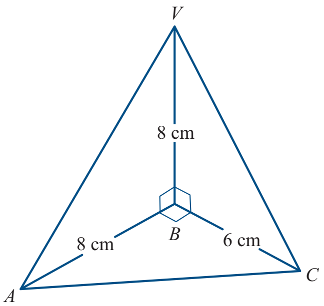
- **a** *OA*
- **b** *OB*
- **c** *AB*

16 From the top of a communications tower, the angles of depression of two points, *A* and *B*, on a horizontal line through the foot of the tower are 30° and 40°, respectively. The distance between the points is 100 m. Find the:



- **a** distance *AT*
- **b** distance *BT*
- **c** height of the tower

17 Angles *VBC*, *VBA* and *ABC* are right angles. Find the:



- **a** distance *VA*
- **b** distance *VC*
- **c** distance *AC*
- **d** magnitude of angle *VCA*

CHAPTER

3

Exponential functions and logarithms

Objectives

- To define and understand **exponential functions**.
- To **sketch** graphs of the various types of exponential functions.
- To understand the rules for manipulating **exponential** and **logarithmic expressions**.
- To solve **exponential equations**.
- To evaluate **logarithmic expressions**.
- To solve equations using **logarithmic methods**.
- To understand and use a range of **exponential models**.
- To **sketch** graphs of exponential functions.
- To **sketch** graphs of logarithmic functions.
- To **apply** exponential and logarithmic functions to solving problems.

The function, $y = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1, is called an **exponential function** (or index function).

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n 1-minute intervals (assuming that all organisms are still alive).

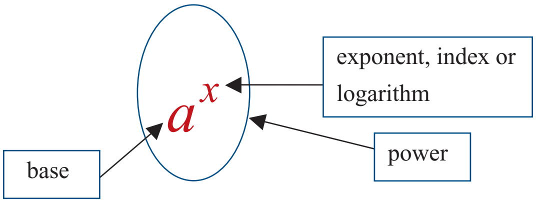
n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus, P defines a function that has the rule $P = 2^n$, which is an exponential (or index) function.

3.1 Rules for exponents (indices)

We shall review the rules for manipulating exponential expressions, a^x, where a is a real number called the **base**, and x is a real number called the **exponent** or **index**.

Values of a^x are **powers** of the **base**.



Some powers of base 2 are:

x	0	1	2	3	4
2^x (power)	1	2	4	8	16

Multiplication, $a^m \times a^n$

If m and n are positive integers,

then $$a^m = \underset{\leftarrow\ m \text{ factors}\ \rightarrow}{a \times a \times a \ldots \times a}$$

and $$a^n = \underset{\leftarrow n \text{ factors} \rightarrow}{a \times a \ldots \times a}$$

$$\begin{aligned} \therefore \quad a^m \times a^n &= \underset{\leftarrow\ m \text{ factors}\ \rightarrow}{(a \times a \times a \ldots \times a)} \times \underset{\leftarrow n \text{ factors} \rightarrow}{(a \times a \ldots \times a)} \\ &= \underset{\leftarrow m+n \text{ factors} \rightarrow}{(a \times a \times a \ldots \times a)} \\ &= a^{m+n} \end{aligned}$$

Rule 1 $a^m \times a^n = a^{m+n}$

To multiply two numbers in exponent form with the **same base**, **add** the exponents.

i.e. $a^m \times a^n = a^{m+n}$

Example 1

Simplify:

a $2^3 \times 8^4$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^2$

Solution

a
$$\begin{aligned} 2^3 \times 8^4 &= 2^3 \times (2^3)^4 \\ &= 2^3 \times 2^3 \times 2^3 \times 2^3 \times 2^3 \\ &= 2^{15} \end{aligned}$$

b
$$\begin{aligned} x^2y^3 \times x^4y &= x^2 \times x^4 \times y^3 \times y \\ &= x^6y^4 \end{aligned}$$

c
$$\begin{aligned} 2^x \times 2^{x+2} &= 2^{x+x+2} \\ &= 2^{2x+2} \end{aligned}$$

d
$$\begin{aligned} 3a^2b^3 \times 4a^3b^2 &= 3 \times 4a^2 \times a^3 \times b^3 \times b^2 \\ &= 12a^5b^5 \end{aligned}$$

Division: $a^m \div a^n$

If m and n are positive integers and $m > n$,

then
$$\begin{aligned} a^m \div a^n &= \frac{\overset{\leftarrow m \text{ factors} \rightarrow}{\cancel{a} \times \cancel{a} \times \cancel{a} \ldots a}}{\underset{\leftarrow n \text{ factors} \rightarrow}{\cancel{a} \times \cancel{a} \times \cancel{a} \ldots a}} \\ &= \underset{\leftarrow (m-n) \text{ factors} \rightarrow}{a \times a \times a \ldots a} \qquad \text{(by cancelling)} \\ &= a^{m-n} \end{aligned}$$

Rule 2 $a^m \div a^n = a^{m-n}$

To divide two numbers in exponent form with the **same base, subtract** the exponents.

i.e. $a^m \div a^n = a^{m-n}$

Note that **Rule 1** and **Rule 2** also hold for negative indices m, n for $a \neq 0$.
For example:

$$2^4 \times 2^{-3} = 2^{4\,+\,-3} = 2^1$$
$$2^5 \div 2^{-1} = 2^{5\,-\,(-1)} = 2^{5\,+\,1} = 2^6$$

Example 2

Simplify:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

Solution

a $$\begin{aligned}\frac{x^4y^3}{x^2y^2} &= x^{4-2}y^{3-2}\\ &= x^2y\end{aligned}$$

b $$\begin{aligned}\frac{b^{4x} \times b^{x+1}}{b^{2x}} &= b^{4x+x+1-2x}\\ &= b^{3x+1}\end{aligned}$$

c $$\begin{aligned}\frac{16a^5b \times 4a^4b^3}{8ab} &= \frac{64a^9b^4}{8ab}\\ &= 8a^8b^3\end{aligned}$$

Raising the power to the index *n*, $(a^m)^n$

Consider the following:

$$\begin{aligned}(2^3)^2 &= 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3\times 2}\\ (4^3)^4 &= 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3\times 4}\\ (a^2)^5 &= a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2\times 5}\end{aligned}$$

In general, $(a^m)^n = a^{m \times n}$.

Rule 3 $(a^m)^n = a^{mn}$

To raise the power of a to an index, **multiply** the indices or exponents.

i.e. $(a^m)^n = a^{m \times n}$

Products and quotients

Consider $(ab)^n$.

$$\begin{aligned}(ab)^n &= (ab) \times (ab) \times \ldots \times (ab)\\ &\quad \longleftarrow \quad n \text{ factors} \quad \longrightarrow\\ &= (a \times a \times \ldots a) \times (b \times b \times \ldots b)\\ &\quad \leftarrow n\text{ factors}\rightarrow \quad \leftarrow n \text{ factors} \rightarrow\\ &= a^nb^n\end{aligned}$$

Rule 4 $(ab)^n = a^nb^n$

Consider $\left(\dfrac{a}{b}\right)^n$.

$$\begin{aligned}\left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \cdots \frac{a}{b}\\ &\quad \leftarrow n \text{ factors} \rightarrow\\ &\quad \leftarrow n \text{ factors} \rightarrow\\ &= \frac{a \times a \times \ldots a}{b \times b \times \ldots b}\\ &\quad \leftarrow n \text{ factors}\rightarrow\\ &= \frac{a^n}{b^n}\end{aligned}$$

Rule 5 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example 3

Simplify the following, expressing with positive exponents:

a 8^{-2} **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **d** $\frac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a
$$\begin{aligned} 8^{-2} &= \frac{1}{8^2} \\ &= \frac{1}{(2^3)^2} \\ &= \frac{1}{2^6} \end{aligned}$$

b
$$\begin{aligned} \left(\frac{1}{2}\right)^{-4} &= \frac{1}{2^{-4}} \\ &= 2^4 \end{aligned}$$

c
$$\begin{aligned} \frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} &= \frac{3^{-3} \times 2^4 \times 3^4 \times 3^{-3} \times 2^{-3} \times 2^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}} \\ &= \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}} \\ &= 3^6 \end{aligned}$$

d
$$\begin{aligned} \frac{3^{2n} \times 6^n}{8^n \times 3^n} &= \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} \\ &= \frac{3^n \times 3^n}{2^{2n}} \\ &= \left(\frac{3}{2}\right)^{2n} \end{aligned}$$

Raising to the index 0, a^0 for $a \neq 0$

Consider $\frac{a^m}{a^m}$.

$$\begin{aligned} \frac{a^m}{a^m} &= \left(\frac{\cancel{a} \times \cancel{a} \times \cancel{a} \ldots m \text{ factors}}{\cancel{a} \times \cancel{a} \times \cancel{a} \ldots m \text{ factors}}\right) \\ a^{m-m} &= 1 \qquad \text{by cancelling} \\ a^0 &= 1 \end{aligned}$$

Rule 6 $a^0 = 1$

Example 4

Simplify each of the following, using index rules:

a $\dfrac{3^3 \times 3^1}{3^4}$ **b** $\dfrac{4 \times (-2)^3}{32}$ **c** $\dfrac{a^{-m}}{a^m \times a^{-2m}}$

Solution

a
$$\begin{aligned}\frac{3^3 \times 3^1}{3^4} &= \frac{3^{3+1}}{3^4}\\ &= \frac{3^4}{3^4}\\ &= 3^0\\ &= 1\end{aligned}$$

b
$$\begin{aligned}\frac{4 \times (-2)^3}{32} &= \frac{2^2 \times (-1)^3 \times 2^3}{2^5}\\ &= \frac{-1 \times 2^{2+3}}{2^5}\\ &= -1 \times 2^{2+3-5}\\ &= -1 \times 2^0\\ &= -1\end{aligned}$$

c
$$\begin{aligned}\frac{a^{-m}}{a^m \times a^{-2m}} &= \frac{a^{-m}}{a^{-m}}\\ &= a^0\\ &= 1\end{aligned}$$

Raising to a negative index, a^{-n}, $a \neq 0$

Consider a^{-n}.

$$\begin{aligned}a^{-n} &= a^{0-n}\\ &= \frac{a^0}{a^n}\\ &= \frac{1}{a^n}\end{aligned}$$

Rule 7 $a^{-n} = \dfrac{1}{a^n}, \quad a \neq 0$

This rule should be used to express all answers in positive exponent form.

Example 5

Simplify each of the following, using index rules:

a $3^{-1} \times 2^{-3}$ **b** $(-5)^{-2}$ **c** $\dfrac{a^{-x}}{b^{-y}}$

Solution

a
$$\begin{aligned}3^{-1} \times 2^{-3} &= \frac{1}{3^1} \times \frac{1}{2^3}\\ &= \frac{1}{3 \times 8}\\ &= \frac{1}{24}\end{aligned}$$

b
$$\begin{aligned}(-5)^{-2} &= \frac{1}{(-5)^2}\\ &= \frac{1}{25}\end{aligned}$$

c
$$\begin{aligned}\frac{a^{-x}}{b^{-y}} &= a^{-x} \times \frac{1}{b^{-y}}\\ &= \frac{1}{a^x} \times \frac{1}{\frac{1}{b^y}}\\ &= \frac{1}{a^x} \times \frac{b^y}{1}\\ &= \frac{b^y}{a^x}\end{aligned}$$

Note: Shifting a power from the numerator to the denominator, or vice versa, *changes the sign* of the index.

Exercise 3A

1 Simplify:

a $x^2 \times x^3$ **b** $2 \times x^3 \times x^4 \times 4$ **c** $\frac{x^5}{x^3}$ **d** $\frac{4x^6}{2x^3}$

e $(a^3)^2$ **f** $(2^3)^2$ **g** $(xy)^2$ **h** $(x^2y^3)^2$

i $\left(\frac{x}{y}\right)^3$ **j** $\left(\frac{x^3}{y^2}\right)^2$ **k** $(-2)^0$ **l** $\left(\frac{2}{3}\right)^0$

m $\left(\frac{2}{3}\right)^{-1}$ **n** $\frac{5^{-2}}{(2^{-1})^3}$

2 Simplify:

Example 1

a $x^3 \times x^4 \times x^2$ **b** $2^4 \times 4^3 \times 8^2$ **c** $3^4 \times 9^2 \times 27^3$

d $(q^2p)^3 \times (qp^3)^2$ **e** $a^2b^{-3} \times (a^3b^2)^3$ **f** $(2x^3)^2 \times (4x^4)^3$

g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$ **h** $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

3 Simplify:

Example 2

a $\frac{x^3y^5}{xy^2}$ **b** $\frac{16a^5b \times 4a^4b^3}{8ab}$

c $\frac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$ **d** $\frac{(-3x^2y^3)^2}{(2xy)^3} \times \frac{4x^4y^3}{(xy)^3}$

Example 3

4 Simplify each of the following, expressing with positive exponents:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\frac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\frac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\frac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\frac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

5 Simplify each of the following, expressing your answer in positive exponent form:

Example 4

a $\frac{4^n \times 8^{n+1}}{2^3 \times 32^n}$ **b** $3^{4n} \times 9^{2n} \times 27^{3n}$ **c** $\frac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\frac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\frac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\frac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\frac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\frac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\frac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

Example 5

j $\frac{2^{-1}}{3^{-4n}}$ **k** $\frac{3^{-n} \times 5^{2n}}{5^{3n} \times 9^{-n}}$ **l** $\frac{8^{2n} \times 9^{-n} \times 4}{27^n \times 2^{3(2n+1)}}$

6 Simplify and evaluate:

a $\frac{(8^3)^4}{(2^{12})^2}$ **b** $\frac{(125)^3}{(25)^2}$ **c** $\frac{(81)^4 \div 27^3}{9^2}$

MAPS

7 Given $x^0 = 1$ and $0^p = 0$, use a graphics calculator to approximate 0^0.

3.2 Rational exponents

$a^{\frac{1}{n}}$, where n is a natural number, is defined to be the n^{th} root of a, which is denoted $\sqrt[n]{a}$. If $a \geq 0$ then $a^{\frac{1}{n}}$ is defined for all $n \in N$. If $a < 0$ then $\sqrt[n]{a}$ is only defined for n odd. (Remember that only real numbers are being considered.)

$$a^{\frac{1}{n}} = \sqrt[n]{a} \text{ with } \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots

Given: $\sqrt[2]{a} = \sqrt{a}$

$$\begin{aligned} \sqrt{a} \times \sqrt{a} &= a \\ a^{\frac{1}{2}} \times a^{\frac{1}{2}} &= a \\ \therefore \sqrt{a} &= a^{\frac{1}{2}} \end{aligned}$$

Using this notation for cube roots

Given:

$$\begin{aligned} \sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} &= a \\ a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} &= a \\ \therefore \sqrt[3]{a} &= a^{\frac{1}{3}} \end{aligned}$$

Furthermore, the expression a^x can be defined for rational exponents (fractions); that is, when $x = \frac{m}{n}$, where m and n are integers, by defining:

Rule 8 $\quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Example 6

Evaluate, without using a calculator:

a $16^{\frac{5}{2}}$ **b** $9^{-\frac{1}{2}}$ **c** $64^{-\frac{2}{3}}$

Solution

a $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

b $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

c $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(64^{\frac{1}{3}}\right)^2} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

To summarise:

$$\begin{aligned} a^m \times a^n &= a^{m+n} \\ a^m \div a^n &= a^{m-n} \\ (a^m)^n &= a^{m \times n} \\ (ab)^n &= a^n b^n \\ \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \\ a^{-n} &= \frac{1}{a^n}, \quad a \neq 0 \\ a^{\frac{m}{n}} &= \left(a^{\frac{1}{n}}\right)^n = (a^m)^{\frac{1}{n}}; \text{ i.e. } a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \end{aligned}$$

Note: These rules are also applicable for all rational exponents, where m and n are fractions.

Example 7

Simplify:

a $\dfrac{3^{\frac{1}{4}}\times\sqrt{6}\times\sqrt[4]{2}}{16^{\frac{3}{4}}}$

b $(x^{-2}y)^{\frac{1}{2}}\times\left(\dfrac{x}{y^{-3}}\right)^{4}$

Solution

a $$\frac{3^{\frac{1}{4}}\times\sqrt{6}\times\sqrt[4]{2}}{16^{\frac{3}{4}}}=\frac{3^{\frac{1}{4}}\times 3^{\frac{1}{2}}\times 2^{\frac{1}{2}}\times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^{3}}$$
$$=\frac{3^{\frac{1}{4}}\times 3^{\frac{1}{2}}\times 2^{\frac{1}{2}}\times 2^{\frac{1}{4}}}{2^{3}}=\frac{3^{\frac{3}{4}}\times 2^{\frac{3}{4}}}{2^{3}}$$
$$=\frac{3^{\frac{3}{4}}}{2^{\frac{12}{4}-\frac{3}{4}}}=\frac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$$

b $$(x^{-2}y)^{\frac{1}{2}}\times\left(\frac{x}{y^{-3}}\right)^{4}=x^{-1}y^{\frac{1}{2}}\times\frac{x^{4}}{y^{-12}}$$
$$=x^{3}\times y^{\frac{25}{2}}$$

Exercise 3B

Example 6 **1** Evaluate:

a $125^{\frac{2}{3}}$ **b** $243^{\frac{3}{5}}$ **c** $81^{-\frac{1}{2}}$ **d** $64^{\frac{2}{3}}$

e $\left(\dfrac{1}{8}\right)^{\frac{1}{3}}$ **f** $32^{-\frac{2}{5}}$ **g** $125^{-\frac{2}{3}}$ **h** $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ **j** $10\,000^{\frac{3}{4}}$ **k** $81^{\frac{3}{4}}$ **l** $\left(\dfrac{27}{125}\right)^{\frac{1}{3}}$

Example 7 **2** Simplify:

a $\sqrt[3]{4}\times\sqrt[5]{32}$ **b** $\sqrt[3]{a^{2}b}\div\sqrt{ab^{3}}$ **c** $(a^{-2}b)^{3}\times\left(\dfrac{1}{b^{-3}}\right)^{\frac{1}{2}}$

d $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}}\times 15^{\frac{3}{2}}}$ **e** $2^{\frac{3}{2}}\times 4^{-\frac{1}{4}}\times 16^{-\frac{3}{4}}$ **f** $\left(\dfrac{x^{3}y^{-2}}{3^{-3}y^{-3}}\right)^{-2}\div\left(\dfrac{3^{-3}x^{-2}y}{x^{4}y^{-2}}\right)^{2}$

g $\left(\sqrt[5]{a^{2}}\right)^{\frac{3}{2}}\times\left(\sqrt[3]{a^{5}}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ **b** $(x-1)^{2}\sqrt{x-1}$ **c** $(x^{2}+1)\sqrt{x^{2}+1}$

d $(x-1)\sqrt[3]{x-1}$ **e** $\dfrac{1}{\sqrt{x-1}}+\sqrt{x-1}$ **f** $(5x^{2}+1)\sqrt[3]{5x^{2}+1}$

3.3 Solving exponential equations and inequations

Method 1

Given $a^x = a^y$

$\therefore x = y$ (by equating the exponents)

When both sides of the equation have the **same base**, the **exponents are equal**.

Example 8

Find the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$ **c** $5^{2x-4} = 25^{-x+2}$

Solution

a

$$\begin{aligned} 4^x &= 256 \\ 4^x &= 4^4 \\ \therefore x &= 4 \end{aligned}$$

b

$$\begin{aligned} 3^{x-1} &= 81 \\ 3^{x-1} &= 3^4 \\ x-1 &= 4 \\ \therefore x &= 5 \end{aligned}$$

c

$$\begin{aligned} 5^{2x-4} &= 25^{-x+2} \\ &= (5^2)^{-x+2} \\ &= 5^{-2x+4} \\ \therefore 2x-4 &= -2x+4 \\ 4x &= 8 \\ \therefore x &= 2 \end{aligned}$$

Example 9

Solve $9^x = 12 \times 3^x - 27$.

Solution

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$

Then $y^2 = 12y - 27$

$$y^2 - 12y + 27 = 0$$

$$(y-3)(y-9) = 0$$

Therefore, $y - 3 = 0$ or $y - 9 = 0$

$y = 3$ or $y = 9$

Hence $3^x = 3^1$ or $3^x = 3^2$

$\therefore x = 1$ or $x = 2$

Method 2

Using technology

When both sides of the equation cannot be expressed with the same base.

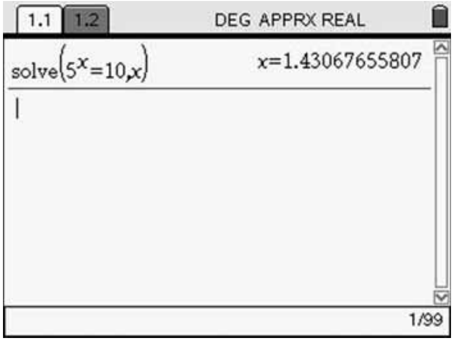
Example 10

Solve $5^x = 10$, correct to 2 decimal places.

Solution

Using the TI-Nspire:

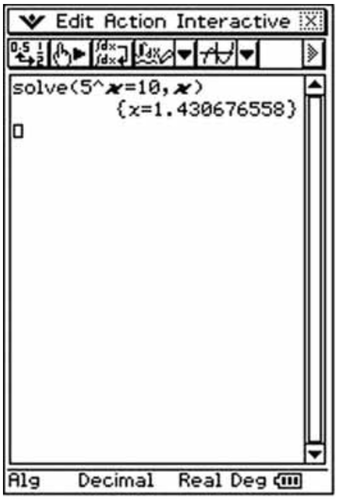
1. Set to Approximate.
2. Press (menu) and select *solve* from the Algebra menu.
3. Type $\mathbf{5^x = 10, x)}$ then press (enter).



```
solve(5^x=10,x)                x=1.43067655807
```

Using the ClassPad:

1. Set to Decimal.
2. From the Action menu select *solve* from the Advanced submenu.
3. Type $\mathbf{5^x = 10, x)}$ then press (EXE).



```
solve(5^x=10,x)
          {x=1.430676558}
```

Answer: $x = 1.43$ (correct to 2 decimal places).

Solution of inequalities

Method 1

Example 11

Solve:

a $16^x > 2$ **b** $2^{-3x+1} < \frac{1}{16}$

Solution

a

$$\begin{aligned} 2^{4x} &> 2^1 \\ \Leftrightarrow 4x &> 1 \\ \Leftrightarrow x &> \frac{1}{4} \end{aligned}$$

b

$$\begin{aligned} 2^{-3x+1} &< 2^{-4} \\ \Leftrightarrow -3x+1 &< -4 \\ \Leftrightarrow -3x &< -5 \\ \Leftrightarrow x &> \frac{5}{3} \end{aligned}$$

Note: Multiplication or division by a negative number *reverses* the inequality.

Method 2

Using technology

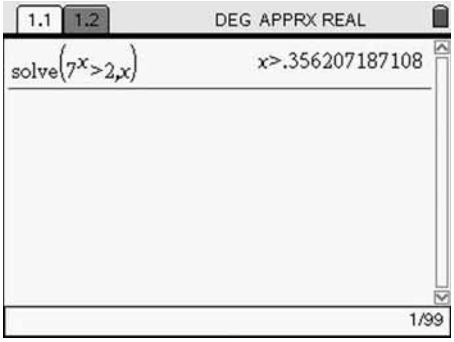
Example 12

Solve $7^x > 2$, correct to 2 decimal places.

Solution A

Using the TI-Nspire:

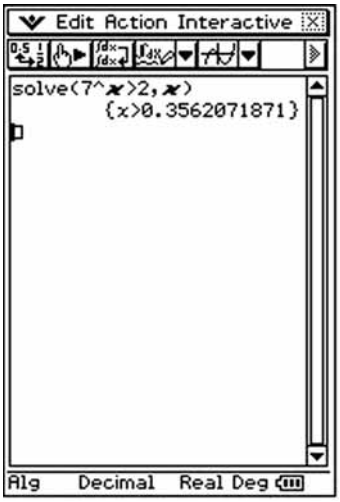
1. Set to Approximate.
2. Type **solve($7^x > 2, x$)** then press (enter).



```
solve(7^x>2,x)        x>.356207187108
```

Using the ClassPad:

1. Set to Decimal.
2. Type **solve($7^x > 2, x$)** then press (EXE).



```
solve(7^x>2,x)
          {x>0.3562071871}
```

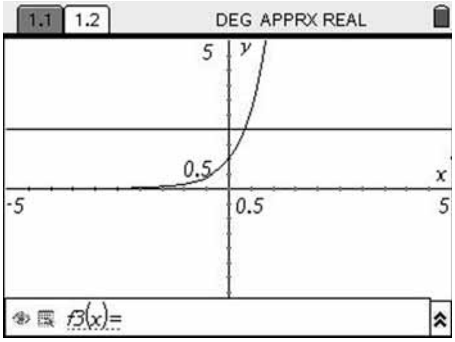
Answer: $x > 0.36$ (correct to 2 decimal places).

Solution B

Using the TI-Nspire:

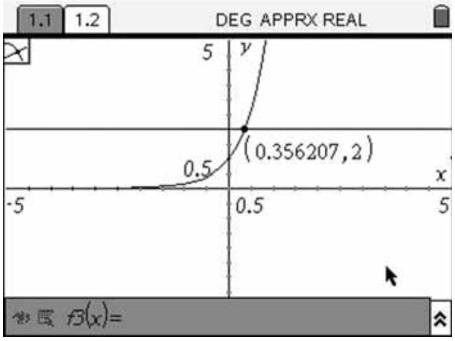
1 Type $\mathbf{7^x}$ into $f1(x)$ and press (enter).

2 Type **2** into $f2(x)$ and press (enter).



3 To calculate the point of intersection, press (menu) and select *Intersection Point(s)* from the Points & Lines submenu.

4 Move the cursor to the point of intersection to display its coordinates.



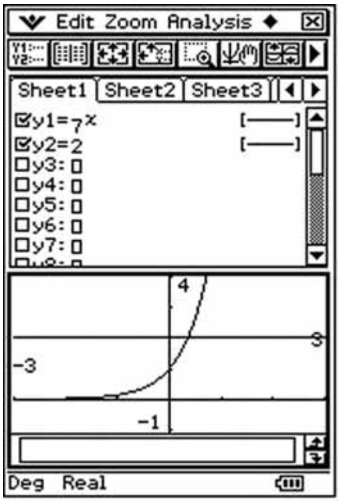
(0.356207, 2)

Using the ClassPad:

1 Type $\mathbf{7^x}$ into $y1$ and press (EXE).

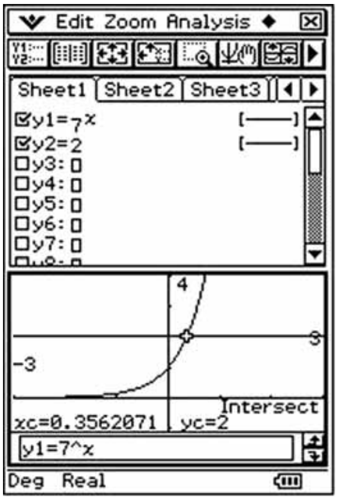
2 Type **2** into $y2$ and press (EXE).

3 Tap ⊞ to sketch the two functions.



☑y1=7^x
☑y2=2
☐y3: ☐
☐y4: ☐
☐y5: ☐
☐y6: ☐
☐y7: ☐

4 To calculate the point of intersection tap Analysis and select *Intersect* from the G-Solve submenu.



☑y1=7^x
☑y2=2
☐y3: ☐
☐y4: ☐
☐y5: ☐
☐y6: ☐
☐y7: ☐

xc=0.3562071 yc=2

y1=7^x

Answer: $x > 0.36$ (correct to 2 decimal places).

Exercise 3C

Example 8 **1** Solve for x:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$

d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$

g $16^x = 256$ **h** $4^{-x} = \frac{1}{64}$ **i** $5^{-x} = \frac{1}{125}$

2 Solve for n:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$

d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$

g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$

j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$

l $125^{4-n} = 5^{6-2n}$ **m** $4^{2-n} = \frac{1}{2048}$

3 Solve the exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^x = 243$ **c** $(27 \times 3^x)^2 = 27^x \times 3^{\frac{1}{2}}$

Example 9 **4** Solve for x, without using your calculator:

a $4(2^{2x}) = 8(2^x) - 4$ **b** $8(2^{2x}) - 10(2^x) + 2 = 0$

c $3 \times 2^{2x} - 18(2^x) + 24 = 0$ **d** $9^x - 4(3^x) + 3 = 0$

Example 10 **5** Use the graphics calculator to solve, correct to 2 decimal places:

a $2^x = 5$ **b** $4^x = 6$ **c** $10^x = 18$ **d** $10^x = 56$

Example 11 **6** Solve for x:

a $7^x > 49$ **b** $8^x > 2$ **c** $25^x \leq 5$ **d** $3^{x+1} < 81$

e $9^{2x+1} < 243$ **f** $4^{2x+1} > 64$ **g** $3^{2x-2} \leq 81$

Example 12 **7** Use a graphics calculator to solve, correct to 2 decimal places.

a $5^x \leq 10$ **b** $6^x \geq 4$ **c** $3^{-x} < 8$ **d** $\left(\frac{2}{3}\right)^{1-x} > 2$

MAPS

8 Use algebraic methods to solve for x: $0.25^{-x} - 1 = 2 \times 0.25^x$.
Validate your answer using your calculator.

MAPS

9 Use algebraic methods to solve for x: $\left(\frac{5}{3}\right)^{(1-2x)} \geq 0.216$.
Validate your answer using your calculator.

MAPS

10 Solve for x, without using your calculator:

a $7^{x+1} - 7^{x-1} = 336\sqrt{7}$ **b** $3^x - 3^{2-x} = 8$

3.4 Graphs of exponential functions

Two types of graphs will be examined.

Case 1: Graphs of $y = a^x$, $a > 1$

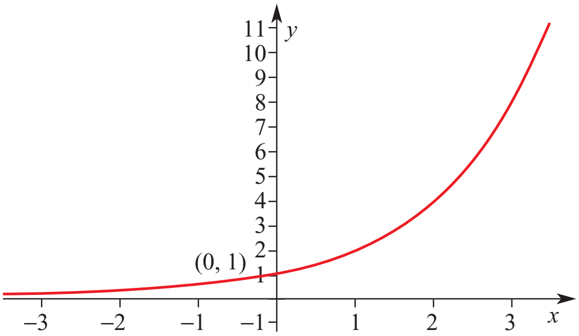
Example 13

Use the given table to plot the graph of $y = 2^x$, showing axes intercepts. State the equation of the **asymptote**. Use a graphics calculator to validate your plot. (See Example 16.)

Solution

x	-2	-1	0	1	2
$y = 2^x$	0.25	0.5	1	2	4

The x-axis, $y = 0$, is the asymptote, and the y-axis intercept is (0, 1).



Example 14

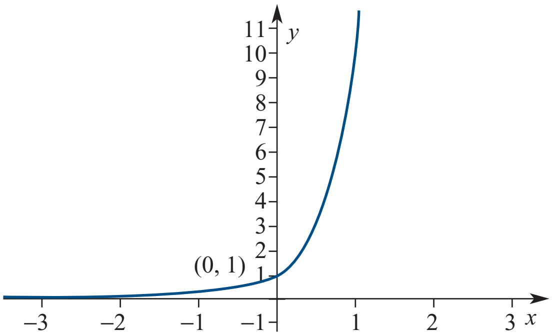
Use the given table to plot the graph of $y = 10^x$, showing axes intercepts. State the equation of the asymptote. Use a graphics calculator to validate your plot.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	0.316	1	3.16	10

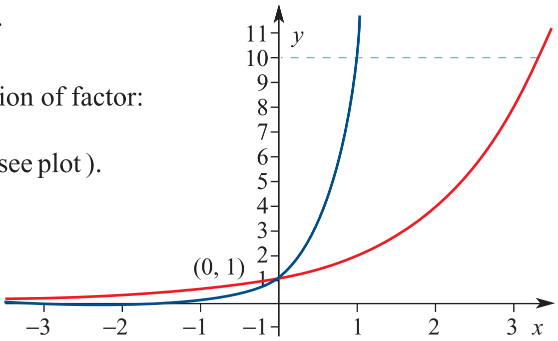
The x-axis, $y = 0$, is the asymptote, and the y-axis intercept is (0, 1).

It is worth noting at this stage that for a and b positive numbers > 1, there is a positive number k such that $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$. Using technology to solve $2^k = 10$, graphically, gives $k = 3.321928\ldots.$



Hence, $10^x = (2^{3.321928...})^x = 2^{(3.321928...)x}$.
This means that the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor:

$$k = \frac{1}{3.32928} \ldots \text{from the } y\text{-axis (see plot)}.$$



This shows that all graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis.

For $y = a^x$ the larger the base a the closer to the y-axis for any y value ($y \neq 1$ and $y > 0$).

This will be discussed again later in the chapter.

Case 2: Graphs of $y = a^x$, $0 < a < 1$

Example 15

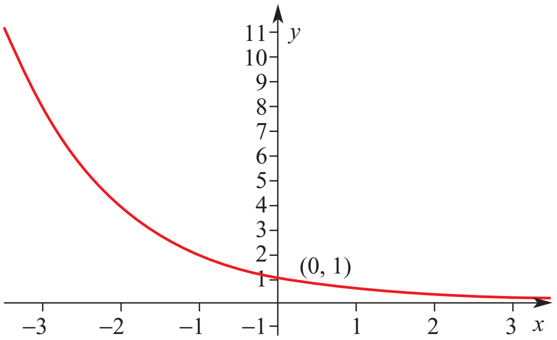
Use the given table to plot the graph of $y = \left(\frac{1}{2}\right)^x$, showing axes intercepts. State the equation of the asymptote. Use a graphics calculator to validate your plot.

Solution

x	−3	−2	−1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x = 2^{-x}$	8	4	2	1	0.5	0.25	0.125

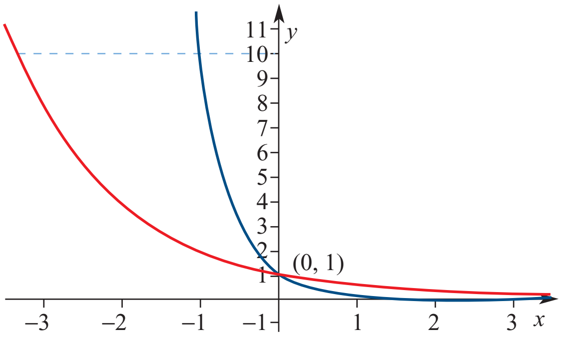
The x-axis is an asymptote, and the y-axis intercept is (0, 1).

For $0 < a < 1$, $y = a^x$ is equivalent to $y = b^{-x}$, where $b = \dfrac{1}{a}$.



The graph of $y = a^{-x}$ is obtained from the graph $y = a^x$ by a reflection in the y-axis. For example, the graph of $y = \left(\frac{1}{2}\right)^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis and vice versa.

For $y = a^{-x}$ the larger the base a the closer to the y-axis for any y value ($y \neq 1$ and $y > 0$).



Using technology

Example 16

Plot the graph of $y = 2^x$ and, hence, find the value of:

a y when $x = 2.1$, correct to 3 decimal places

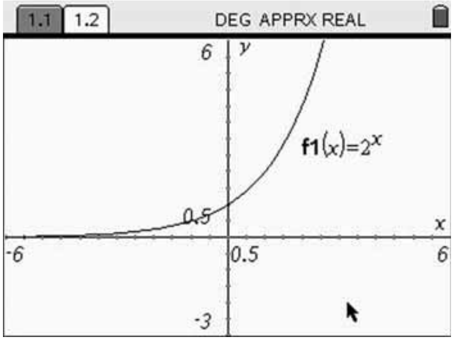
b x when $y = 9$

Solution

Using the TI-Nspire:

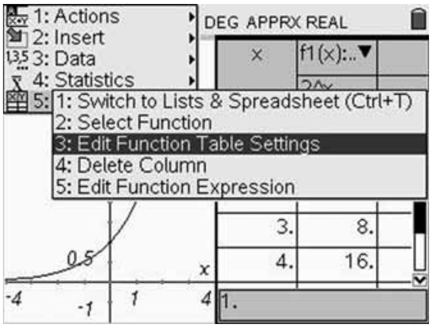
a

1 Type $\mathbf{2^x}$ into $f1(x)$ and press enter.



2 Press ctrl T to view a table of values.

3 Press menu and select *Edit Function Table Settings* from the Function Table submenu.

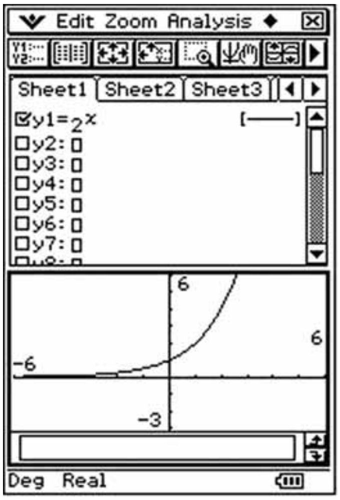


Using the ClassPad:

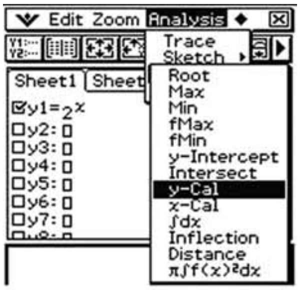
a

1 Type $\mathbf{2^x}$ into $y1$ and press EXE.

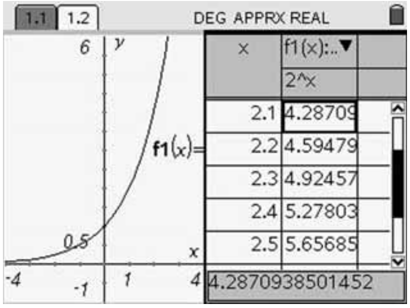
2 Tap to sketch the graph.



3 Tap Analysis and select *y-Calc* from the G-Solve submenu.



Input the following:
Table Start: 2.1
Table Step: 0.1
Then press enter.

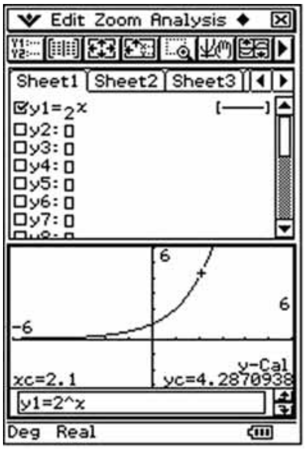


x	f1(x):= 2^x
2.1	4.28709
2.2	4.59479
2.3	4.92457
2.4	5.27803
2.5	5.65685

4.2870938501452

Therefore, $y = 4.287$ when $x = 2.1$

When prompted by the x value, type **2.1** and tap OK.



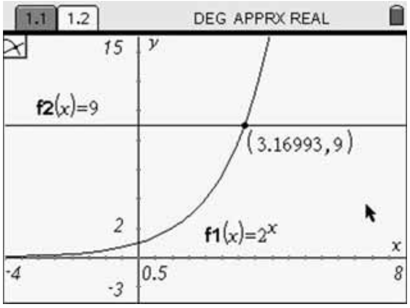
xc=2.1 yc=4.2870938
y1=2^x

b

1 Type **9** into $f2(x)$ and press enter.

2 To calculate the point of intersection press menu and select *Intersection Point(s)* from the Points & Lines submenu.

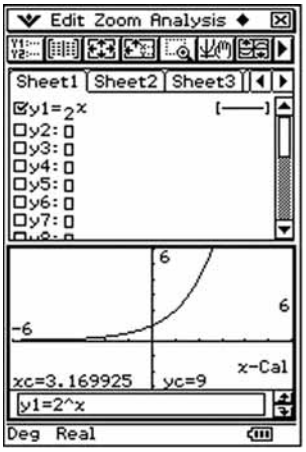
3 Move the cursor to the point of intersection to display its coordinates.



b

1 Tap Analysis and select *x-Calc* from the G-Solve submenu.

2 When prompted by the y value, type **9** and tap OK.



xc=3.169925 yc=9
y1=2^x

Therefore, when $y = 9$, $x = 3.170$, correct to 3 decimal places.

Transformations of exponential functions

Example 17

Sketch the graphs of each of the following functions. Give equations of asymptotes and y-axis intercepts of each of the functions. (*Note*: x-axis intercepts need not be given.)

a $y = 2^x + 3$ **b** $y = 2 \times 3^x + 1$ **c** $y = -3^x + 2$

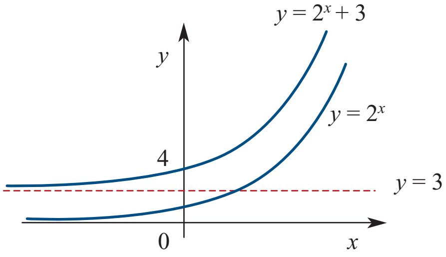
Solution

a For the function $y = 2^x + 3$, the corresponding graph is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$, with equation $y = 0$, is transformed to the asymptote with equation $y = 3$ for the graph of $y = 2^x + 3$.

When $x = 0$, $y = 2^0 + 3 = 4$.

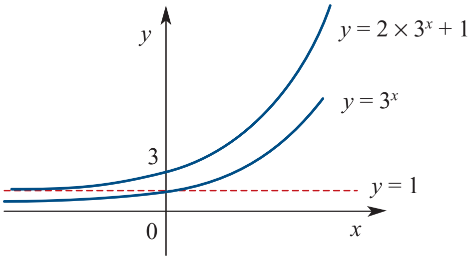
Hence, the y-axis intercept is 4.



b For the function $y = 2 \times 3^x + 1$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 1$ for the graph of $y = 2 \times 3^x + 1$.

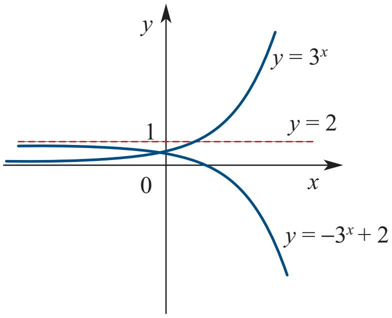
The y-axis intercept is given by $x = 0$, $y = 2 \times 3^0 + 1 = 3$.



c For the function $y = -3^x + 2$, the corresponding graph is obtained by transforming the graph of $y = 3^x$ by reflection in the x-axis, followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$, with equation $y = 0$, is transformed to the asymptote $y = 2$ for the graph of $y = -3^x + 2$.

The y-axis intercept is given by $x = 0$, $y = -3^0 + 2 = 1$.



Example 18

Sketch:

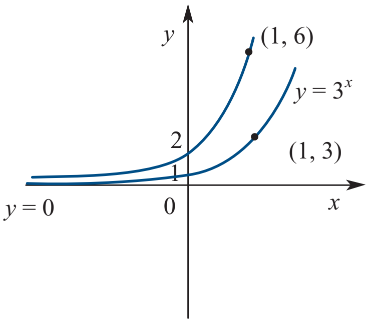
a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = 3^{\frac{x}{2}}$ **d** $y = -3^{2x} + 4$

Solution

a The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis.

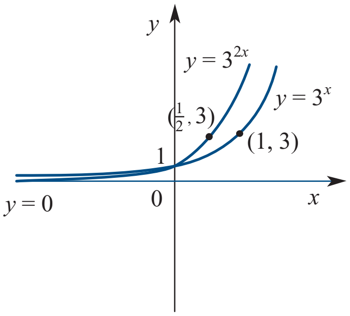
The horizontal asymptote for both graphs has equation $y = 0$.

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
2×3^x	$\frac{2}{9}$	$\frac{2}{3}$	2	6	18



b The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis. For example:

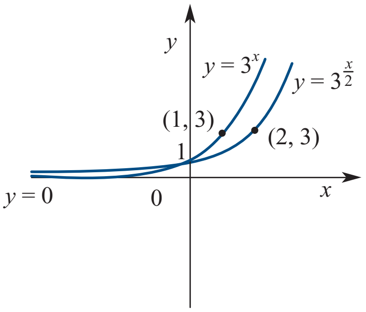
x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$			3	9
3^{2x}		$\frac{1}{9}$	$\frac{1}{3}$	3	9	



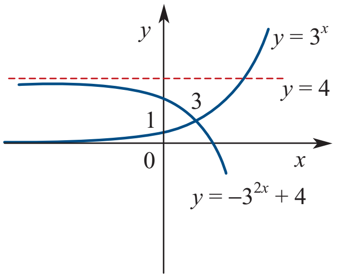
The horizontal asymptote for both graphs has equation $y = 0$.

c The graph of $y = 3^{\frac{x}{2}}$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the y-axis. For example:

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
3^x		$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	3	
$3^{\frac{x}{2}}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$			$\sqrt{3}$	3



d The graph of $y = -3^{2x} + 4$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis followed by a translation of 4 units in the positive direction of the y-axis.

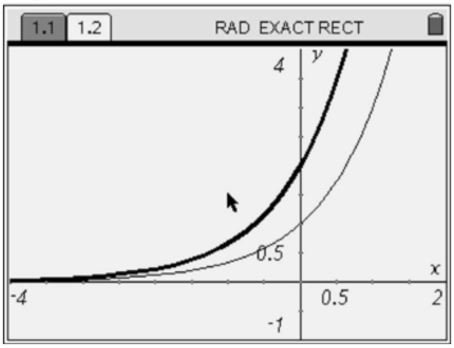


Using technology

Using the TI-Nspire:

a

1. Select the Graphs & Geometry application.
2. Type **3^*x*** into $f1(x)$ then press enter.
3. Type **2** × **3^*x*** into $f2(x)$ then press enter. Make this a bold line.



Using the ClassPad:

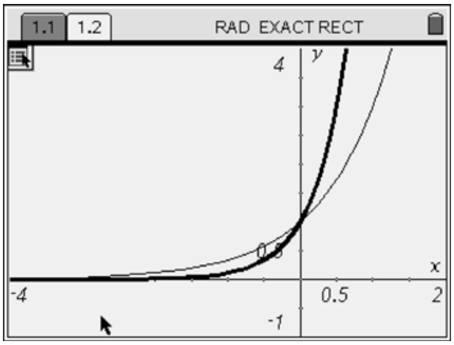
a

1. Select the Graphs and Tables application by tapping on Graph&Tab.. .
2. Type **3^*x*** into $y1$ then press EXE.
3. Type **2** × **3^*x*** into $y2$ then press EXE. Make this a bold line.

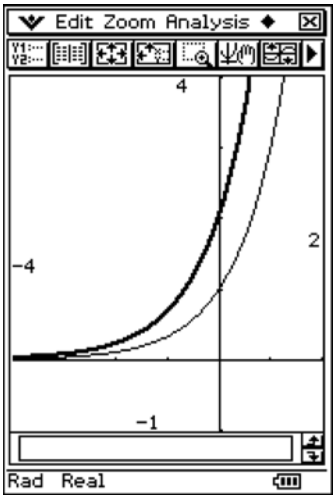
b

1 Type **3^*x*** into $f1(x)$ then press enter.

2 Type **3^(2*x*)** into $f2(x)$ then press enter.
Make this a bold line.



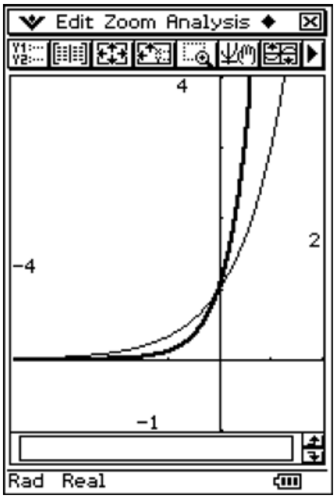
4 Tap to sketch the two functions.



b

1 Type **3^*x*** into $y1$ then press EXE.

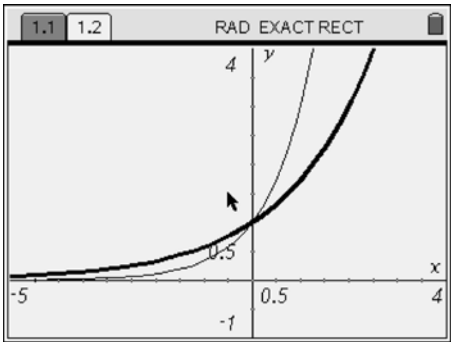
2 Type **3^(2*x*)** into $y2$ then press EXE.
Make this a bold line. Now tap .



c

1 Type **3^*x*** into $f1(x)$ then press enter.

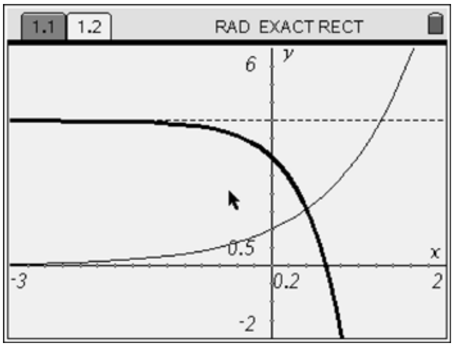
2 Type **3^(*x*/2)** into $f2(x)$ then press enter. Make this a bold line.



d

1 Type **3^*x*** into $f1(x)$ then press enter.

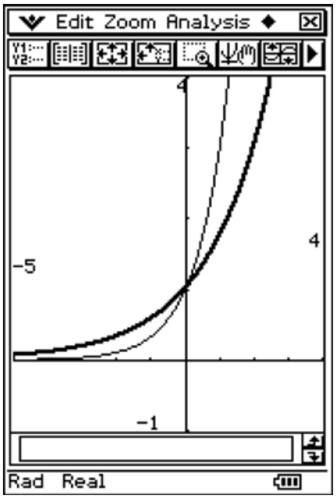
2 Type **−3^(2*x*) + 4** into $f2(x)$ and then press enter. Make this a bold line.



c

1 Type **3^*x*** into $y1$ then press EXE.

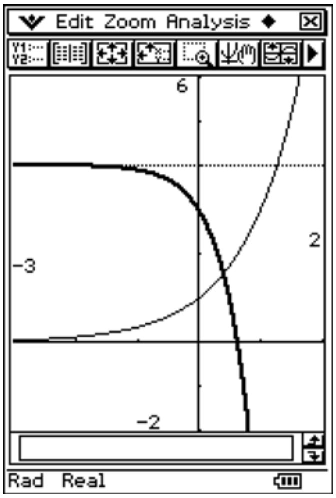
2 Type **3^(*x*/2)** into $y2$ then press EXE. Make this a bold line. Now tap .



d

1 Type **3^*x*** into $y1$ then press EXE.

2 Type **3^(2*x*) + 4** into $y2$ then press EXE. Make this a bold line. Now tap .



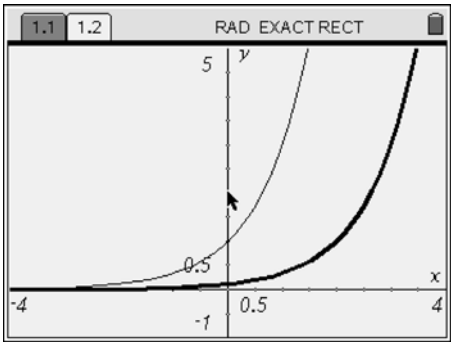
Extra example relating to an x translation

Using the TI-Nspire:

1 Select the Graphs & Geometry application.

2 Type **3^*x*** into $f1(x)$ then press enter.

3 Type **3^(*x* – 2)** into $f2(x)$ then press enter. Make this a bold line.



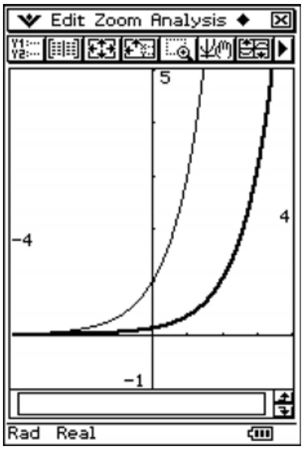
Using the ClassPad:

1 Select the Graphs and Tables application by tapping on Graph&Tab.

2 Type **3^*x*** into $y1$ then press EXE.

3 Type **3^(*x* – 2)** into $y2$ then press EXE. Make this a bold line.

4 Tap to sketch the two functions.



Exercise 3D

Examples 13–15 **1** Using your calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ **b** $y = 2.4^x$ **c** $y = 0.9^x$ **d** $y = 0.5^x$

Examples 13–15 **2** Using your calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ **b** $y = 5 \times 3^x$ **c** $y = -2 \times 3^x$ **d** $y = -5 \times 3^x$

Example 16 **3** Plot, using your calculator, the graph of $y = 2^x$ for $-4 \leq x \leq 4$ and, hence, find the solution of the equation $2^x = 14$.

4 Plot, using your calculator, the graph of $y = 10^x$ for $-0.4 \leq x \leq 0.8$ and, hence, find the solution of the equation $10^x = 6$.

Examples 17,18 **5** Sketch the graphs of these functions. Give equations of asymptotes and state y-axis intercepts. (*Note*: x-axis intercepts need not be given.) Validate, using your calculator.

a $y = 3 \times 2^x + 2$ **b** $y = 3 \times 2^x - 3$ **c** $y = -3^x - 2$

d $y = -2 \times 3^x + 2$ **e** $y = \left(\frac{1}{2}\right)^x + 2$ **f** $y = -2 \times 3^x - 2$

Examples 17,18

6 Sketch each of the following. Validate, using your calculator.

a $y = 2 \times 5^x$ **b** $y = 3^{3x}$ **c** $y = 5^{\frac{x}{2}}$ **d** $y = -3^{2x} + 2$

3.5 Logarithms

Many functions are of the type $a^y = x$. For example:

- a slide rule scale
- the Richter scale for the strength of an earthquake
- the decibel scale measuring sound loudness
- certain population growth
- a FM radio dial scale
- measure of the acidity of a solution
- fading at the end of a recorded song

For these functions x is the independent variable and y is the dependent variable (as per previous work). In such functions, x is often known and y is to be calculated. An efficient and direct evaluation of y requires expressing y as a function of x; that is, $y = f(x)$.

This function is **log**.

$$\therefore a^y = x \text{ is written as } y = \log_a x.$$

Consider $2^3 = 8$.

To make the index 3 the subject of this statement, it is written in the equivalent and alternative form:

$$3 = \log_2 8$$

Further examples are given:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$.
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$.
- $10^{-2} = 0.01$ is equivalent to $\log_{10} 0.01 = -2$.
- $\frac{1}{16} = 2^{-4} = 0.0625$ is equivalent to $\log_2 0.0625 = -4$.
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$.

In general:

$$a^y = x \text{ is equivalent to } y = \log_a x.$$

i.e. y is the **index** to which a is raised to equate to x.

Similarly,

$$a^x = y \text{ is equivalent to } x = \log_a y.$$

i.e. x is the **index** to which a is raised to equate to y.

Example 19

Without the aid of a calculator, evaluate:

a $\log_2 32$ **b** $\log_3 81$ **c** $\log_{10} 0.1$

Solution

a Let $\log_2 32 = y$.

$$2^y = 32$$
$$2^y = 2^5$$

Therefore, $y = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$.

$$3^x = 81$$
$$3^x = 3^4$$

Therefore, $x = 4$, giving $\log_3 81 = 4$.

c Let $\log_{10} 0.1 = y$.

$$10^y = 0.1$$
$$10^y = 10^{-1}$$

Therefore, $y = -1$, giving $\log_{10} 0.1 = -1$.

Laws of logarithms

Let $x = a^m$ and $y = a^n$ where x, y and a are positive real numbers.

$\therefore m = \log_a x$ and $n = \log_a y$

1 $xy = a^m \times a^n$

$$= a^{m+n}$$

$\therefore \log(xy) = m + n$

but $m = \log_a x$ and $n = \log_a y$

$\therefore \log_a(xy) = \log_a x + \log_a y$

$$\log_a(xy) = \log_a x + \log_a y \qquad \textbf{Rule 1}$$

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10}(1000)\\ &= 3\end{aligned}$$

2
$$\begin{aligned}\frac{x}{y} &= \frac{a^m}{a^n} \\ &= a^{m-n}\end{aligned}$$

$$\begin{aligned}\therefore \log_a\left(\frac{x}{y}\right) &= m - n \\ &= \log_a x - \log_a y\end{aligned}$$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ **Rule 2**

For example:

$$\begin{aligned}\log_2 12 - \log_2 3 &= \log_2 \frac{12}{3} \\ &= \log_2 4 \\ &= 2\end{aligned}$$

3 Given: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ and $x = 1$

$$\begin{aligned}\log_a\left(\frac{1}{y}\right) &= \log_a 1 - \log_a y \\ &= 0 - \log_a y \\ &= -\log_a y\end{aligned}$$

$\log_a\left(\frac{1}{y}\right) = -\log_a y$ **Rule 3**

For example:

$$\begin{aligned}\log_{10}\left(\frac{1}{100}\right) &= -\log_{10}(100) \\ &= -2\end{aligned}$$

4 Consider
$$\begin{aligned}x^n &= (a^m)^n \\ &= a^{mn}\end{aligned}$$

Taking $\log_a$ of both sides:

$$\begin{aligned}\log_a x^n &= \log_a (a^{mn}) \\ &= nm \\ &= n \log_a x\end{aligned}$$

$\log_a (x^n) = \log_a x$ **Rule 4**

For example:

$$\begin{aligned}\log_3 (25) &= \log_3 5^2 \\ &= 2 \log_3 5\end{aligned}$$

Example 20

Without using a calculator, simplify $2\log_{10} 3 + \log_{10} 16 - 2\log_{10} \frac{6}{5}$.

Solution

$$\begin{aligned}
2\log_{10} 3 + \log_{10} 16 - 2\log_{10} \frac{6}{5} &= \log_{10} 3^2 + \log_{10} 16 - \log_{10} \left(\frac{6}{5}\right)^2 \\
&= \log_{10} 9 + \log_{10} 16 - \log_{10} \frac{36}{25} \\
&= \log_{10}\left(9 \times 16 \times \frac{25}{36}\right) \\
&= \log_{10}(100) \\
&= 2
\end{aligned}$$

Example 21

Solve for x:

a $\log_5 x = 3$

b $\log_5(2x + 1) = 2$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

Solution

a $\log_5 x = 3 \Leftrightarrow x = 5^3 = 125$

b $\log_5 (2x + 1) = 2 \Leftrightarrow 2x + 1 = 5^2$

$$\begin{aligned}
\therefore 2x + 1 &= 25 \\
2x &= 24 \\
x &= 12
\end{aligned}$$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

$$\begin{aligned}
\text{Then,} \quad \log_2\left(\frac{2x+1}{x-1}\right) &= 4 \\
\frac{2x+1}{x-1} &= 2^4 \\
\Leftrightarrow \qquad 2x + 1 &= 16\,(x - 1) \\
\text{Then,} \qquad 17 &= 14x \\
\text{Hence,} \qquad \frac{17}{14} &= x
\end{aligned}$$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

Therefore, $\log_3((x - 1)(x + 1)) = 1$, which implies $x^2 - 1 = 3$ and $x = \pm 2$. But the expression is not defined for $x = -2$. Therefore, $x = 2$.

Exercise 3E

Example 19

1 Without using a calculator, evaluate:

a $\log_3 27$ **b** $\log_5 625$ **c** $\log_2\left(\frac{1}{128}\right)$

d $\log_4\left(\frac{1}{64}\right)$ **e** $\log_x x^4$ **f** $\log_2 0.125$

g $\log_{10} 10\,000$ **h** $\log_{10} 0.000\,001$ **i** $-3\log_5 125$

j $-4\log_{16} 2$ **k** $2\log_3 9$ **l** $-4\log_{16} 4$

2 Use the stated rule to give an equivalent expression in simplest form.

a $\log_2 10 + \log_2 a$ **b** $\log_{10} 5 + \log_{10} 2$ Rule 1

c $\log_2 9 - \log_2 4$ **d** $\log_2 10 - \log_2 5$ Rule 2

e $\log_5\left(\frac{1}{6}\right)$ **f** $\log_5\left(\frac{1}{25}\right)$ Rule 3

g $\log_2(a^3)$ **h** $\log_2(8^3)$ Rule 4

Example 20

3 Without using a calculator, simplify:

a $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5$ **b** $\log_2 16 + \log_2 8$

c $\log_2 128 + \log_3 45 - \log_3 5$ **d** $\log_4 32 - \log_9 27$

e $\log_b b^3 - \log_b \sqrt{b}$ **f** $2\log_x a + \log_x a^3$

g $x\log_2 8 + \log_2 8^{1-x}$ **h** $\frac{3}{2}\log_a a - \log_a \sqrt{a}$

Example 21

4 Solve for x:

a $\log_3 9 = x$ **b** $\log_3 x = 3$

c $\log_5 x = -3$ **d** $\log_{10} x = \log_{10} 4 + \log_{10} 2$

e $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$ **f** $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$

g $\log_x 64 = 2$ **h** $\log_5(2x - 3) = 3$

i $\log_3(x + 2) - \log_3 2 = 1$ **j** $\log_x 0.01 = -2$

5 Solve for x:

a $\log_x\left(\frac{1}{25}\right) = -2$ **b** $\log_4(2x - 1) = 3$

c $\log_4(x + 2) - \log_4 6 = 1$ **d** $\log_4(3x + 4) + \log_4 16 = 5$

e $\log_3(x^2 - 3x - 1) = 0$ **f** $\log_3(x^2 - 3x + 1) = 0$

6 If $\log_{10} x = a$ and $\log_{10} y = c$, express $\log_{10}\left(\frac{100x^3y^{-\frac{1}{2}}}{y^2}\right)$ in terms of a and c.

7 Prove that $\log_{10}\left(\frac{ab^2}{c}\right) + \log_{10}\left(\frac{c^2}{ab}\right) - \log_{10}(bc) = 0$.

8 $\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$. Find k.

9 Solve for x:

a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$ **b** $\log_{10}(5x) - \log_{10}(3 - 2x) = 1$

c $3\log_{10}(x - 1) = \log_{10} 8$ **d** $\log_{10}(20x) - \log_{10}(x - 8) = 2$

e $2\log_{10} 5 + \log_{10}(x + 1) = 1 + \log_{10}(2x + 7)$

f $1 + 2\log_{10}(x + 1) = \log_{10}(2x + 1) + \log_{10}(5x + 8)$

3.6 Using logarithms in the solution of exponential equations and inequations

Laws of logarithms

5 Consider $x = a^m$

$m = \log_a x$

Taking $\log_b$ of both sides:

$$\begin{aligned} \log_b x &= \log_b (a^m) \\ &= m \log_b a \\ \frac{\log_b x}{\log_b a} &= m \end{aligned}$$

$$\therefore \log_a x = \frac{\log_b x}{\log_b a}$$ **Rule 5 Change of base rule**

Using technology

The graphics calculator can evaluate $\log_{10}()$, which is commonly written as $\log()$.

The calculator can also evaluate $\log_e()$, which is commonly written as $\ln()$.

$e \approx 2.71828\ldots$ (Euler's number; pronounced 'Oiler') is an important irrational number.

Evaluate $\log_2 5$.

Using $\log_{10}$

Using the TI-Nspire:

1 In the Calculator application press ctrl $\log\ 10^x$. (This gives a template for a logarithm with any base.)

2 Type **10** into the base.

3 Type **5** in between the brackets.

Using the ClassPad:

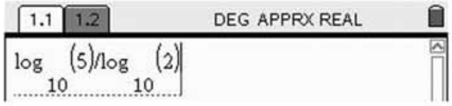
1 In the Main application press Keyboard.

2 Tap log then type **5**).

3 Tap or press ÷, then tap log, then type **2**).

4 Move the cursor to the end of the expression and then press ÷.

5 Repeat steps 1 to 3 but place a **2** in the brackets.



$\log_{10}(5)/\log_{10}(2)$

6 Press enter.

$\dfrac{\log_{10}(5)}{\log_{10}(2)}$ 2.32192809489

4 Press EXE.

log(5)/log(2)
2.321928095

Using log$_e$

Using the TI-Nspire:

1 Set to Approximate mode.

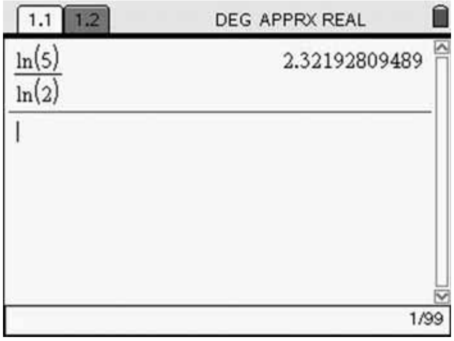
2 Press ctrl ln eˣ.

3 Type **5**).

4 Press ÷.

5 Press ctrl ln eˣ.

6 Type **2**) then press enter.



$\dfrac{\ln(5)}{\ln(2)}$ 2.32192809489

Using the ClassPad:

1 Set to Decimal mode.

2 Press Keyboard.

3 Tap ln then type **5**).

4 Tap or press ÷, then tap ln, then type **2**).

5 Press EXE.

ln(5)/ln(2)
2.321928095

In section 3.3, two methods were shown for solving exponential equations and inequations. Method 3 uses logarithms, as above.

Method 3

Using logarithms

Example 22

Solve for x, if $2^x = 11$.

Solution

Using $\log_{10}$:

$$\begin{aligned} x &= \frac{\log 11}{\log 2} \\ &= \frac{1.0414}{0.3010} \\ &= 3.46 \qquad \text{(to 2 decimal places)} \end{aligned}$$

Using $\log_e$:

$$\begin{aligned} x &= \frac{\ln 11}{\ln 2} \\ &= \frac{2.3979}{0.6931} \\ &= 3.46 \qquad \text{(to 2 decimal places)} \end{aligned}$$

Example 23

Solve $3^{2x-1} = 28$.

Solution

Using $\log_{10}$:

$$\begin{aligned} 2x - 1 &= \frac{\log 28}{\log 3} \\ &\approx \frac{1.4472}{0.4771} \\ &\approx 3.0331 \\ \therefore 2x &\approx 4.0331 \\ \therefore x &= 2.017 \qquad \text{(to 2 decimal places)} \end{aligned}$$

Example 24

Solve $\{x : 0.7^x \geq 0.3\}$.

Solution

Using $\log_e$:

$$\begin{aligned} x &\leq \frac{\ln 0.3}{\ln 0.7} \qquad \text{The inequality is reversed because of division by } \ln(0.7) < 0. \\ &\leq \frac{-1.2040}{-0.3567} \\ \therefore x &\leq 3.38 \qquad \text{(to 2 decimal places)} \end{aligned}$$

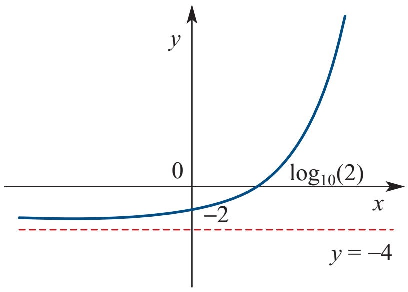
Example 25

Sketch the graph of $y = 2 \times 10^x - 4$, giving the equation of the asymptote and axes intercepts.

Solution

For y-axis intercept, $x = 0$.

$$\begin{aligned} y &= 2 \times 10^0 - 4 \\ &= 2 - 4 \\ &= -2 \end{aligned}$$



The equation of the horizontal asymptote is $y = -4$.

For x-axis intercept, $y = 0$.

$$\begin{aligned} 2 \times 10^x - 4 &= 0 \\ 2 \times 10^x &= 4 \\ 10^x &= 2 \\ \therefore\ x &= \log 2 \\ &= 0.3010 \qquad \text{(correct to 4 decimal places)} \end{aligned}$$

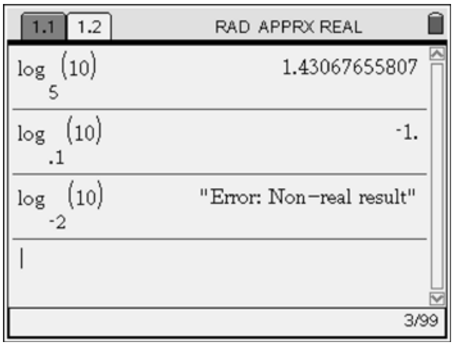
Using technology

Using the TI-Nspire:

1 In the Calculator application press ctrl log 10^x to access the log template.

2 Type the desired number into the base.

3 Type the desired number in between the brackets and then press enter to evaluate the expression.



$\log_5(10)$ 1.43067655807

$\log_{.1}(10)$ -1.

$\log_{-2}(10)$ "Error: Non–real result"

Using the ClassPad:

1 In the main application, press Keyboard then tap the 2D tab.

2 Tap log■□ to access the log template.

3 Type the desired number into the base.

4 Type the desired number in between the brackets and then press EXE to evaluate the expression.

```
log5(10)
                    1.430676558
log0.1(10)
                             -1
log-2(10)
    ERROR:Non-Real in Calc
```

Exercise 3F

Examples 22, 23 **1** Solve each of the following, correct to 2 decimal places:

a $2^x = 7$
b $2^x = 0.4$
c $3^x = 14$
d $4^x = 3$
e $2^{-x} = 6$
f $0.3^x = 2$
g $5^x = 3^{x-2}$
h $8^x = 2005^{x+1}$
i $3^{x-1} = 10$
j $0.2^{x+1} = 0.6$

Example 24 **2** Solve for x. Give values correct to 2 decimal places if necessary.

a $2^x > 8$
b $3^x < 5$
c $0.3^x > 4$
d $3^{x-1} \leq 7$
e $0.4^x \leq 0.3$

Example 25 **3** Sketch the graph of each relation below, giving the equation of the asymptote and axes intercepts. Validate, using your calculator.

a $y = 2^x - 4$
b $y = 2 \times 3^x - 6$
c $y = 3 \times e^x - 5$
d $y = -2 \times 10^x + 4$
e $y = -3 \times 2^x + 6$
f $y = 5 \times 2^x - 6$

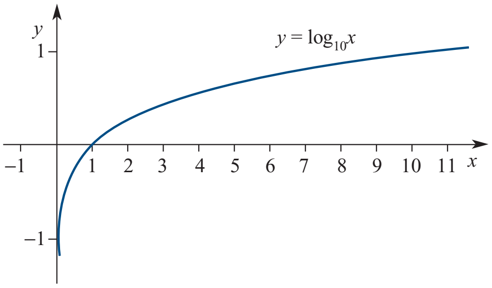
4 During the initial period of its life, a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter of the tree in centimetres, t years after the beginning of this period. The diameter after 1 year is 52 cm and after 3 years is 80 cm. Calculate the values of the constants d_0 and m.

3.7 Graph of $y = \log_a x$, where $a > 1$

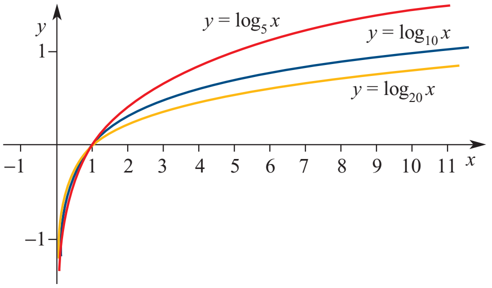
A table of values for $y = \log_{10} x$ is given below. (The values are correct to 2 decimal places.) Use your calculator to check these values.

x	0.1	1	2	3	4	5
$y = \log_{10} x$	−1	0	0.30	0.48	0.60	0.70

Note that $\log_{10} 1 = 0$ as $10^0 = 1$.



For $a > 1$, the graphs of logarithm functions all have this same basic shape.



Example 26

Use a table of values to sketch each of the following. Determine any axes intercepts and the equation of the asymptote.

a $y = 1 + \log_{10} x$ **b** $y = -1 + \log_{10} x$

Solution

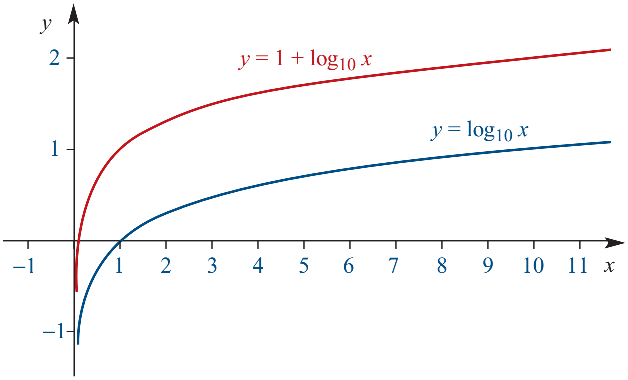
a

x	0.1	1	2	3	4	5
$y = 1 + \log_{10} x$	0	1	1.30	1.48	1.60	1.70

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore \quad 0 &= 1 + \log_{10} x \\ x &= 10^{-1} \\ &= 0.1 \end{aligned}$$

The asymptote is $x = 0$.



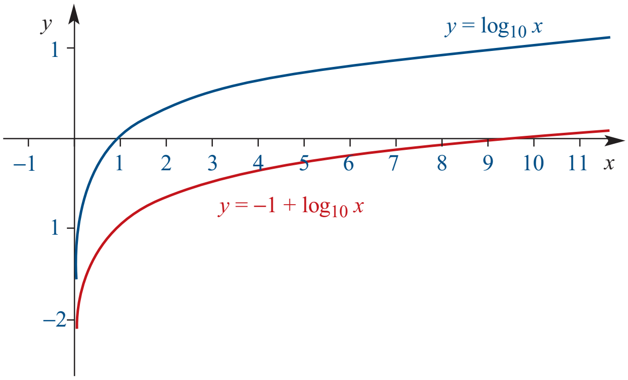
b

x	0.1	1	2	3	4	10
$y = -1 + \log_{10} x$	−2	−1	−0.7	−0.52	−0.4	0

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore \quad 0 &= -1 + \log_{10} x \\ x &= 10^{1} \\ &= 10 \end{aligned}$$

The asymptote is $x = 0$.



In general:

If $y = d + \log_a x$, then the plot $y = \log_a x$ is shifted d units from the x-axis parallel to the y-axis.

Using technology

Using the TI-Nspire:

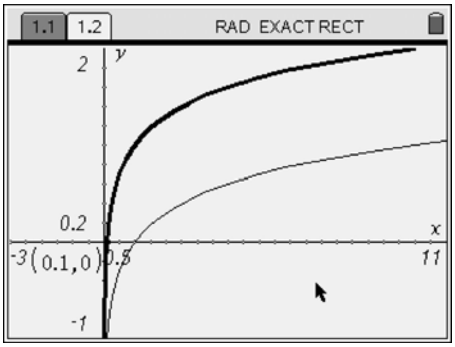
a

1 Select the Graphs & Geometry application.

2 Type **$\log_{10}(x)$** into $f1(x)$ then press enter.

3 Type **$1 + \log_{10}(x)$** into $f2(x)$ then press enter. Make this a bold line.

4 Press menu and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.

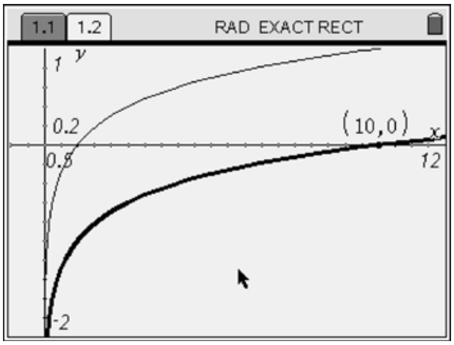


b

1 Type **$\log_{10}(x)$** into $f1(x)$ then press enter.

2 Type **$-1 + \log_{10}(x)$** into $f2(x)$ then press enter. Make this a bold line.

3 Press menu and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



Using the ClassPad:

a

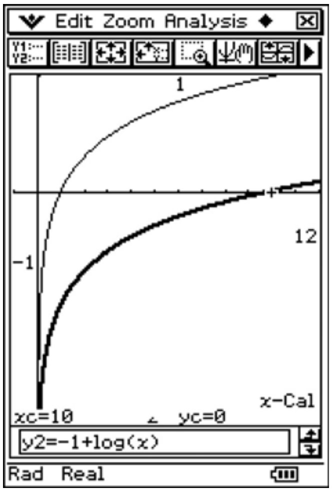
1 Select the Graphs and Tables application by tapping on Graph&Tab.. .

2 Type **$\log_{10}(x)$** into $y1$ then press EXE.

3 Type **$1 + \log_{10}(x)$** into $y2$ then press EXE. Make this a bold line.

4 Tap 📈 to sketch the two functions.

5 Tap Analysis and select x-*Calc* from the G-Solve submenu to calculate the axes intercepts.



xc=10 yc=0

y2=-1+log(x)

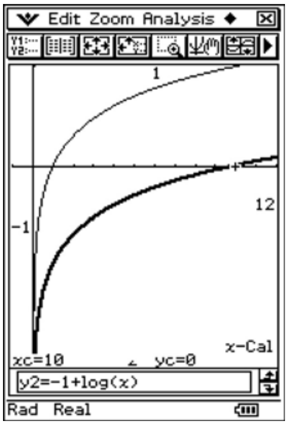
b

1 Type **$\log_{10}(x)$** into $y1$ then press EXE.

2 Type **$-1 + \log_{10}(x)$** into $y2$ then press EXE. Make this a bold line.

3 Tap 📈 to sketch the two functions.

4 Tap Analysis and select x-*Calc* from the G-Solve submenu to calculate the axes intercepts.



xc=10 yc=0

y2=-1+log(x)

Example 27

Use a table of values to sketch each of the following. Determine any axes intercepts and the equation of the asymptote.

a $y = 2\log_2 x$ **b** $y = -2\log_2 x$ **c** $y = -\frac{1}{2}\log_2 x$

Solution

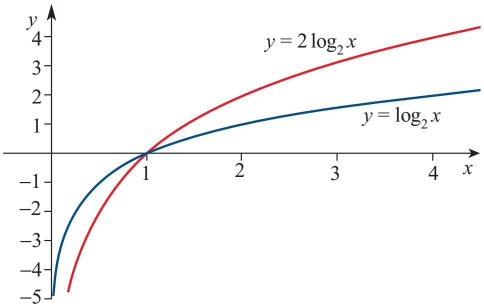
a

x	1	2	4
$y = 2\log_2 x$	0	2	4

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore\ 0 &= 2\log_2 x \\ x &= 2^0 \\ &= 1 \end{aligned}$$

The asymptote is $x = 0$.



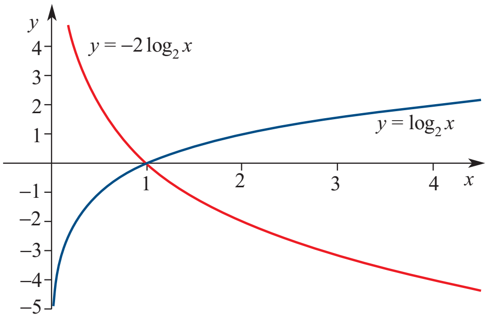
b

x	1	2	4
$y = -2\log_2 x$	0	−2	−4

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore\ 0 &= -2\log_2 x \\ x &= 2^0 \\ &= 1 \end{aligned}$$

The asymptote is $x = 0$.



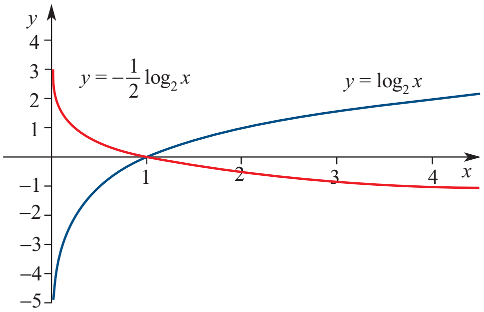
c

x	1	2	4
$y = -\frac{1}{2}\log_2 x$	0	$-\frac{1}{2}$	-1

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore 0 &= -\frac{1}{2}\log_2 x \\ x &= 2^0 \\ &= 1 \end{aligned}$$

The asymptote is $x = 0$.



Note: If $y = k \log_a x$, then the plot $y = \log_a x$ is dilated k units from the x-axis parallel to the y-axis.

Using technology

Using the TI-Nspire:

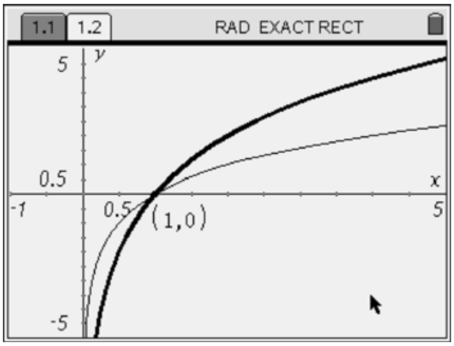
a

1 Select the Graphs & Geometry application.

2 Type $\mathbf{log_2(x)}$ into $f1(x)$ then press (enter).

3 Type $\mathbf{2\ log_2(x)}$ into $f2(x)$ then press (enter). Make this a bold line.

4 Press (menu) and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



Using the ClassPad:

a

1 Select the Graphs and Tables application by tapping on Graph&Tab... .

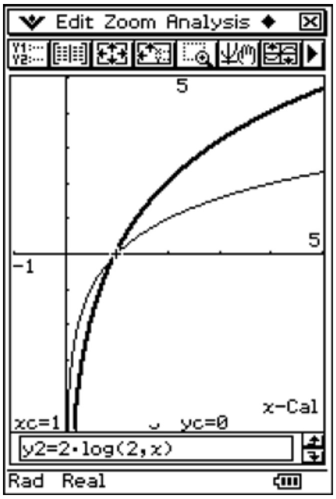
2 Type $\mathbf{log_2(x)}$ into $y1$ then press (EXE).

3 Type $\mathbf{2\ log_2(x)}$ into $y2$ then press (EXE). Make this a bold line.

4 Tap ⊞ to sketch the two functions.

5 Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.

Note: To obtain $\log_2(x)$, press (Keyboard) then tap the 2D tab and select the log template.

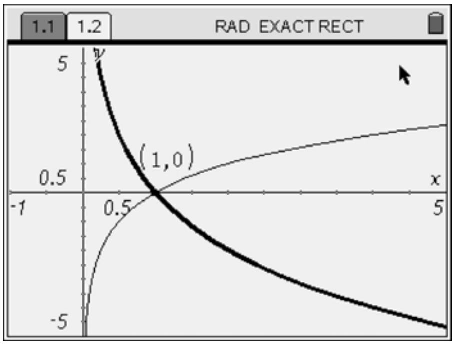


xc=1 yc=0

y2=2·log(2,x)

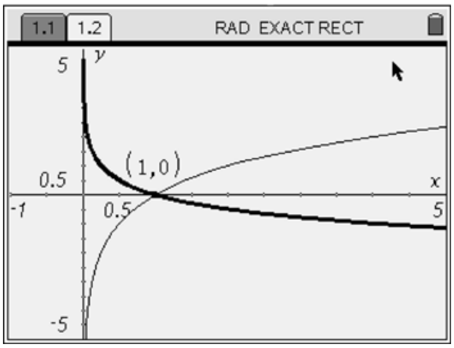
b

1. Type **$\log_2(x)$** into $f1(x)$ then press (enter).
2. Type **$-2\log_2(x)$** into $f2(x)$ then press (enter). Make this a bold line.
3. Press (menu) and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



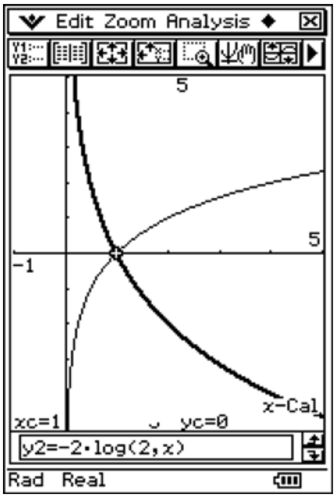
c

1. Type **$\log_2(x)$** into $f1(x)$ then press (enter).
2. Type **$-(1/2)\log_2(x)$** into $f2(x)$ then press (enter). Make this a bold line.
3. Press (menu) and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



b

1. Type **$\log_2(x)$** into $y1$ then press (EXE).
2. Type **$-2\log_2(x)$** into $y2$ then press (EXE). Make this a bold line.
3. Tap ⊞ to sketch the two functions.
4. Tap Analysis and *select x-Calc* from the G-Solve submenu to calculate the axes intercepts.

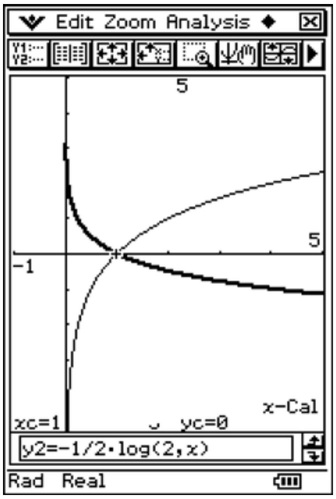


xc=1 yc=0

y2=-2·log(2,x)

c

1. Type **$\log_2(x)$** into $y1$ then press (EXE).
2. Type **$-(1/2)\log_2(x)$** into $y2$ then press (EXE). Make this a bold line.
3. Tap ⊞ to sketch the two functions.
4. Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.



xc=1 yc=0

y2=-1/2·log(2,x)

Example 28

Use a table of values to sketch each of the following. Determine any axes intercepts and the equation of the asymptote.

a $y = \log_3(x - 1)$ **b** $y = \log_3(x + 2)$

Solution

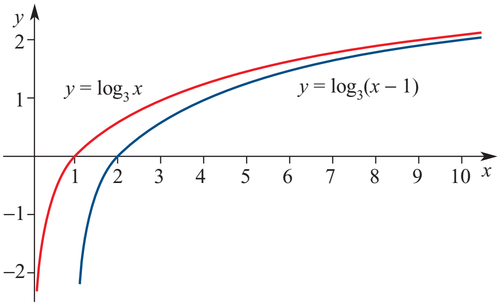
a

x	2	4	10
$y = \log_3(x - 1)$	0	1	2

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore 0 &= \log_3(x - 1) \\ x - 1 &= 3^0 \\ x - 1 &= 1 \\ x &= 2 \end{aligned}$$

The asymptote is $x = 1$.



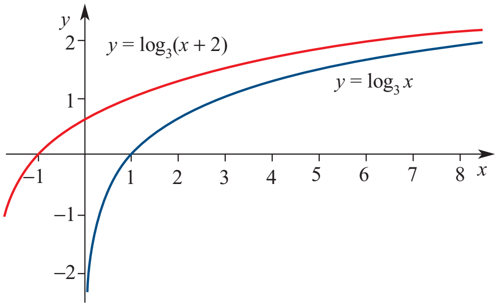
b

x	−1	1	7
$y = \log_3(x + 2)$	0	1	2

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore 0 &= \log_3(x + 2) \\ x + 2 &= 3^0 \\ x + 2 &= 1 \\ x &= -1 \end{aligned}$$

The asymptote is $x = -2$.



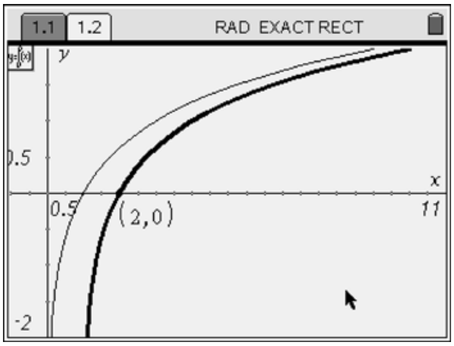
Note: If $y = \log_a(x + c)$ then the plot $y = \log_a x$ is shifted c units from the y-axis and parallel to the x-axis.

Using technology

Using the TI-Nspire:

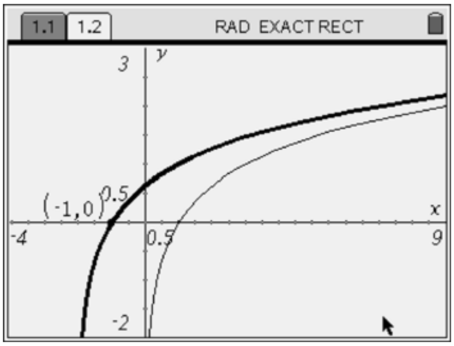
a

1. Select the Graphs & Geometry application.
2. Type **$\log_3(x)$** into $f1(x)$ then press enter.
3. Type **$\log_3(x-1)$** into $f2(x)$ then press enter. Make this a bold line.
4. Press menu and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



b

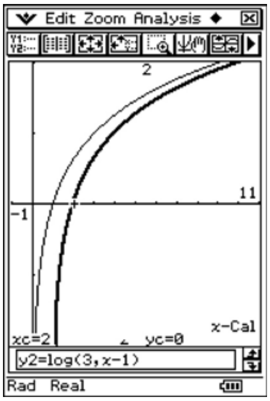
1. Type **$\log_3(x)$** into $f1(x)$ then press enter.
2. Type **$\log_3(x+2)$** into $f2(x)$ then press enter. Make this a bold line.
3. Press menu and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



Using the ClassPad:

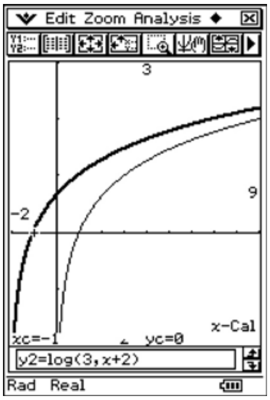
a

1. Select the Graphs and Tables application by tapping on Graph&Tab.
2. Type **$\log_3(x)$** into $y1$ then press EXE.
3. Type **$\log_3(x-1)$** into $y2$ then press EXE. Make this a bold line.
4. Tap ▨ to sketch the two functions.
5. Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.



b

1. Type **$\log_3(x)$** into $y1$ then press EXE.
2. Type **$\log_3(x+2)$** into $y2$ then press EXE. Make this a bold line.
3. Tap ▨ to sketch the two functions.
4. Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.



Example 29

Use a table of values to sketch each of the following. Determine any axes intercepts and the equation of the asymptote.

a $y = \log_2\left(\frac{x}{4}\right)$ **b** $y = \log_5(2x)$

Solution

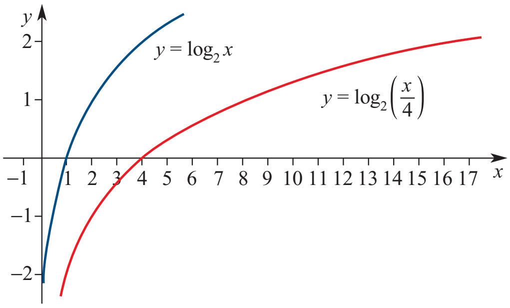
a

x	1	2	4	8	16
$y = \log_2\left(\frac{x}{4}\right)$	−2	−1	0	1	2

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore 0 &= \log_2\left(\frac{x}{4}\right) \\ \frac{x}{4} &= 2^0 \\ \frac{x}{4} &= 1 \\ x &= 4 \end{aligned}$$

The asymptote is $x = 0$.



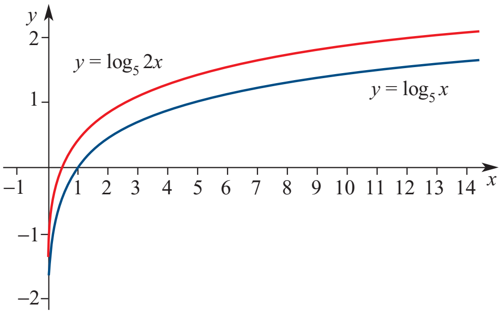
b

x	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{25}{2}$
$y = \log_5(2x)$	−1	0	1	2

At x-intercept, $y = 0$.

$$\begin{aligned} \therefore 0 &= \log_5(2x) \\ 2x &= 5^0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

The asymptote is $x = 0$.



Note: If $y = \log_a(bx)$, then the plot $y = \log_a x$ is dilated $\frac{1}{b}$ units from the y-axis parallel to the x-axis.

Using technology

Using the TI-Nspire:

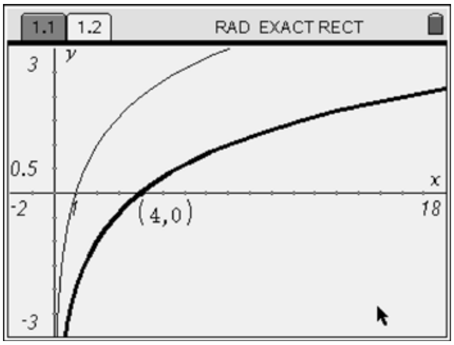
a

1 Select the Graphs & Geometry application.

2 Type **$\log_2(x)$** into $f1(x)$ then press enter.

3 Type **$\log_2(x/4)$** into $f2(x)$ then press enter. Make this a bold line.

4 Press menu and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



(4, 0)

b

1 Type **$\log_5(x)$** into $f1(x)$ then press enter.

2 Type **$\log_5(2x)$** into $f2(x)$ then press enter. Make this a bold line.

Using the ClassPad:

a

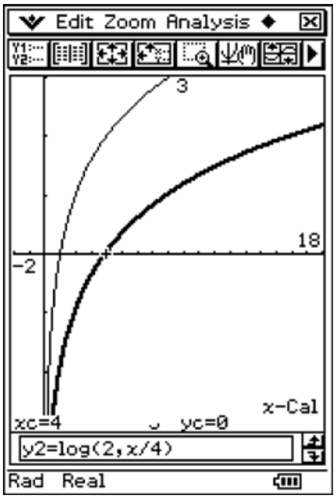
1 Select the Graphs and Tables application by tapping on Graph&Tab.

2 Type **$\log_2(x)$** into $y1$ then press EXE.

3 Type **$\log_2(x/4)$** into $y2$ then press EXE. Make this a bold line.

4 Tap to sketch the two functions.

5 Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.



xc=4 yc=0

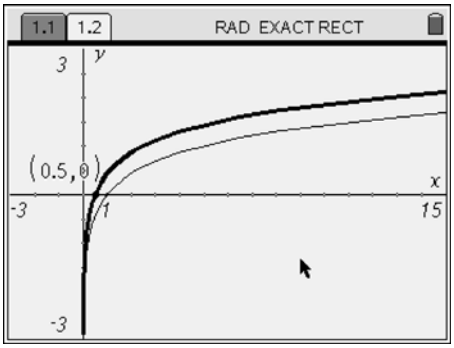
y2=log(2,x/4)

b

1 Type **$\log_5(x)$** into $y1$ then press EXE.

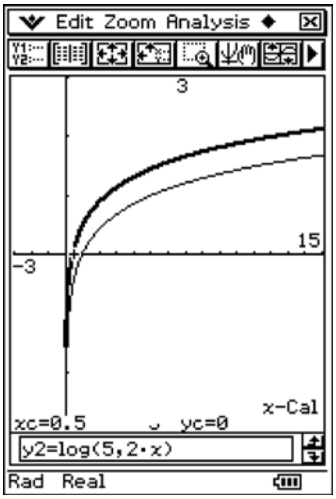
2 Type **$\log_5(2x)$** into $y2$ then press EXE. Make this a bold line.

3 Press (menu) and select *Intersection Point(s)* from the Points & Lines submenu to calculate the axes intercepts.



3 Tap 📈 to sketch the two functions.

4 Tap Analysis and select *x-Calc* from the G-Solve submenu to calculate the axes intercepts.



xc=0.5 yc=0

y2=log(5,2·x)

Using technology

Using the TI-Nspire:

Note: At the time of writing, the current operating system does not support list graphing.

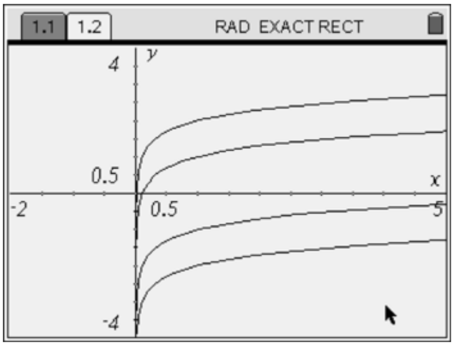
a

Type **log(*x*) − 2** into $f1(x)$ and press (enter).

Type **log(*x*) − 1** into $f2(x)$ and press (enter).

Type **log(*x*) + 1** into $f3(x)$ and press (enter).

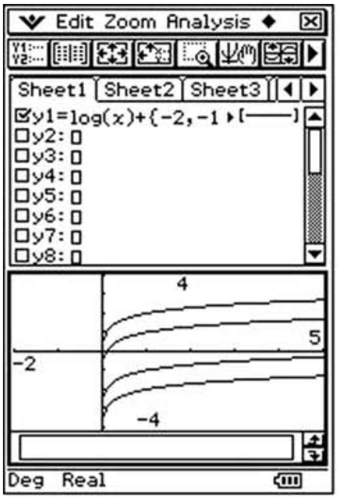
Type **log(*x*) + 2** into $f4(x)$ and press (enter).



Using the ClassPad:

a

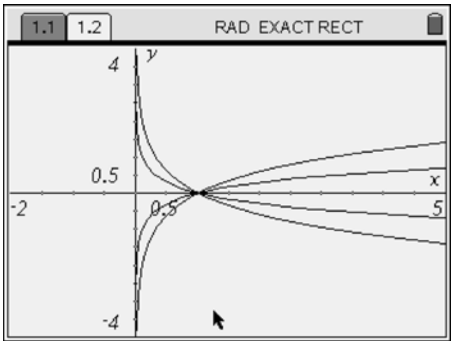
Type **log(*x*) + {−2, −1, 1, 2}** into $y1$ and press (EXE). Tap 📈 to see all graphs.



☑y1=log(x)+{−2,−1▸[
☐y2:□
☐y3:□
☐y4:□
☐y5:□
☐y6:□
☐y7:□
☐y8:□

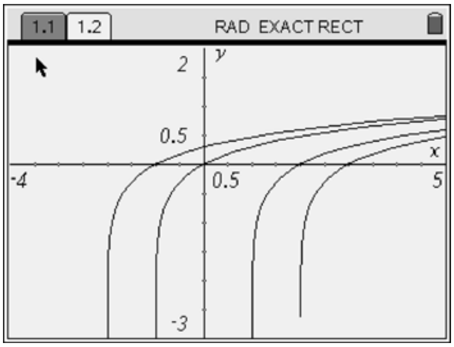
b

Type $-2\log(x)$ into $f1(x)$ and press enter.
Type $-\log(x)$ into $f2(x)$ and press enter.
Type $\log(x)$ into $f3(x)$ and press enter.
Type $2\log(x)$ into $f4(x)$ and press enter.



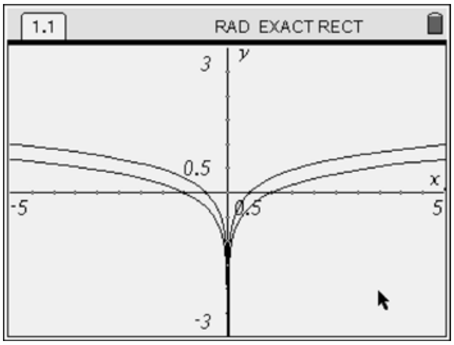
c

Type $\log(x-2)$ into $f1(x)$ and press enter.
Type $\log(x-1)$ into $f2(x)$ and press enter.
Type $\log(x+1)$ into $f3(x)$ and press enter.
Type $\log(x+2)$ into $f4(x)$ and press enter.



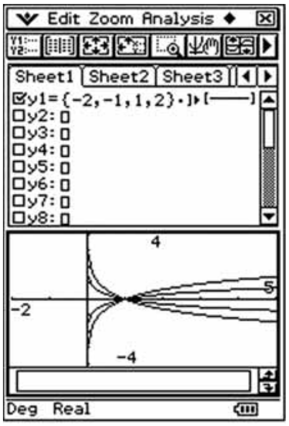
d

Type $\log(-2x)$ into $f1(x)$ and press enter.
Type $\log(-x)$ into $f2(x)$ and press enter.
Type $\log(x)$ into $f3(x)$ and press enter.
Type $\log(2x)$ into $f4(x)$ and press enter.



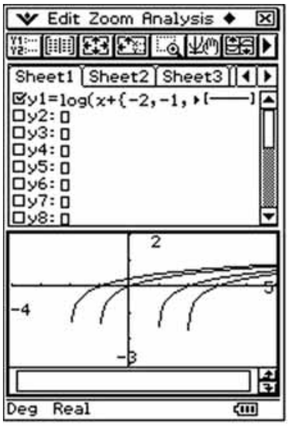
b

Type $\{-2, -1, 1, 2\}\log(x)$ into $y1$ and press EXE. Tap ⊞ to see all graphs.



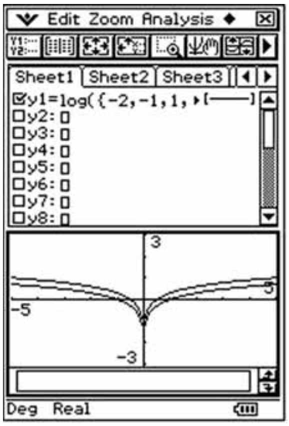
c

Type $\log(x) + \{-2, -1, 1, 2\})$ into $y1$ and press EXE. Tap ⊞ to see all graphs.



d

Type $\log(\{-2, -1, 1, 2\}x)$ into $y1$ and press EXE. Tap ⊞ to see all graphs.



Exercise 3G

Examples 26, 27

1 Use a table of values to sketch each of the following. Determine any axes intercepts and the equation of the asymptote.

a $y = 1 + \log_2 x$
b $y = -2 + \log_5 x$
c $y = \frac{1}{2} + \log_3 x$
d $y = 3\log_2 x$
e $y = -2\log_3 x$
f $y = 0.5 \times \log_{10} x$

Example 19

2 **a** Express as a log function:
  i $y = 10^{0.5x}$
  ii $y = 10^{3x}$
b Express as an exponential function:
  i $y = 3\log_{10} x$
  ii $y = 2\log_{10} 3x$

Example 19

3 Find an equivalent log or exponential function for:

a $y = 3^x + 2$
b $y = \log_2(x - 3)$
c $y = 4 \times 3^x + 2$
d $y = 5^x - 2$
e $y = \log_2(3x)$
f $y = \log_2\left(\frac{x}{3}\right)$
g $y = \log_2(x + 3)$
h $y = 5 \times 3^x - 2$

Examples 28, 29

4 Sketch each of the following. State the equation of the asymptote and the axes intercepts. Validate, using your calculator:

a $y = \log_2(x - 4)$
b $y = \log_2(x + 3)$
c $y = \log_2(2x)$
d $y = \log_2(x + 2)$
e $y = \log_2\left(\frac{x}{3}\right)$
f $y = \log_2(-2x)$

5 Use a graphics calculator to solve each of the following equations, correct to 2 decimal places:

a $2^{-x} = x$
b $\log_{10}(x) + x = 0$

6 Use a graphics calculator to plot the graphs of $y = \log_{10}(x^2)$ and $y = 2\log_{10}(x)$ for $-10 \leq x \leq 10, x \neq 0$.

7 On the same set of axes plot the graph of $y = \log_{10}(\sqrt{x})$ and $y = \frac{1}{2}\log_{10}(x)$ for $0 < x \leq 10$.

8 Use a graphics calculator to plot the graphs of $y = \log_{10}(2x) + \log_{10}(3x)$ and $y = \log_{10}(6x^2)$.

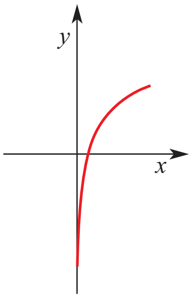
MAPS

9 Find a and k, such that the graph of $y = a10^{kx}$ passes through the points (2, 6) and (5, 20).

MAPS

10 A plot of the function $y = \log_a(x)$ is shown.
Use the grid to plot:

a $y_1 = \log_a(ax)$
b $y_2 = \log_{\sqrt{a}}(x)$



11 Use a graphics calculator to plot $y = \ln(2x - 6)$ and $y = \log\left(\frac{1}{2}x + 1\right)$. Examine these plots to establish how the plot of $y = \log_a x$ is transformed to $y = \log_a(bx - c)$.

3.8 Exponential models and applications

Fitting data

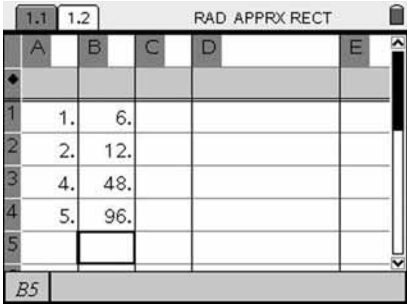
Using technology

Using the TI-Nspire:

1 Enter into the Lists & Spreadsheet application.

2 Type the x coordinates **1**, **2**, **4** and **5** into the first column.

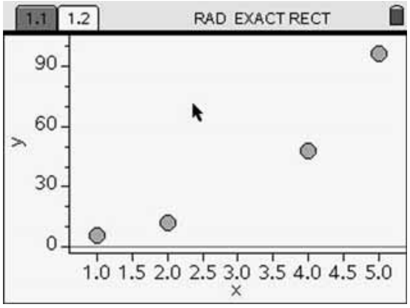
3 Type the y coordinates **6**, **12**, **48** and **96** into the second column.



	A	B	C	D	E
1	1.	6.			
2	2.	12.			
3	4.	48.			
4	5.	96.			
5					

4 Give column A the name **x**, and give column B the name **y**.

5 Highlight both columns, press menu, then select Quick Graph from the Data submenu.

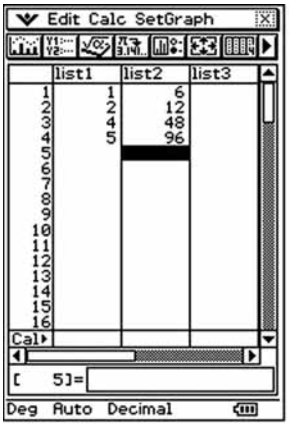


Using the ClassPad:

1 Enter into the Statistics application.

2 Type the x coordinates **1**, **2**, **4** and **5** into list1.

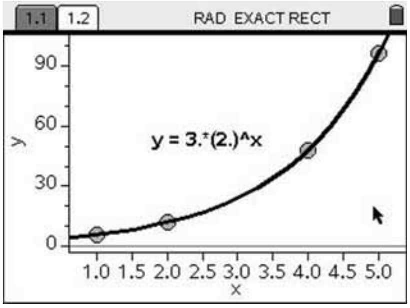
3 Type the y coordinates **6**, **12**, **48** and **96** into list2.



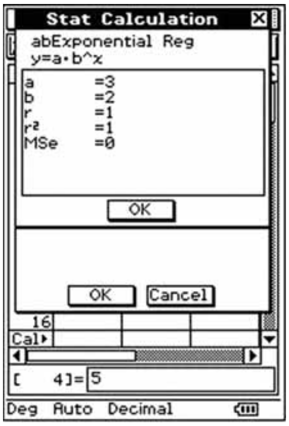
	list1	list2	list3
1	1	6	
2	2	12	
3	4	48	
4	5	96	
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			

4 To perform an Exponential regression on the data, enter into the Calc menu and then tap *abExponential Reg*.

6 To perform an Exponential Regression, press (menu), enter into the Actions menu and select *Show Exponential* from the Regression submenu.

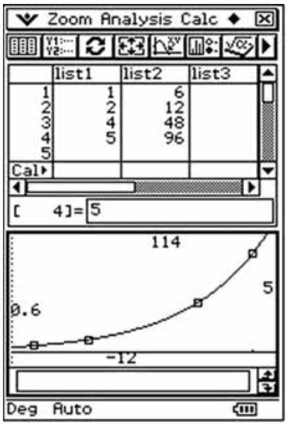


5 Ensure the following is set:
XList: list1 and YList: list2
Tap OK.



```
Stat Calculation
abExponential Reg
y=a·b^x
a       =3
b       =2
r       =1
r²      =1
MSe     =0
```

6 Tap OK to view the regression line.



	list1	list2	list3
1	1	6	
2	2	12	
3	4	48	
4	5	96	
5			

Thus, the curve has the equation $y = 3 \times 2^x$.

There are many practical situations in which the relationship between variables is exponential.

Paper folding

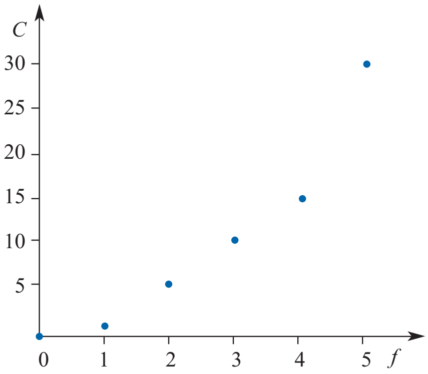
Example 30

Take a rectangular piece of paper measuring approximately 30 cm × 6 cm. Fold the paper in half, successively in the same direction, until you have folded it five times. Tabulate the times folded, f, and the creases in the paper, C.

Solution

No. times folded, f	0	1	2	3	4	5
Creases, C	0	1	3	7	15	31

The rule connecting C and f is $C = 2^f - 1,\ f = 0, 1, 2, \ldots$.



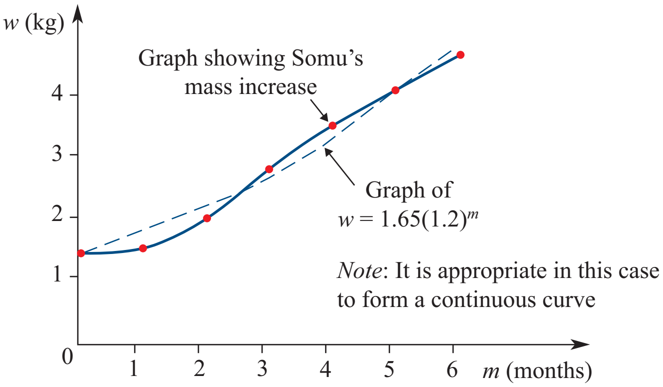
Example 31

The table below shows the increase in weight of Somu, the orang-utan, born at the Eastern Plains Zoo. Draw a graph to show Somu's mass for the first 6 months.

Months, m	0	1	2	3	4	5	6
Mass, w (kg)	1.65	1.7	2.2	3.0	3.7	4.2	4.8

Solution

Plotting these values:



Note: It is appropriate in this case to form a continuous curve

The graph of the exponential function $w = 1.65(1.2)^m$, $0 \le m \le 6$ is plotted on the same set of axes.

The table of values is

m	0	1	2	3	4	5	6
w	1.65	1.98	2.38	2.85	3.42	4.1	4.93

It can be seen from the graphs that the exponential model, $w = 1.65(1.2)^m$ approximates to the actual mass and would be a useful model to predict mass for any future orang-utan births at the zoo. This model describes a growth rate, for the first 6 months, of 120% per month.

Using technology

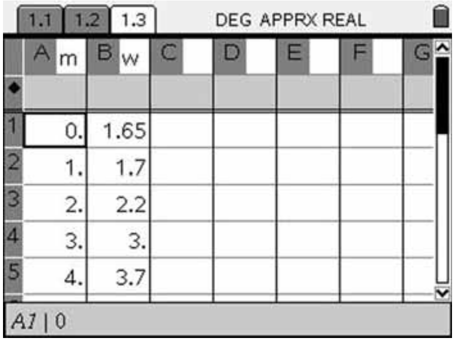
Using the TI-Nspire:

1 Press ⌂ and enter into the Lists & Spreadsheet application.

2 Give column A the name **m**.

3 Give column B the name **w**.

4 Enter the data into their respective columns.

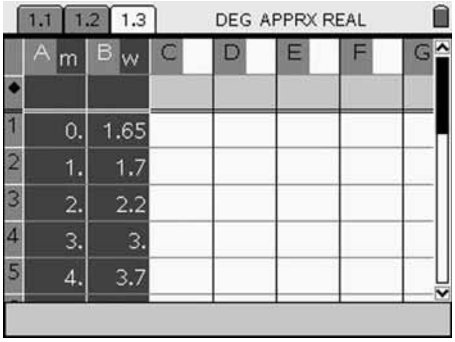


A m	B w	C	D	E	F	G
0.	1.65					
1.	1.7					
2.	2.2					
3.	3.					
4.	3.7					

A1 | 0

5 Move the cursor to the very top of column A until it is highlighted.

6 Hold CAPS and press the right arrow on the navigation pad. (Both columns should now be highlighted.)



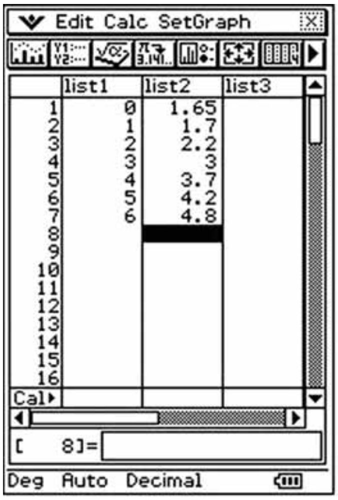
A m	B w	C	D	E	F	G
0.	1.65					
1.	1.7					
2.	2.2					
3.	3.					
4.	3.7					

Using the ClassPad:

1 Select the Statistics application by tapping on Statistics.

2 Enter the *m* values from the table into list1.

3 Enter the *w* values from the table into list2.

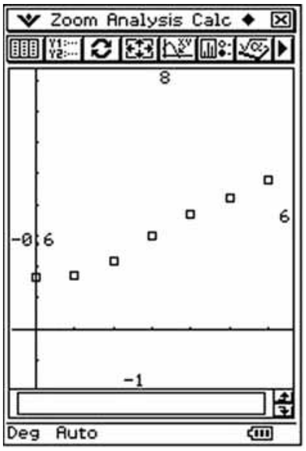


	list1	list2	list3
1	0	1.65	
2	1	1.7	
3	2	2.2	
4	3	3	
5	4	3.7	
6	5	4.2	
7	6	4.8	
8			
9			
10			
11			
12			
13			
14			
15			
16			
Cal▸			

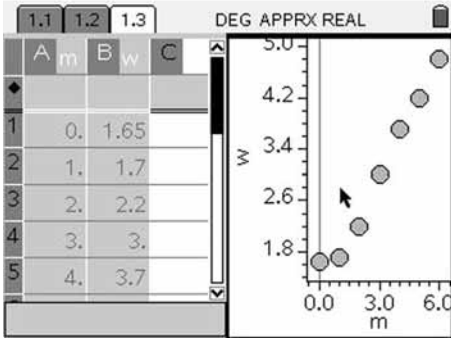
[8]=

4 Tap SetGraph and ensure the Settings of Graph 1 are correct.

5 Tap ▦ to plot the points, then tap Resize for a full-screen plot.



7 Press menu and select *Quick Graph* from the Data submenu.



	A m	B w	C
◆			
1	0.	1.65	
2	1.	1.7	
3	2.	2.2	
4	3.	3.	
5	4.	3.7	

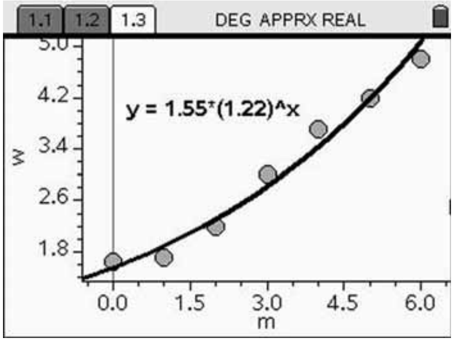
8 Press menu and enter into the Actions submenu.

9 Select *Show Exponential* from the Regression submenu.

10 To view the graph in full, do the following: Press ctrl tab, ctrl K, ctrl clear then ctrl home.

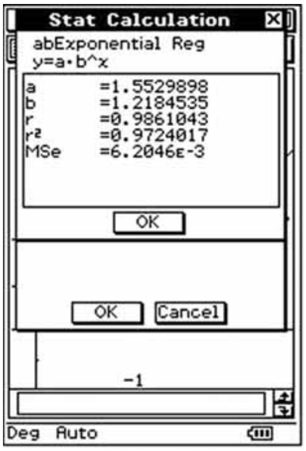
Now go to:

Page Layout→*Select Layout*→1: *Layout 1*.



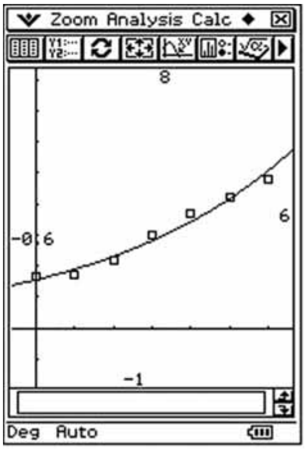
6 Tap Calc and press EXE on abExponential Reg.

7 Tap OK once you have checked that the information displayed is correct.



```
Stat Calculation
abExponential Reg
y=a·b^x
a    =1.5529898
b    =1.2184535
r    =0.9861043
r²   =0.9724017
MSe  =6.2046E-3
```

8 Tap OK to see the regression line that is drawn.



Note: Using a calculator generates the exponential regression curve $y = 1.55(1.22)^x$, which differs from the curve $w = 1.65(1.2)^m$ drawn initially to agree with Somu's mass at birth.

Example 32

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at 11% per annum whereas that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let $P =$ population of grey kangaroos at the start.

$\therefore$ Number of grey kangaroos after n years $= P(1.11)^n$,

And number of red kangaroos after n years $= 10P(0.95)^n$.

When the proportions are reversed:

$$\begin{aligned} P\,(1.11)^n &= 10 \times \left[10P\,(1.95)^n\right] \\ (1.11)^n &= 100\,(0.95)^n \end{aligned}$$

Taking the $\log_{10}$ of both sides gives:

$$\begin{aligned} \log_{10}(1.11)^n &= \log_{10} 100\,(0.95)^n \\ n\log_{10}(1.11) &= \log_{10} 100 + n\log_{10} 0.95 \\ n \times 0.04532 &= 2 + n\,(-0.0223) \\ N &= \frac{2}{0.0676} \\ &= 29.6 \end{aligned}$$

Hence, the proportions of kangaroo populations will be reversed by the 30th year.

Exponential growth

Examples 31 and 32 are examples of exponential change. In the following, A is a variable subject to exponential change.

Let A be the quantity at time t. Then $A = A_0a^t$, where A_0 is a positive constant and a is a real number.

If $a > 1$, the model represents **growth**. If $a < 1$, the model represents **decay**.

Physical situations in which this is applicable include:

- the growth of cells
- population growth
- continuously compounded interest
- radioactive decay
- cooling of materials

Consider an amount of money, \$10 000, invested at a rate of 5% per annum but compounded continually. That is, it is compounded at every instant.

If there are n compound periods in a year, the interest paid per period is $\frac{5}{n}$%. Therefore, the amount of the investment, A, at the end of the year is

$$A = 10\,000\left(1 + \frac{5}{100n}\right)^n = 10\,000\left(1 + \frac{1}{20n}\right)^n$$

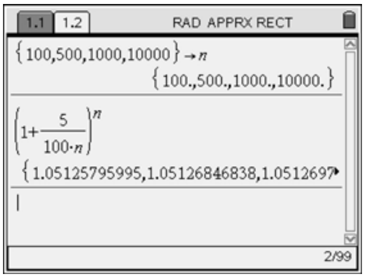
Using technology

Using the TI-Nspire:

1 Enter into the Calculator application.

2 Type **{100, 500, 1000, 10000}**→ ***n*** and then press enter.

3 Type (**1** + **5**/(**100*n***))^***n*** then press enter.



```
{100,500,1000,10000}→n
                {100.,500.,1000.,10000.}
(1+5/(100·n))^n
 {1.05125795995,1.05126846838,1.0512697▸
```

Using the ClassPad:

1 Enter into the Main application.

2 Type **{100, 500, 1000, 10000}**⇒ ***n*** and then press EXE.

3 Type (**1** + **5**/(**100*n***))^***n*** then press EXE.

```
{100,500,1000,10000}⇒n
     {100,500,1000,10000}
(1+5/(100n))^n
{1.05125796,1.051268468▸
```

3.9 Modelling and problem solving

Exercise 3H

Example 30

1 Find an exponential model of the form $y = ab^x$ to fit the following data:

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

Example 30

2 Find an exponential model of the form $p = ab^t$ to fit the following data:

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

Example 31

3 A sheet of paper, 0.2 mm thick, is cut in half and one piece is stacked on top of the other.

a If this process is repeated, complete the table below.

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	⋮
⋮	⋮	⋮
10		

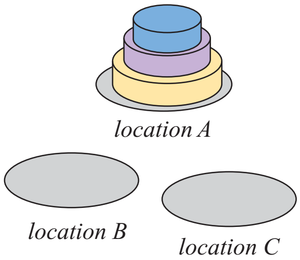
b Write a formula that shows the relationship between T and n.

c Draw a graph of T against n for $n \leq 10$.

d What would be the total thickness, T, after 30 cuts?

4 This problem is based on the so-called 'Tower of Hanoi' puzzle. Given a number of discs of varying sizes, the problem is to move a pile of discs (where n, the number of discs, is ≥ 1) to a second location (if starting at A then to either B or C), according to the following rules:

- Only one disc can be moved at a time.
- A total of only three locations can be used to 'rest' discs.
- A larger-sized disc cannot be placed on top of a smaller disc.
- The task must be completed in the least number of moves possible.



a Use two coins to complete the puzzle. First repeat with three coins and then four coins, and thus complete the table.

No. discs, n	1	2	3	4
Minimum number of moves, M	1			

b Find the formula that shows the relationship between M and n. Use your formula to extend the table of values for $n = 5$, 6 and 7.

c Plot the graph of M against n.

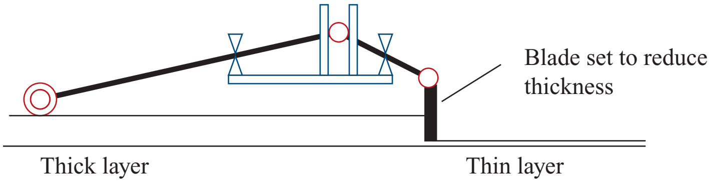
d Investigate, for both $n = 3$ and 4, whether there is a pattern for the number of times each particular disc is moved.

5 To control an advanced electronic machine 2187 different switch positions are required. There are two kinds of switches available:

- Switch 1: these can be set to nine different positions.
- Switch 2: these can be set to three different positions.

If n of switch 1 type and $n + 1$ of switch 2 type are used, calculate the value of n to give the required number of switch positions.

6 Research is being done to investigate the durability of paints of different thicknesses. The automatic machine, shown in the diagram, is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate, reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting, the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $(n - 1)$ times. At the end of this second stage express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $(n - 3)$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

7 A lonely bush hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves in the sun after use and then stores the dried tea in an airtight box. He estimates that after each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.

The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

a Write expressions for the amount of caffeine when the packet of tea leaves is re-used for the first, second, third and nth times.

b Do the same for the amount of tannin remaining.

c Find the number of times he can re-use the tea leaves if a 'tea' containing more than three times as much tannin as caffeine is undrinkable.

8 In 1880 two trackers, Jack Noble and Gary Owens, whose tribal names were Wannamutta and Werrannallee, helped the police capture Ned Kelly. For their services they were to be paid 50 pounds. They were never paid and the total value of monies owed to them, with interest, is now worth \$45 million. What average per annum interest rate is being applied to calculate the present value of the monies owed?

9 A measure of the acidity of a solution (p) is given by the function $p = \log\left(\frac{1}{H}\right)$, where H is the concentration of the hydrogen ions in the solution, in moles per litre. Recently, the local residents of the old mining town of Mt Morgan believed that the acidity level in a nearby popular swimming hole was above the acceptable value. The Health Department standard states that $H < 3 \times 10^{-10}$ is unsafe. If the measure of acidity (p) in the swimming hole was 10.9, should the Department be concerned?

10 A share portfolio consists of shares in company A plus company B. The value of shares (\$) in each company is modelled by:

$$A = 300 \times 10^{-0.03\,T}$$
$$B = 170 \times \log(5(T + 1))$$

where T is the number of months since purchase.

Use your calculator to **investigate** the *maximum total value* of the portfolio in the *short- and long-term future*. Briefly **state** any assumptions and **identify** the effect on your solution.

11 Police arrive at a university's Mathematics Department at 8 p.m. to investigate the mysterious death of the Dean of Mathematics. At this time, the temperature of the corpse was 33° C. The temperature of the crime scene had remained steady at 23° C. The human body temperature is 37° C. The police make use of the following equation to determine the time that has elapsed since death:

$$37 = 23 + 10 \times 2.718^{-0.2378t}$$

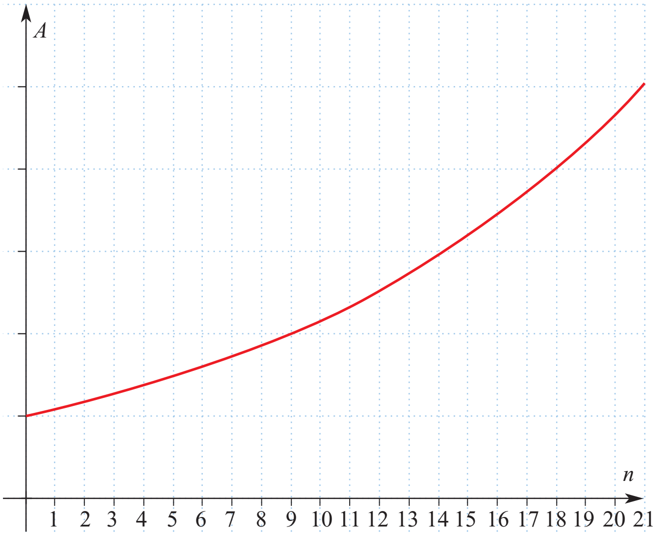
In this instance, t is the time (in hours) from 8 p.m.

Police have three suspects who were in the building at the following times on the day of the murder:

Senior Lecturer	12 noon to 4:30 p.m.
Associate Professor	4:45 p.m. to 6:40 p.m.
Long-serving Professor	6:45 p.m. to 7:20 p.m.

Determine who the prime suspect should be by establishing algebraically the time of death. Validate your results and suggest any relevant assumptions.

Example 32 **12**



The plot above illustrates the growth in value amount, A, of an investment property over a period of n years.

Note: A increases by a constant percentage (%) of the previous year's value in each successive year; that is, $A = A_0 k^n$, where k is the growth factor.

- **a** Determine the value of n, when the amount A doubles.
- **b** Evaluate the growth factor k, correct to 3 decimal places.
- **c** Use the answer to part **b** to calculate the percentage (%) increase from one year to the next.

13 The mid-points of the sides of a square with side 20 are joined to form a second square. The mid-points of the sides of this square are joined to give a third square and so on. Calculate the area of the tenth square.

14 A seismograph is an instrument that measures an earthquake's intensity, using the Richter scale. The Richter number (R) is defined as:

$$R = \log\left(\frac{I}{I_0}\right)$$

where I is the earthquake's intensity and I_0 is a constant number.

The 1985 Mexico City earthquake measured $R = 8.1$. The 1989 San Francisco earthquake measured $R = 7.1$. Compare the intensities of these two earthquakes.

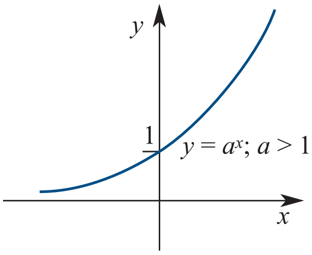
15 The number of words typed per minute (N) on a word processor is a function of the number of brief tutorial sessions (t).

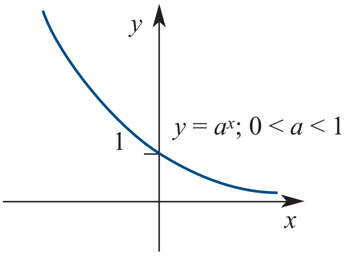
$$t = -144 \log\left(1 - \frac{N}{90}\right)$$

Evaluate the expected number of words typed per minute for a person who has 40 brief tutorial sessions. Discuss the shape of the graph as it relates to the number of tutorial sessions.

Chapter summary

- To multiply two numbers in exponential form with the same base, **add** the exponents:
  $a^m \times a^n = a^{m+n}$
- To divide two numbers in exponential form with the same base, **subtract** the exponents:
  $a^m \div a^n = a^{m-n}$
- To raise the power of a to another power, **multiply** the exponents:
  $(a^m)^n = a^{m \times n}$
- If $a^x = a^y$ then equate the indices (exponents)
  $\therefore x = y$.
- $a^0 = 1$ and $a^1 = a$
- For rational exponents: $a^{\frac{1}{n}} = \sqrt[n]{a}$
  and $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$
  and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$
- Groups of exponential functions
  Examples:



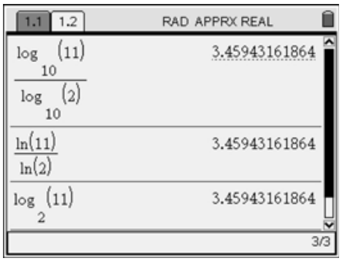


- For $a > 0$ and $a \neq 1$, x real and $0 < x$, $x = a^m$ is equivalent to $m = \log_a x$.
- Laws of logarithms:
  1 $\log_a (xy) = \log_a x + \log_a y$
  2 $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
  3 $\log_a \left(\dfrac{1}{y}\right) = -\log_a y$
  4 $\log_a (x^n) = n \log_a x$
  5 $\log_a (x) = \dfrac{\log_b x}{\log_b a}$
- $\log_{10}(x) \equiv \log(x)$ and $\log_e(x) \equiv \ln(x)$
- Exponential equations can be solved by taking logarithms of both sides.

  For example, if $2^x = 11$, then $x = \dfrac{\log_{10} 11}{\log_{10} 2}$ or, equivalently, $x = \dfrac{\log_e 11}{\log_e 2}$.

- Using technology, solve $2^x = 11$.

Using the TI-Nspire:



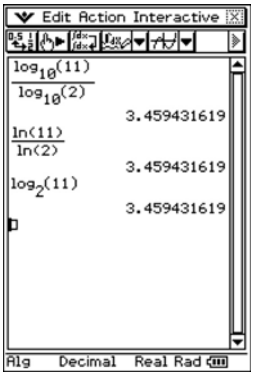
$\frac{\log_{10}(11)}{\log_{10}(2)}$ 3.45943161864

$\frac{\ln(11)}{\ln(2)}$ 3.45943161864

$\log_2(11)$ 3.45943161864

3/3

Using the ClassPad:



$\frac{\log_{10}(11)}{\log_{10}(2)}$

3.459431619

$\frac{\ln(11)}{\ln(2)}$

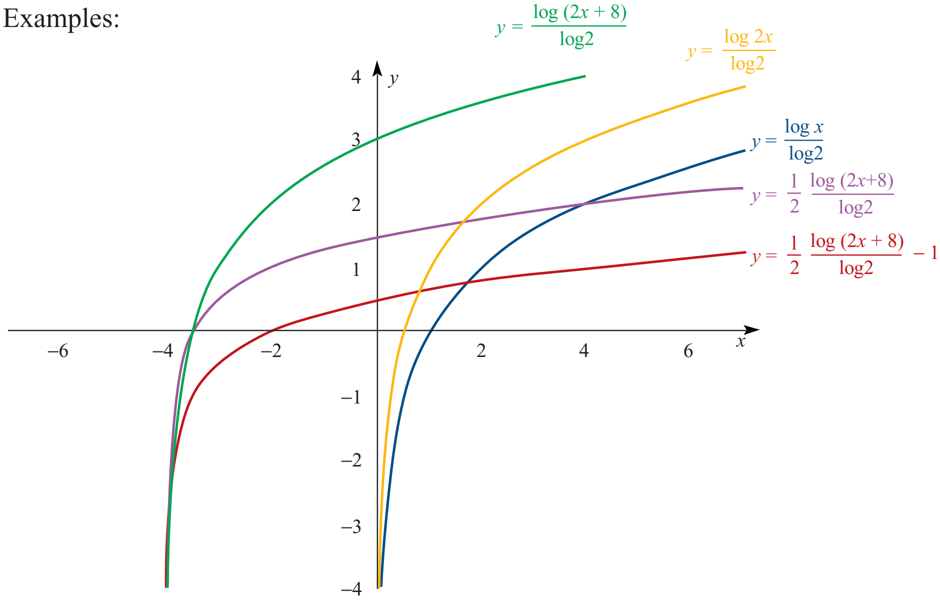
3.459431619

$\log_2(11)$

3.459431619

- Families of logarithmic functions

Examples:



Multiple-choice questions

1 $8x^3 \div (4x^{-3}) =$

A 2 **B** $2x^0$ **C** $2x^6$ **D** $2x^{-1}$ **E** $\frac{2}{x^9}$

2 The expression $\frac{a^2b}{(2ab^2)^3} \div \frac{ab}{16a^0} =$

A $\frac{2}{a^2b^6}$ **B** $\frac{2a^2}{b^6}$ **C** $2a^2b^6$ **D** $\frac{2}{ab^6}$ **E** $\frac{1}{128ab^5}$

3 If $\log_{10}(x-2) - 3\log_{10} 2x = 1 - \log_{10} y$, then y is equal to:

A $\frac{80x^3}{x-2}$ **B** $1 + \frac{8x^3}{x-2}$ **C** $\frac{60x}{x-2}$

D $1 + \frac{6x}{x-2}$ **E** $1 - \frac{x-2}{8x^3}$

4 The solution of the equation $5 \times 2^{5x} = 10$ is x equals:

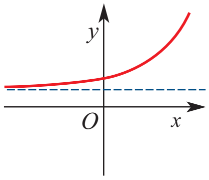
A $\frac{1}{2}$ **B** $\frac{1}{5}$ **C** $\frac{1}{5}\log_2 10$ **D** $\frac{1}{2}\log_2 5$ **E** $\frac{1}{5}2^5$

5 The equation of the asymptote of $y = 3\log_2(5x) + 2$ is:

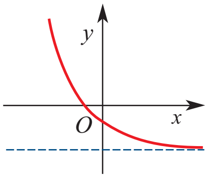
A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 5$ **E** $y = 2$

6 Which of the following graphs could be the graph of the function $f(x) = 2^{ax} + b$, where a and b are positive?

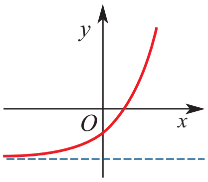
A



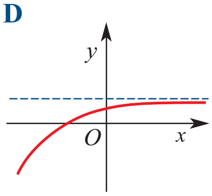
B



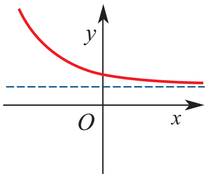
C



D



E



7 Which one of the following functions has a graph with a vertical asymptote with equation $x = b$?

A $y = \log_2(x - b)$ **B** $y = \frac{1}{x + b}$ **C** $y = \frac{1}{x + b} - b$

D $y = 2^x + b$ **E** $y = 2^{x - b}$

8 The expression $\frac{2mh}{(3mh^2)^3} \div \frac{mh}{81m^2}$ is equal to:

A $\frac{6}{mh^6}$ **B** $\frac{6m^2}{h^6}$ **C** $6m^2h^6$ **D** $\frac{6}{m^2h^6}$ **E** $\frac{1}{128mh^5}$

9 Without using a calculator, $e^{-\ln e}$ equals:

A -1 **B** $-e$ **C** e **D** $\frac{1}{e}$ **E** $-\frac{1}{e}$

10 The graph of $y = \log_a x$ is transformed to the graph of $y = \frac{1}{2}\log_a(x - 3) + 1$ by:

A a shift of -3 from the x-axis parallel to the y-axis, followed by a dilation of 2 from the y-axis parallel to the x-axis, followed by a shift of $+1$ from the x-axis parallel to the y-axis

B a shift of $+3$ from the y-axis parallel to the x-axis, followed by a dilation of $\frac{1}{2}$ from the x-axis parallel to the y-axis, followed by a shift of $+1$ from the x-axis parallel to the y-axis

C a shift of $+3$ from the y-axis parallel to the x-axis, followed by a dilation of 2 from the x-axis parallel to the y-axis, followed by a shift of $+1$ from the y-axis parallel to the x-axis

D a shift of −3 from the x-axis parallel to the y-axis, followed by a dilation of $\frac{1}{2}$ from the y-axis parallel to the x-axis, followed by a shift of +1 from the x-axis parallel to the y-axis

E a shift of +3 from the y-axis parallel to the x-axis, followed by a dilation of $\frac{1}{2}$ from the x-axis parallel to the y-axis, followed by a shift of −1 from the x-axis parallel to the y-axis

Short-response questions

1 Simplify each of the following, expressing your answer with positive index:

a $\dfrac{a^6}{a^2}$

b $\dfrac{b^8}{b^{10}}$

c $\dfrac{m^3n^4}{m^5n^6}$

d $\dfrac{a^3b^2}{(ab^2)^4}$

e $\dfrac{6a^8}{4a^2}$

f $\dfrac{10a^7}{6a^9}$

g $\dfrac{8\left(a^3\right)^2}{(2a)^3}$

h $\dfrac{m^{-1}n^2}{(mn^{-2})^3}$

i $\left(p^{-1}q^{-2}\right)^2$

j $\dfrac{\left(2a^{-4}\right)^3}{5a^{-1}}$

k $\dfrac{6a^{-1}}{3a^{-2}}$

l $\dfrac{a^4+a^8}{a^2}$

2 Use logarithms to solve each of the following equations, correct to 2 decimal places:

a $2^x = 7$

b $2^{2x} = 7$

c $10^x = 2$

d $10^x = 3.6$

e $10^x = 110$

f $10^x = 1010$

g $2^{5x} = 100$

h $2^x = 0.1$

3 Evaluate each of the following:

a $\log_2 64$

b $\log_{10} 100$

c $\log_{10} 10^7$

d $\log_a a^2$

e $\log_4 1$

f $\log_3 27$

g $\log_2 \dfrac{1}{4}$

h $\log_{10} 0.001$

i $\log_2 16$

4 Express each of the following as single logarithms:

a $\log_{10} 2 + \log_{10} 3$

b $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$

c $2\log_{10} a - \log_{10} b$

d $2\log_{10} a - 3 - \log_{10} 25$

e $\log_{10} x + \log_{10} y - \log_{10} x$

f $2\log_{10} a + 3\log_{10} b - \log_{10} c$

5 Solve each of the following for x:

a $3^x(3^x - 27) = 0$

b $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$

d $2^{2x} - 12 \cdot 2^x + 32 = 0$

6 Sketch the graph of:

a $y = 2 \cdot 2^x$

b $y = -3 \cdot 2^x$

c $y = 5 \cdot 2^{-x}$

d $y = 2^{-x} + 1$

e $y = 2^x - 1$

f $y = 2^x + 2$

7 Solve the equation $\log_{10} x + \log_{10}(2x) - \log_{10}(x+1) = 0$.

8 Given $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$.

9 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

10 **a** Given that $\log_p 7 + \log_p k = 0$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

11 Solve:

- **a** $2 \times 4^{a+1} = 16^{2a}$ (for a)
- **b** $\log_2 y^2 = 4 + \log_2 (y + 5)$ (for y)

12 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch had 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year and the second at the rate of 6% per year.

- **a** Write expressions for the brightness of each batch after n years.
- **b** A person bought some carpet from the first batch when it was 1 year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

13 The populations (in millions), p and q, of two neighbouring states in America, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since 1950.

- **a** Plot the graphs of both functions, using technology.
- **b** Find when the population of state P is:
  - **i** equal to the population of state Q
  - **ii** is twice the population of state Q

14 The value of shares in company X increased linearly over a 2-year period, according to the model $x = 0.8 + 0.17t$, where t is the number of months from the beginning of January 2006 and \$$x$ is the value of the shares at time t.

The value of shares in company Y increased over the same period of time, according to the model $y = 10^{0.03t}$, where \$$y$ is the value of these shares at time t months.

The value of shares in a third company, company Z, increased over the same period according to the model $z = 1.7 \log_{10}(5(x + 1))$, where \$$z$ is the value of the shares at time t months.

Use a graphics calculator to sketch the graphs of each of the functions on the one screen.

- **a** Find the values of the shares in each of the three companies at the end of June 2006.
- **b** Find the values of the shares in the three companies at the end of September 2007.
- **c** During which months were the shares in company X more valuable than the shares in company Y?
- **d** For how long and during which months were the shares in company X the most valuable?

15 In 2000 in a game park in Africa, it was estimated that there were approximately 700 wildebeests and that their population was increasing at 3% per year. At the same time in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a graphics calculator to plot the graphs of each function.

a After how many years is the number of wildebeests greater than the number of zebras?

b It is also estimated that there were 1000 antelopes and their numbers were increasing by 50 per year. After how many years is the number of antelopes greater than the number of zebras?

16 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements are taken.

Time (min)	3	6	9	12	15	18	21
Temperature (˚C)	71.5	59	49	45.5	34	28	23.5

a Find an exponential model to fit the data collected.

b Use this model to estimate:

- **i** the initial temperature of the liquid
- **ii** the temperature of the liquid after 25 minutes

It is suspected that one of the temperature readings is incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

- **i** the initial temperature of the liquid
- **ii** the temperature of the liquid at $t = 12$

e If the room temperature is 15°C, find the approximate time at which the cooling of the liquid ceased.

17 The curve with equation $y = ab^x$ passes through the points (1, 1) and (2, 5).

a Use algebra to find the values of a and b.

b Let $b^x = 10^z$.

- **i** Take logarithms of both sides (base 10) to find z as an expression of x.
- **ii** Find the value of k and a, such that $y = a10^{kx}$ passes through the points (1, 1) and (2, 5).

18 **a** Find an exponential model of the form $y = a \cdot b^x$ to fit the data below.

x	0	2	4	5	10
y	2	5	13	20	200

b Express the model you have found in part **a** in the form $y = a \cdot 10^{kx}$.

c Hence, find an expression for x in terms of y.

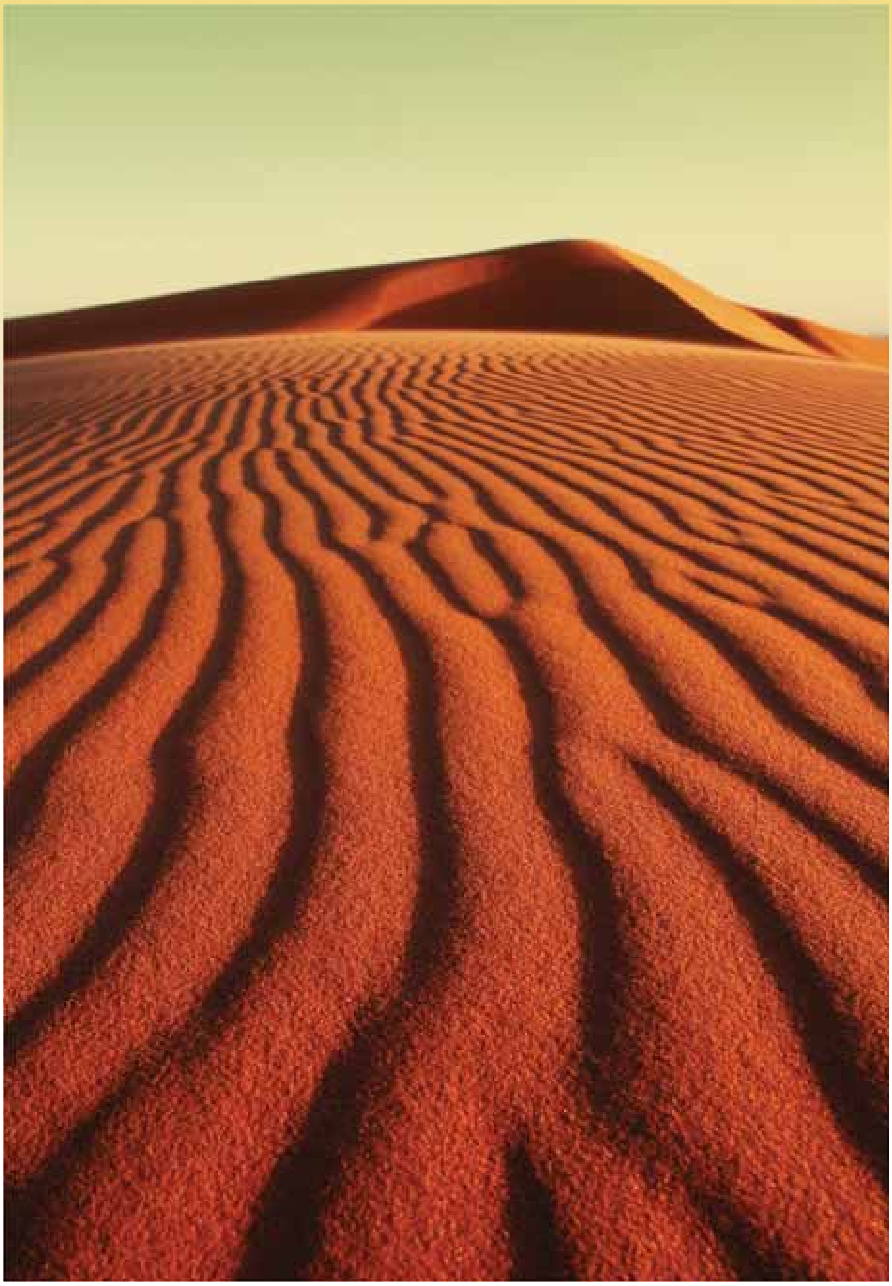
CHAPTER

4

Functions

Objectives

- To reinforce the notion that, fundamentally, graphs consist of dots.
- To understand an approach to **modelling**, including **life-related contexts**.
- To understand the terms **independent variable** and **dependent variable**.
- To understand the terms **discrete variable** and **continuous variable**.
- To find the **domain** and **range** of a function.
- To represent functions as **mappings**.
- To understand the terms **relation** and **function**.
- To understand the terms **continuous**, **discontinuous** and **discrete functions**.
- To use $f(x)$ notation.
- To understand **hybrid functions**.
- To understand **maximal** and **restricted domains**.
- To find **inverse functions**.



On the use of technology

Microsoft Excel is used to generate the data for graphing in the first section of this chapter. It may be used to help complete the work in Exercise 4A.

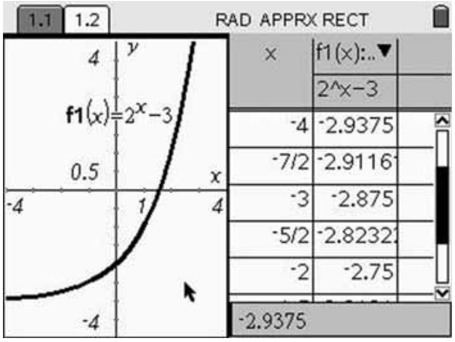
Using the TI-Nspire:

1. Use the Graphs & Geometry application to sketch functions.
2. To sketch a graph, enter the function into $f1(x) =$ then press enter.
3. To view a table of values for a graph press ctrl T. When the table of values is active, press menu and select *Edit Function Table* from the Function Table submenu to change the start value and the increment of the table.
4. When the graphs page is active, press menu and select *Window Settings* from the Window submenu to make changes.

Using the ClassPad:

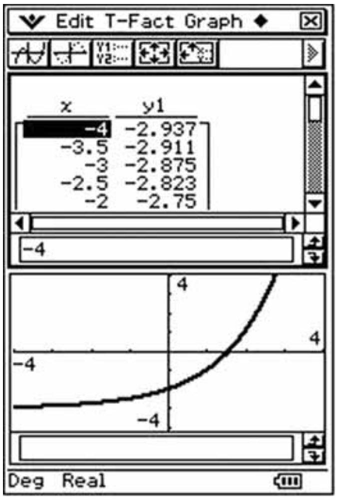
1. To enter into the Graph & Table application tap .
2. To sketch a graph, enter the function into $y1 =$, press EXE then tap .
3. To view a table of values tap .
4. To change the starting value, ending value and the increment of the table tap .
5. To change the Window settings tap .

Sketching the graph of $y = 2^x - 3$, where $X_{min} = Y_{min} = -4$ and $X_{max} = Y_{max} = 4$, and constructing a table of values for x beginning at –4 in increments of 0.5 we have:



x	f1(x):..
	2^x-3
-4	-2.9375
-7/2	-2.9116
-3	-2.875
-5/2	-2.8232
-2	-2.75

-2.9375



x	y1
-4	-2.937
-3.5	-2.911
-3	-2.875
-2.5	-2.823
-2	-2.75

-4

4.1 Introduction to more complex graphs

Example 1

Draw neat and accurate graphs of the equations $y = 2^x - 3$ and $y = \log_3(x + 2)$ on the same Cartesian plane. (Use a 2 cm grid.)

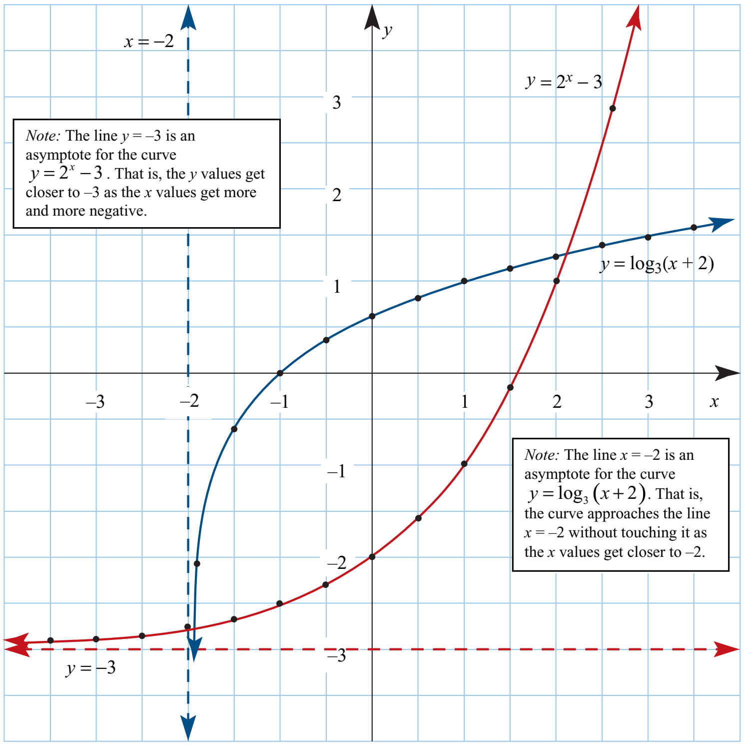
Solution

$y = 2^x - 3$

x	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5	3
y	−2.91	−2.88	−2.82	−2.75	−2.65	−2.5	−2.29	−2	−1.59	−1	−0.17	1	2.66	5

$$\begin{aligned} y &= \log_3(x + 2) \\ &= \log(x + 2) / \log(3) \qquad \text{change of base} \end{aligned}$$

x	−2	−1.9	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y	####	−2.1	−0.63	0	0.37	0.63	0.83	1	1.14	1.26	1.37	1.46	1.55	1.63



Note: The line $y = -3$ is an asymptote for the curve $y = 2^x - 3$. That is, the y values get closer to −3 as the x values get more and more negative.

Note: The line $x = -2$ is an asymptote for the curve $y = \log_3(x + 2)$. That is, the curve approaches the line $x = -2$ without touching it as the x values get closer to −2.

Note: The lines $y = -3$ and $x = -2$ are **boundaries** for the respective curves; that is, they show where the edges of the curves are.

Example 2

Using a 2 cm grid, draw neat and accurate graphs of the following equations on the same set of axes:

a $4x - 2y = 5$

b $y^2 = x + 3$

Solution

a

$$\begin{aligned} 4x - 2y &= 5 \\ 2y &= 4x - 5 \\ y &= 2x - 2.5 \end{aligned}$$

$m = 2$, y-intercept $= -2.5$

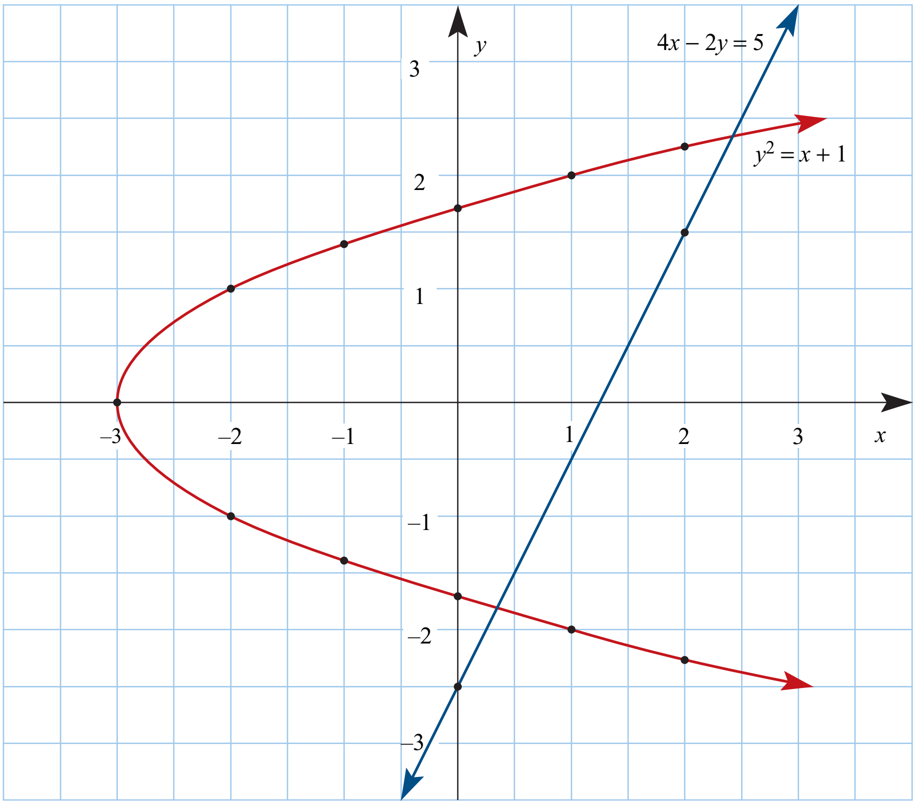
b

$$\begin{aligned} y^2 &= x + 3 \\ y &= \pm\sqrt{x+3} \\ y &= \sqrt{x+3} \quad \text{or} \quad y = -\sqrt{x+3} \\ y &= \sqrt{x+3} \end{aligned}$$

x	−4	−3	−2	−1	0	1	2	3	4
y	####	0	1	1.41	1.73	2	2.24	2.45	2.65

$y = -\sqrt{x+3}$

x	−4	−3	−2	−1	0	1	2	3	4
y	####	0	−1	−1.4	−1.7	−2	−2.2	−2.4	−2.6



Example 3

Draw a neat and accurate graph of the equation $y = \dfrac{6}{x+2}$ on the Cartesian plane. (Use a 1 cm grid.)

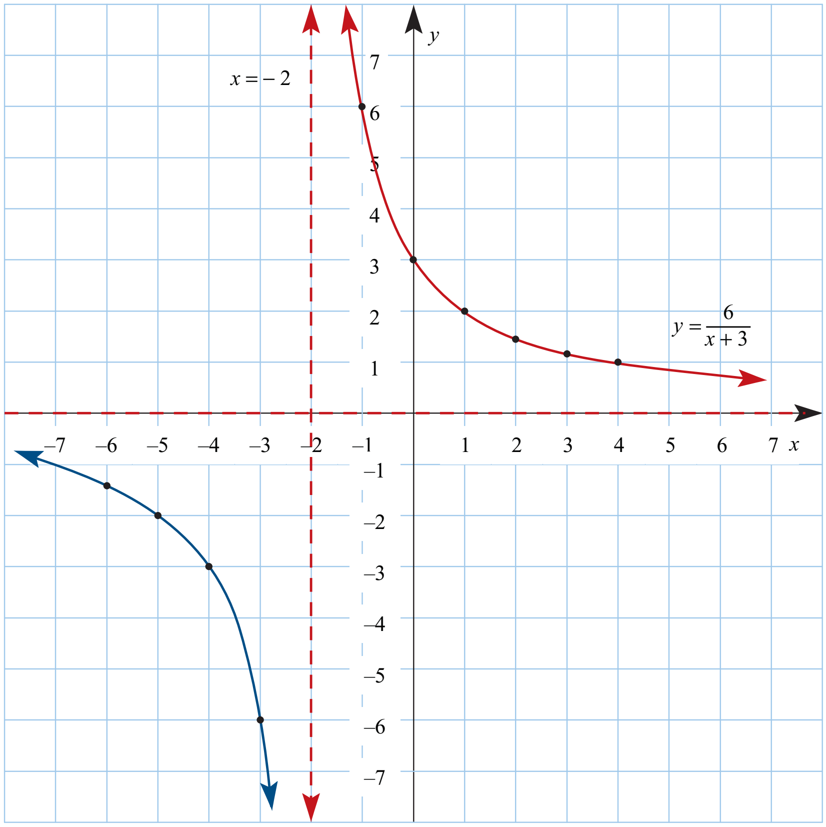
Solution

$$y = \frac{6}{x+2}$$

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
y	-1.5	-2	-3	-6	# # # #	6	3	2	1.5	1.2	1	0.86	0.75

x	-3	-2.75	-2.5	-2.25	-2	-1.75	-1.5	-1.25	-1
y	-6	-8	-12	-24	# # # # #	24	12	8	6

Note that y is not defined for $x = -2$.



Note: When a y value cannot be calculated, calculate y values for other points near the x value concerned. The dotted vertical line $x = -2$ shown above is referred to as an asymptote and shows the **boundary** between the separate parts of the graph. The x-axis, $y = 0$, is also an asymptote for the curve shown above.

Exercise 4A

1 Using a 2 cm grid, draw neat and accurate graphs of the following equations on the number plane:

a $y = 2x - 3$ **b** $y = x^2 - 3x + 1$ **c** $y = (x + 1)(2 - x)$ **d** $y = \sqrt{9 - x^2}$

Example 1 2 Using a 1 cm grid, draw neat and accurate graphs of the following equations on the number plane:

a $y = 3^x - 5$ **b** $y = \log_2(x + 3)$ **c** $y = 2^x + 1$ **d** $y = \log_4(x - 2)$

Example 2 3 Using a 1 cm grid, draw neat and accurate graphs of the following equations on the number plane:

a $y^2 = x^2 - 9$ **b** $x^2 = y^2$ **c** $x^2 + y^2 = 25$ **d** $x^2 - y^2 + 9 = 0$

Example 3 4 Using a 1 cm grid, draw neat and accurate graphs of the following equations on the number plane:

a $y = \dfrac{4}{x - 1}$ **b** $y = \dfrac{6}{2 - x}$ **c** $y = \dfrac{4}{x^2 - 1}$ **d** $y = \dfrac{4}{x^2 + 1}$

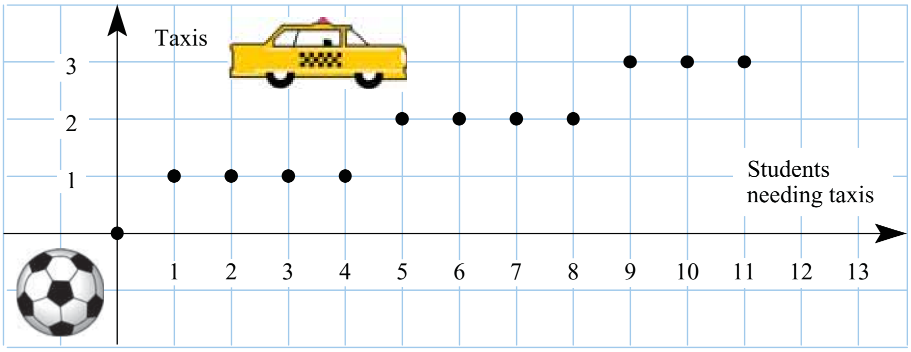
4.2 Modelling life-related situations as graphs

Example 4

The U16 soccer team plays mid-week fixtures and needs transport to the games. Sometimes some of the parents come to assist with transport. Those who don't get parent transport take taxis. The reserves always go with the coach in her car.

Draw a graph of 'Number of taxis' vs 'Number of students needing taxis'. Use a 1 cm grid (i.e. 1 cm = 1 unit).

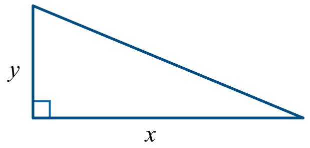
Solution



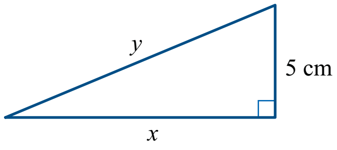
Note: Graphing is not always a process of dot-to-dot. The dots are joined when it is possible to also have the dots in between. Joining the dots with lines is done only if the dots in between are also relevant. In the U16 soccer team example (i.e. Example 4), the graph is a set of dots with no lines joining them.

Example 5

a Use the Cartesian plane to show all possible ordered pairs (x, y) for which the triangle shown has an area of 6 m^2.



b Use the Cartesian plane to show all possible ordered pairs (x, y) for the triangle shown.



Solution

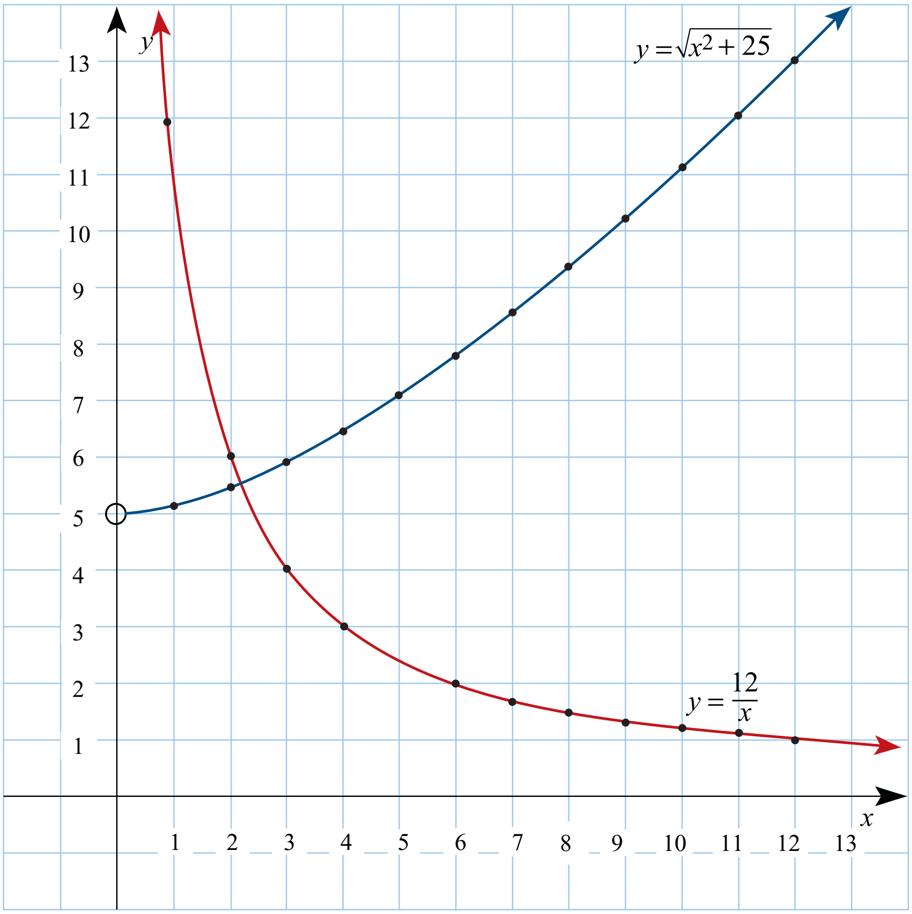
a $A = \frac{1}{2} \times b \times h \qquad \therefore 6 = \frac{1}{2} \times x \times y \qquad \text{i.e. } y = \frac{12}{x} \qquad (\textit{Note}: x > 0.)$

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	#	12	6	4	3	2.4	2	1.7	1.5	1.3	1.2	1.0	1

b Using Pythagoras' theorem:

$$y^2 = x^2 + 5^2 \qquad \therefore y = \sqrt{x^2 + 25} \qquad (\textit{Note}: x > 0 \text{ and } y > 0.)$$

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	#	5.0	5.3	5.8	6.4	7.0	7.8	8.6	9.4	10	11	12	13



Note: A small open circle is drawn around the point (0, 5) in part **b** to show that it should not be included in the graph. Although the value of x in the triangle can be slightly bigger than zero, it cannot be exactly zero. $x = 0$ is a **boundary point** (i.e. an end point) and, in this case, it is not included, so the circle is open.

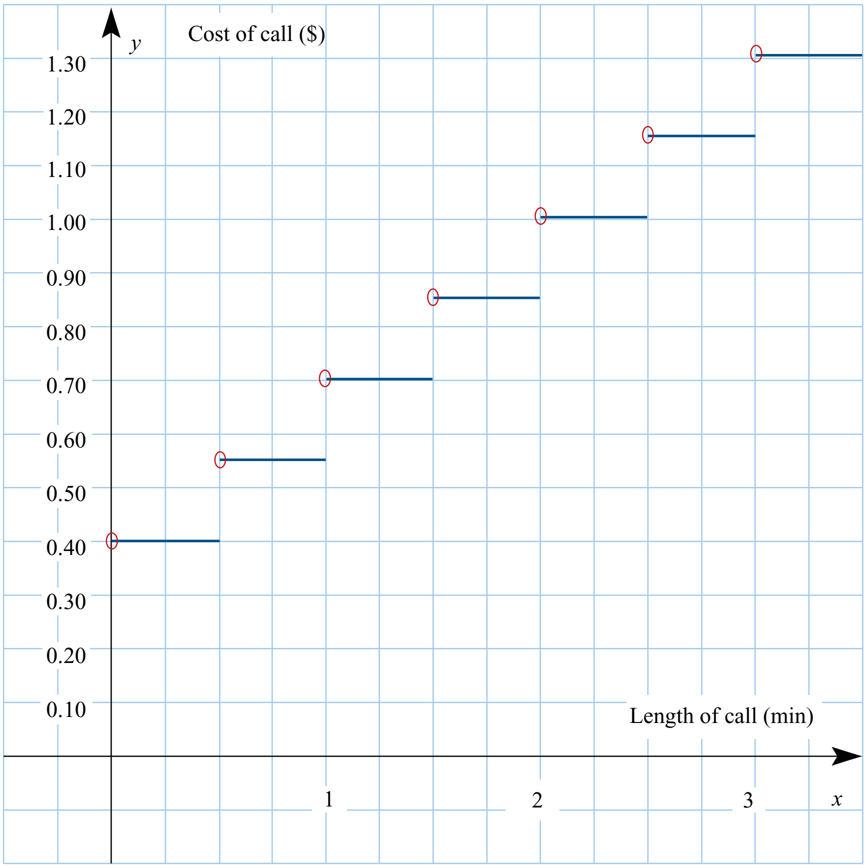
Example 6

The AAPT standard mobile call rate to mobiles and landlines in Australia in December 2006 was:

- 15c per 30 seconds
- 25c per call flagfall

Draw a graph of 'Cost of call' vs 'Length of call'. (Use 4 cm = 1 minute, 1 cm = \$0.10.)

Solution

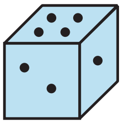


Note: Again, open circles are used to show **boundary points** that are not included. In the case above, a 1-minute call will cost $0.55 and a call lasting 1 minute and 1 second will cost $0.70. Therefore, it is necessary to show the boundary point (1, 0.70) with an open circle.

Exercise 4B

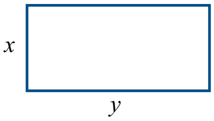
Example 4 **1** A supermarket is offering a special on eggs: $2 per carton, limit of 5. Draw a graph of 'Cost' against 'Number of cartons purchased'. Use a 1 cm grid.

2 A die is rolled and the uppermost face and the face touching the table are noted as an ordered pair. Graph all possible points on the Cartesian plane.

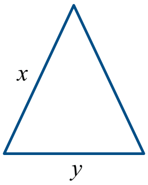


Example 5 **3** **a** Use the Cartesian plane to show all possible ordered pairs (x, y) for which the rectangle shown has a perimeter of 16 m.

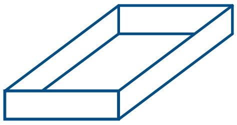
b Use the Cartesian plane to show all possible ordered pairs (x, y) for which the rectangle shown has an area of 12 m^2.

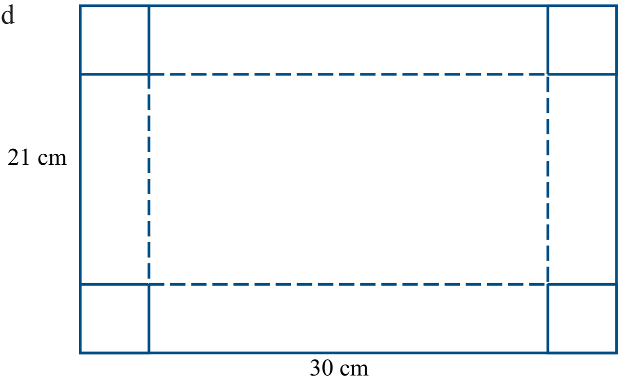


4 Use the Cartesian plane to show all possible ordered pairs (x, y) for which the isosceles triangle shown has a total perimeter of 16 m.



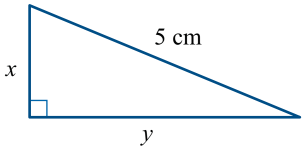
5 A chocolate tray is to be constructed from a sheet of A4 paper, where it is cut along the heavy lines and folded along the dotted lines. Sketch a graph showing 'Volume of the tray' against 'Length of each cut'.



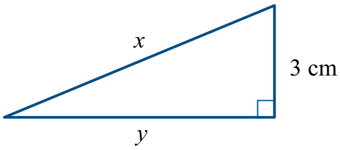


6 Sketch on the Cartesian plane all possible ordered pairs (x, y) for each triangle.

a



b



Example 6 **7** An Australia Post rates table for DL (220 × 110 mm) sized letters, which are between 5 and 20 mm thick, is shown.

Draw a graph of 'Cost of postage' vs 'Weight of letter'.

Postage rates: DL (220 × 110 mm) between 5 and 20 mm thick

Weight	Cost of postage
Up to 50 g	$1.00
Over 50 g to 125 g	$1.00
Over 125 g to 250 g	$1.45
Over 250 g to 500 g	$2.45

8 The rates table for the short-term car park at the domestic terminal at Brisbane Airport is given. The short-term car park is for parking up to 24 hours.

Draw a graph of the fee against the time spent in the car park.

Rates for short-term parking at domestic terminal

Hours	Fee
0.5	$5.00
1	$8.00
2	$10.00
3	$12.00
4	$14.00
5	Plus $2 per hour up to maximum $36 per day

4.3 Terminology of functions

A **relation** is a set of ordered pairs. These sets of ordered pairs can be infinite and have been represented as tables of values, graphs on the Cartesian plane or equations in your earlier study of Mathematics.

The first number in an ordered pair is called the **independent variable**. The second number is the **dependent variable**. The value of the dependent variable 'depends' on the value of the independent variable.

The **domain** is the set of all possible values of the independent variable. The **range** is the set of all possible values of the dependent variable; that is, the domain is usually the set of x values possible and the range is the set of y values possible.

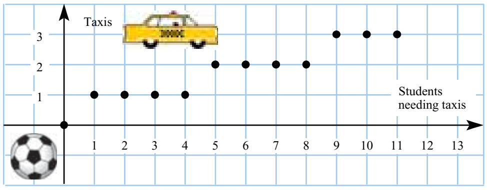
A **mapping** is a diagram that shows the links between the **elements** (numbers) of the domain and the elements (numbers) of the range.

A **discrete variable** is a variable for which the elements of the domain can be listed. Discrete variables often involve whole numbers and are usually the result of counting.

A **continuous variable** is a variable for which there is an infinite number of values of the variable between any two given values. Continuous variables always involve decimal numbers and are usually the result of measuring.

Example 7

Referring back to Example 4, a solution to the U16 soccer team problem is given.



a What are the independent and dependent variables used in the relation graphed?

b State the domain and range of the relation.

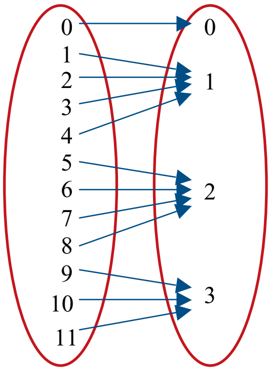
c Draw a mapping diagram for this relation.

Solution

a The independent variable is 'Students needing taxis'.
The dependent variable is 'Taxis'.

b Domain $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
Range $= \{0, 1, 2, 3\}$

c



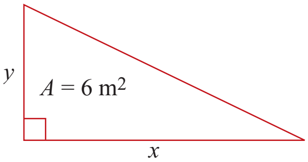
Example 8

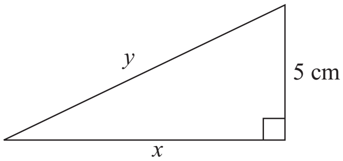
Referring back to Example 5, a solution is given.

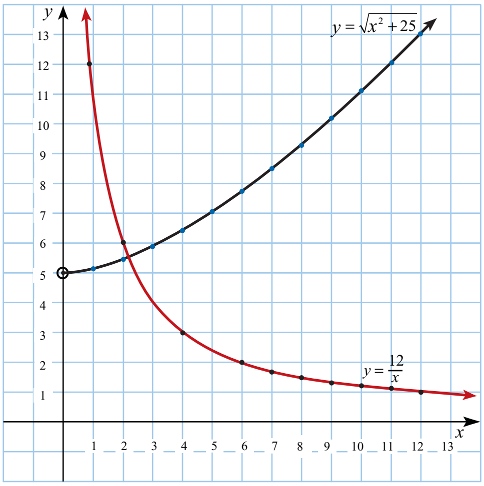
i What are the independent and dependent variables used in the relations graphed?

ii State the domain and range of each function.

iii State an algebraic equation for each function.







Solution

a **i** Assuming the independent variable is the base, then the dependent variable is the height.

ii The domain is $x > 0$. The range is $y > 0$.

iii $y = \dfrac{12}{x}, \quad x > 0.$

b **i** Assuming the independent variable is the base, then the dependent variable is the hypotenuse.

ii The domain is $x > 0$. The range is $y > 5$.

iii $y = \sqrt{x^2 + 25}, \quad x > 0$.

Note: In Example 8 it was necessary to assume that one of the variables was the independent variable in both parts **a** and **b** because there was no natural independent variable.

Example 9

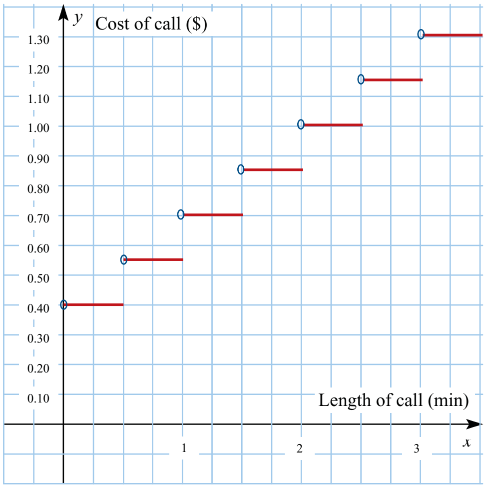
A solution to Example 6 is shown.

The AAPT standard mobile call rate to mobiles and landlines in Australia in December 2006 was:

- 15c per 30 seconds
- 25c per call flagfall

i What are the independent and dependent variables used in the relation graphed?

ii State the domain and range of the relation.



Solution

i The independent variable is ‘Length of call’.
The dependent variable is ‘Cost of call’.

ii The domain is $x > 0$.
The range is $\{0.40, 0.55, 0.70, 0.85, 1.00, \ldots\ldots.\}$.

Example 10

Classify each of the variables in Examples 7 to 9 as being either discrete or continuous.

Solution

Discrete	Continuous
Students needing taxis (Example 7)	Base (Both Examples 8a and 8b)
Taxis (Example 7)	Height (Example 8a)
Cost of call (Example 9)	Hypotenuse (Example 8b)
	Length of call (Example 9)

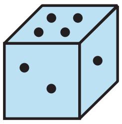
Exercise 4C

*Questions **1–8** refer to questions **1–8** in Exercise **4B**.*

Example 7 **1** **a** What are the independent and dependent variables used in the relation?

b State the domain and range of the relation.

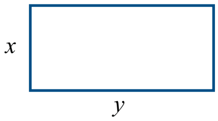
c Draw a mapping diagram for this relation.



2 **a** What are the independent and dependent variables used in the relation?

b State the domain and range of the relation.

c Draw a mapping diagram for this relation.

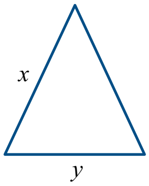


Example 8 **3** Answer the following questions for each rectangle:

i What are the independent and dependent variables used in the relations?

ii State the domain and range of each relation.

iii State an algebraic equation for each relation.



4 **a** What are the independent and dependent variables used in the relation?

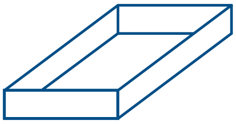
b State the domain and range of the relation.

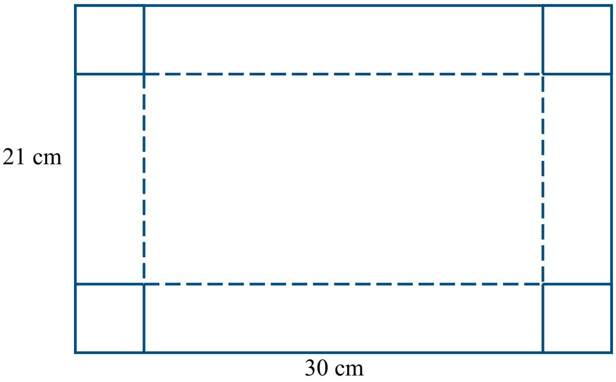
c State an algebraic equation for the relation.

5 **a** What are the independent and dependent variables used in the relation?

b State the domain and range of the relation.

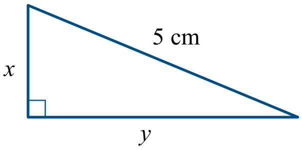
c State an algebraic equation for the relation.



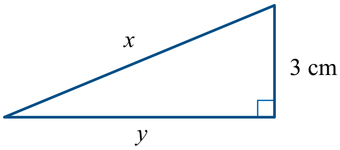


6 Answer the following questions for each triangle:

a



b



i What are the independent and dependent variables used in the relations?

ii State the domain and range of each relation.

iii State an algebraic equation for each relation.

Example 9 **7** **a** What are the independent and dependent variables used in the relation?

b State the domain and range of the relation.

Postage rates: DL (220 × 110 mm)
between 5 and 20 mm thick

Weight	Cost of postage
Up to 50 g	\$1.00
Over 50 g to 125 g	\$1.00
Over 125 g to 250 g	\$1.45
Over 250 g to 500 g	\$2.45

8 **a** What are the independent and dependent variables used in the relation?

b State the domain and range of the relation.

Rates for short-term parking
at domestic terminal

Hours	Fee
0.5	\$5.00
1	\$8.00
2	\$10.00
3	\$12.00
4	\$14.00
5	Plus \$2 per hour up to maximum \$36 per day

Example 10 **9** Classify each of the variables in questions **1–8** as being either discrete or continuous.

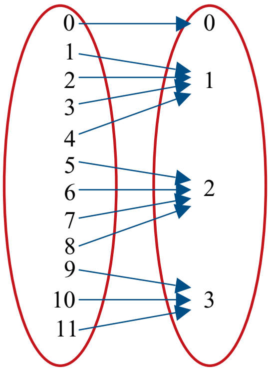
4.4 Relations and functions

For the purpose of this course, a **relation** is a set of ordered pairs; and a **function** is a relation for which no two values of the independent variable are the same. Three consequences of this are:

- In a list of ordered pairs of a function, there will be no repeated x values.
- In a mapping of a function, an element of the domain has only one arrow starting from it.
- Any **vertical line** must not cross the graph more than once.

Example 11

The mapping of the U16 soccer team problem is shown.
Does the mapping represent a function?



Solution

Yes, because each element of the domain has only one arrow starting from it.

Note: When you know how many students need taxis, it is clear how many taxis to order.

Example 12

Classify each of these equations as being either ‘functions’ or ‘not’.

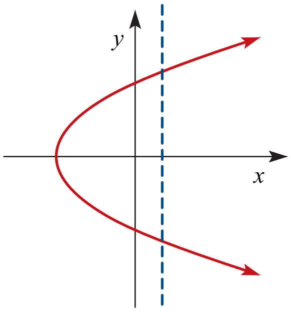
a $4x - 2y = 5$

b $y^2 = x + 3$ (Note that these are the equations from Example 2.)

Solution

a Yes, $4x - 2y = 5$ is a function because no vertical line crosses the graph of it more than once.

b No, $y^2 = x + 3$ is not a function because it fails the vertical line test. The line $x = 1$ crosses the graph twice (shown right).



Exercise 4D

1 Classify each of these relations as being either ‘functions’ or ‘not’.

a

x	y
1	3
2	1
4	4
6	6
7	8

b

x	y
1	1
2	1
4	2
6	2
7	3

c

x	y
1	3
1	1
2	4
2	6
3	8

d

x	y
3	4
4	3
4	5
5	6
5	8

e

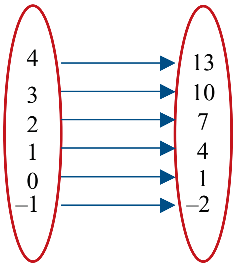
x	y
4	3
3	4
5	4
6	5
8	5

f

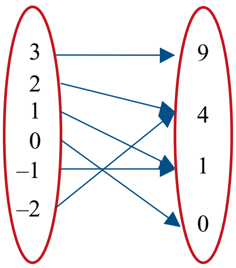
x	y
0	0
1	1
2	2
3	3
4	4

Example 11 **2** Classify each of these relations as being either ‘functions’ or ‘not’.

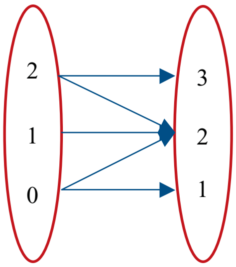
a



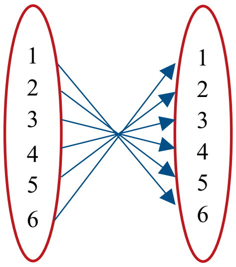
b



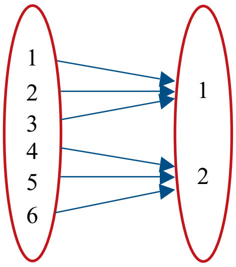
c



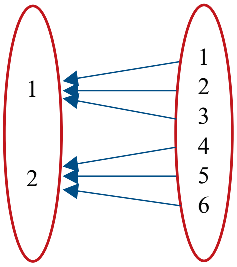
d



e



f



Example 12 **3** Classify each of these relations as being either ‘functions’ or ‘not’.

a $y = 5x + 3$

b $y = x^2 + 3$

c $2y - 3x + 6 = 0$

d $x^2 + y^2 = 36$

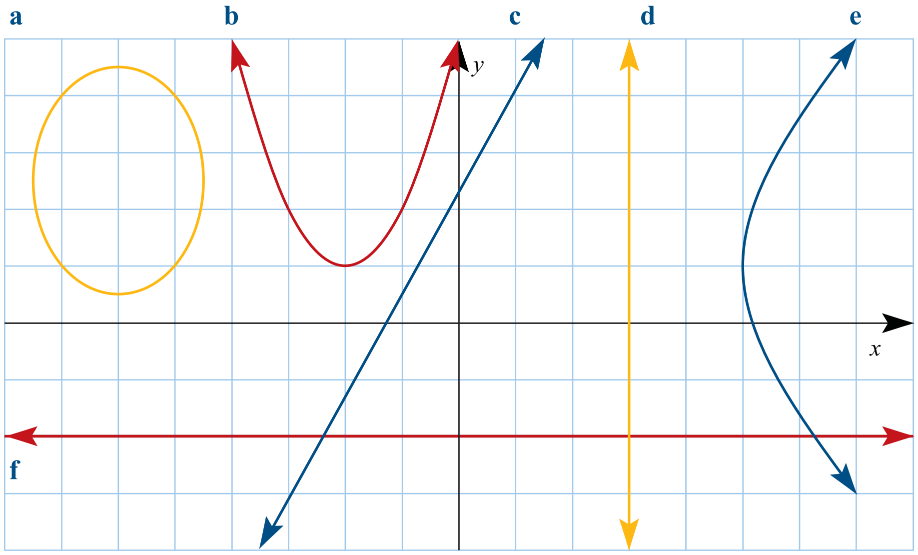
e $y = 6$

f $x^2 - y^2 = 9$

g $y = \sqrt{x^2 - 4}$

h $x = 3$

4 Classify each of these relations as being either 'functions' or 'not'.



4.5 More terminology and function notation

A **continuous function** is best described as one whose graph can be drawn without removing your pencil from the paper.

A **discontinuous function** is a function that is not continuous.

A **discrete function** is a function for which the ordered pairs can be listed in order of the *x* values. They will always appear in a graph as separate dots. Discrete functions are necessarily discontinuous.

Example 13

Classify each of the functions in Examples 1–6 as being either continuous, discontinuous or discrete.

Solution

Continuous	Discontinuous	Discrete
Example 1a,b	Example 3	
Example 2a,b	Example 6 AAPT phone charges	Example 4 Taxi problem
Example 5a,b		

Function notation

In the equation $y = 3x - 5$, the y value is determined by the x value. Said another way, y is a function of x.

The notation $f(x)$, which is read as 'f of x', is used to show this; that is, $f(x) = 3x - 5$ is used instead of $y = 3x - 5$.

Hence, $f(2)$ is the y value obtained when $x = 2$.

Example 14

For $f(x) = 3x + 1$ and $g(x) = x^2 - 1$

a Evaluate: **i** $f(4)$ **ii** $g(-2)$

b Simplify: **i** $f(a - 2)$ **ii** $g(b + 3)$

c Solve: **i** $f(x) = 22$ **ii** $g(x) = 24$

Solution

a **i**
$$\begin{aligned} f(4) &= 3(4) + 1 \\ &= 13 \end{aligned}$$

ii
$$\begin{aligned} g(-2) &= (-2)^2 - 1 \\ &= 3 \end{aligned}$$

b **i**
$$\begin{aligned} f(a - 2) &= 3(a - 2) + 1 \\ &= 3a - 6 + 1 \\ &= 3a - 5 \end{aligned}$$

ii
$$\begin{aligned} g(b + 3) &= (b + 3)^2 - 1 \\ &= b^2 + 6b + 9 - 1 \\ &= b^2 + 6b + 8 \end{aligned}$$

c **i**
$$\begin{aligned} 3x + 1 &= 22 \quad (-\,1) \\ 3x &= 21 \quad (\div 3) \\ x &= 7 \end{aligned}$$

ii
$$\begin{aligned} x^2 - 1 &= 24 \quad (+1) \\ x^2 &= 25 \quad (\sqrt{\ }) \\ x &= \pm 5 \end{aligned}$$

Example 15

For $g(x) = 2 - 4x$ and $h(x) = x^2 + 1$

a Draw neat and accurate graphs (where 2 cm = 1 unit) of the following functions on the same set of axes:

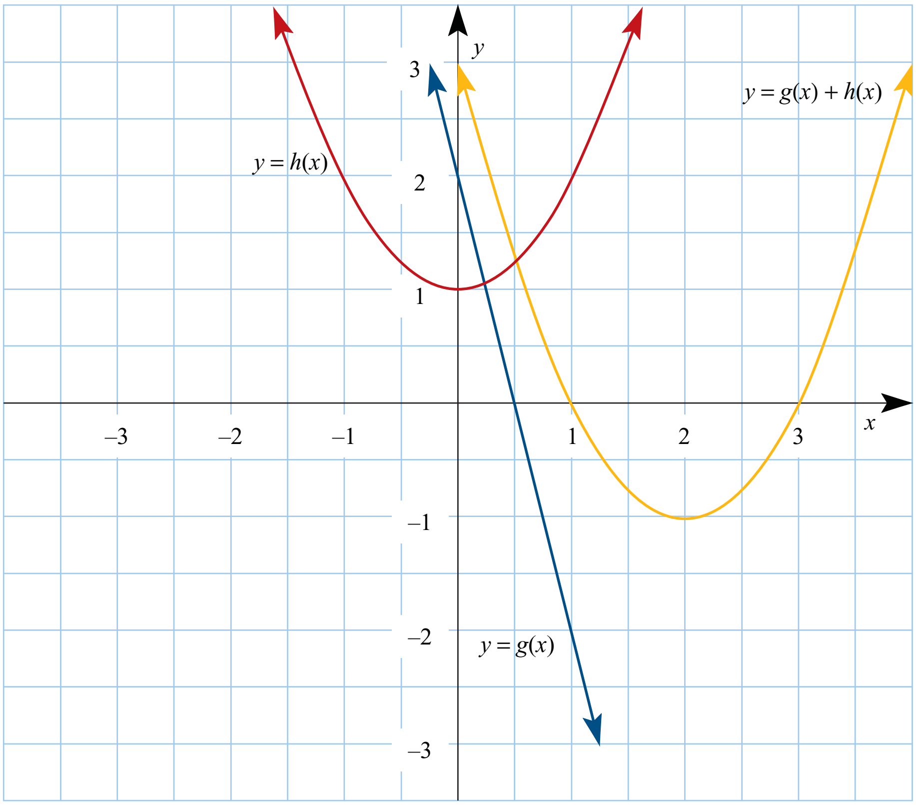
i $y = g(x)$ **ii** $y = h(x)$ **iii** $y = g(x) + h(x)$

b State the domain and range of:

i $y = g(x)$ **ii** $y = h(x)$ **iii** $y = g(x) + h(x)$

Solution

a



b **i** The domain is all real numbers. The range is all real numbers.

ii The domain is all real numbers. The range is $y \geq 1$.

iii The domain is all real numbers. The range is $y \geq -1$.

Exercise 4E

Example 13 **1** Classify each of the functions in Exercise **4B** as being either continuous, discontinuous or discrete.

Example 14 **2** For $f(x) = 2x - 3$ and $g(x) = x^2 + 6$

a Evaluate: **i** $f(5)$ **ii** $g(3)$ **iii** $f(0)$ **iv** $g(-2)$

b Simplify: **i** $f(a + 4)$ **ii** $g(a + 1)$ **iii** $f(1 - 3b)$ **iv** $g(b - 3)$

c Solve: **i** $f(x) = 16$ **ii** $g(x) = 20$

3 For $h(x) = 4 - 3x$ and $f(x) = 2x^2 - 3$

a Evaluate: **i** $f(3)$ **ii** $h(2)$
iii $f(0)$ **iv** $h(-3)$
b Simplify: **i** $h(b - 1)$ **ii** $f(a + 2)$
iii $h(2a + 3)$ **iv** $f(2b + 3)$
c Solve: **i** $h(x) = -3$ **ii** $f(x) = 47$

4 For $g(x) = 5x - x^2$ and $h(x) = 4$

a Evaluate: **i** $g(-3)$ **ii** $h(2)$
b Simplify: **i** $g(a + h)$ **ii** $h(b + 3)$
c Solve: **i** $h(x) = 0$ **ii** $g(x) = h\,(x)$

Example 15 **5** For $f(x) = 3x$ and $g(x) = x^2 - 4$

a Sketch: **i** $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$
b State the domain and range of each of the functions in part **a**.

6 For $f(x) = 2x - x^2$ and $h(x) = 3$

a Sketch: **i** $y = f(x) + h(x)$ **ii** $y = h(x) - f(x)$
b State the domain and range of each of the functions in part **a**.

4.6 Maximal and restricted domain

Maximal domain

When the domain of a relation is not stated explicitly, it is assumed to consist of all real numbers for which the rule has a meaning. We refer to the **implied (maximal)** domain of a relation because the domain is implied by the rule.

Many of the functions studied in this course have a maximal domain of all real numbers. The exceptions studied so far are those involving:

- square root
- fractions with x in the denominator
- log functions

Example 16

State the maximal domain and find the corresponding range of:

a $y = \sqrt{x - 3}$

b $y = \dfrac{1}{2x - 5}$

c $y = \log_2 (x + 4)$

d $y = \dfrac{2x - 3}{5}$

Solution

a $x - 3 \geq 0$

$\therefore$ Domain: $x \geq 3$

Range: $y \geq 0$

b $2x - 5 \neq 0$

$\therefore$ Domain: $x \neq \dfrac{5}{2}$

Range: $y \neq 0$

c $x + 4 > 0$

$\therefore$ Domain: $x > -4$

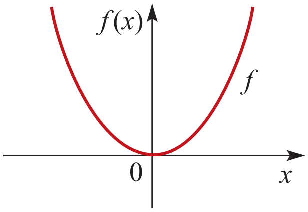
Range: All real numbers

d Domain: All real numbers

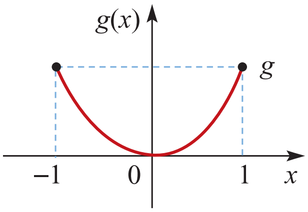
Range: All real numbers

Restriction of a function

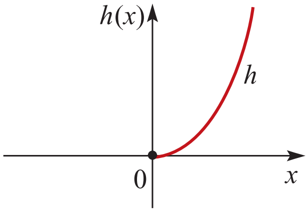
Consider the functions:



$f(x) = x^2$



$g(x) = x^2, -1 \leq x \leq 1$



$h(x) = x^2, x \geq 0$

The different letters, f, g and h, which are used to name the functions, emphasise the fact that there are three different functions even though they each have the same rule. They are different because they are defined for different domains. We call g and h restrictions of f because their domains are subsets of the domain of f.

Example 17

Sketch each of the following on the Cartesian plane:

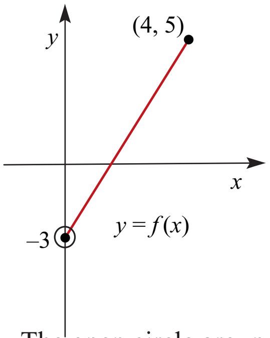
a $f(x) = 2x - 3 \qquad 0 < x \leq 4$

b $g(x) = 2x - x^2 \qquad x \geq 0$

Solution

a $f(0) = -3$

$f(4) = 5$



Note: The open circle around $(0, -3)$ above shows that this point is not included in the graph.

b $0 = 2x - x^2$

$= x(2 - x)$

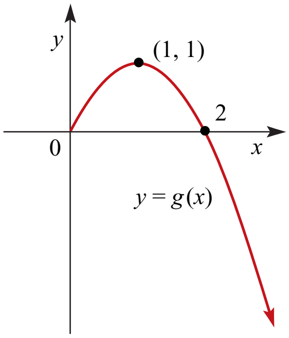
Null factor

$x = 0$ or $2 - x = 0$

$x = 0$ or $x = 2$

$\therefore$ x-intercepts are $x = 0$ and $x = 2$.

Turning point is (1, 1).



Exercise 4F

Example 16 **1** For each of the following, find the maximal domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$ **b** $y = 2\sqrt{x}$ **c** $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$ **e** $y = \dfrac{1}{\sqrt{x}}$ **f** $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$ **h** $y = \log_3(x + 1)$ **i** $y = \sqrt{3 - 2x}$

j $y = \dfrac{1}{x + 2}$ **k** $y = \log_4(x - 5) - 3$ **l** $y = \dfrac{1}{2x - 1}$

m $y = \dfrac{3}{x + 2} - 4$ **n** $y = \dfrac{1}{\sqrt{x - 3}}$ **o** $y = \dfrac{2}{\sqrt{x + 1}}$

Example 17 **2** Sketch each of the following:

a $f(x) = 3x - 2, \quad x \geq 0$ **b** $g(x) = x^2 + x, \quad -1 \leq x \leq 1$

c $h(x) = 3 - 2x, \quad x > -1$ **d** $p(x) = 4, \quad -2 < x < 3$

e $q(x) = \sqrt{4 - x}, \quad x \geq -2$ **f** $r(x) = \sqrt{4 - x^2}, \quad x < 0$

g $v(x) = 2^x + 1, \quad x \geq 0$ **h** $w(x) = \log_{10} x, \quad x > 1$

4.7 Hybrid functions

Functions that have different rules for different subsets of the domain are called **hybrid functions**.

Example 18

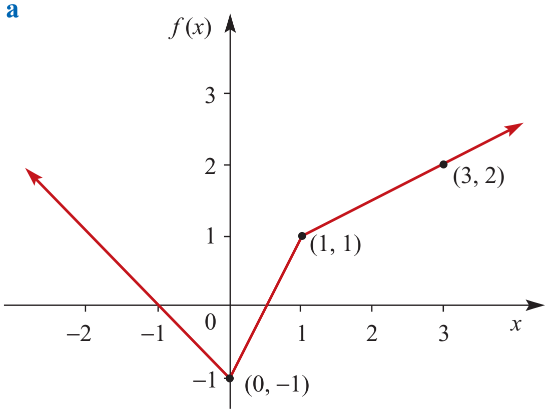
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x \ge 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $y \ge -1$.

Exercise 4G

Example 18 **1** Sketch the graph of each of the following functions and state its range:

a $h(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$ **b** $h(x) = \begin{cases} x - 1, & x \ge 1 \\ 1 - x, & x < 1 \end{cases}$ **c** $h(x) = \begin{cases} -x, & x \ge 0 \\ x, & x < 0 \end{cases}$

d $h(x) = \begin{cases} 1 + x, & x \ge 0 \\ 1 - x, & x < 0 \end{cases}$ **e** $h(x) = \begin{cases} x, & x \ge 1 \\ 2 - x, & x < 1 \end{cases}$

2 **a** Sketch the graph of the function:

$$f(x) = \begin{cases} \frac{2}{3}x + 3, & x < 0 \\ x + 3, & 0 \le x \le 1 \\ -2x + 6, & x > 1 \end{cases}$$

b What is the range of f?

3 Sketch the graph of the function.

$$g(x) = \begin{cases} -x - 3, & x < 1 \\ x - 5, & 1 \leq x \leq 5 \\ 3x - 15, & x > 5 \end{cases}$$

4 a Sketch the graph of the function

$$h(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$$

b State the range of h.

5 a Sketch the graph of the function

$$f(x) = \begin{cases} x + 3, & x < -3 \\ x^2 - 9, & -3 \leq x \leq 3 \\ x - 3, & x > 3 \end{cases}$$

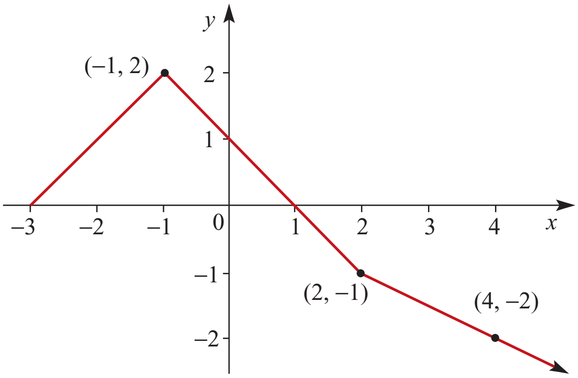
b State the range of f.

6 a Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ x, & x \leq 1 \end{cases}$$

b State the range of f.

7 Specify the function represented by this graph:



4.8 Inverse functions

The function $y = \frac{9}{5}x + 32$ can be used to convert degrees Celsius to degrees Fahrenheit. The reverse, or **inverse** function, $y = \frac{5(x - 32)}{9}$, can be used to convert degrees Fahrenheit back to degrees Celsius.

As a function is a set of ordered pairs, the inverse function is the same set of pairs written in reverse order. In the Celsius/Fahrenheit example, (0, 32) and (100, 212) are both ordered pairs of the function $y = \frac{9}{5}x + 32$. And (32, 0) and (212, 100) are ordered pairs of the inverse function $y = \frac{5(x - 32)}{9}$.

The notation used for inverse functions is $f^{-1}(x)$ and should be read 'inverse f of x'. In this example, $f(x) = \frac{9}{5}x + 32$ and $f^{-1}(x) = \frac{5(x - 32)}{9}$.

Note: The –1 in $f^{-1}(x)$ is not an index. Although $4^{-1} = \frac{1}{4}$, $f^{-1}(x) \neq \frac{1}{f(x)}$ in the vast majority of cases.

Example 19

Find the inverse function of:

a $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$

b $y = 4x + 8$

Solution

a $\{(2, 1), (4, 2), (6, 3), (8, 4)\}$

b $y = 4x + 8$

Inverse function is

$$\begin{aligned} x &= 4y + 8 && (-8) \\ x - 8 &= 4y && (\div 4) \\ \frac{1}{4}x - 2 &= y \end{aligned}$$

i.e. $y = \frac{1}{4}x - 2$ is the inverse function of $y = 4x + 8$.

Example 20

For the function $f(x) = 2x - 3, \quad x \geq 0$

a Find $f^{-1}(x)$.

b Sketch the graph of both $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

c State the range of $y = f^{-1}(x)$.

Solution

a $f(x) = 2x - 3, \quad x \geq 0$

$y = 2x - 3, \quad x \geq 0$

Inverse function is

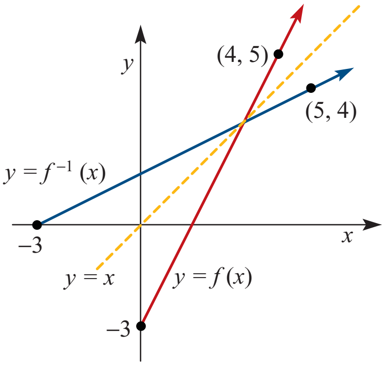
$x = 2y - 3, \quad y \geq 0$

$x + 3 = 2y$

$\frac{1}{2}x + \frac{3}{2} = y, \quad x \geq -3$

$\therefore f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}, \quad x \geq -3$

b



c Range: $y \geq 0$

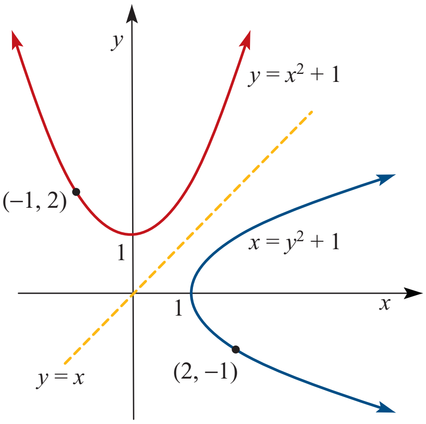
Note: It is necessary to state the domain of $f^{-1}(x)$ in part **a** because the restricted domain is part of it.

Note: The graph of the inverse function is always the reflection of the original function about the line $y = x$.

Restricting the range

For many functions the inverse 'function' does not exist. Although $y = x^2 + 1$ is a function, swapping the domain and range creates a relation that is not a function. Although the graph of $y = x^2 + 1$ passes the vertical line test, the graph of $x = y^2 + 1$ does not.

Therefore, it is necessary to restrict the range in some way to find an inverse function of at least part of the original function.



Example 21

Find an inverse function of:

a $y = x^2 + 1$

b $f(x) = \sqrt{4 - x^2}$

Solution

a $y = x^2 + 1$

Inverse function is

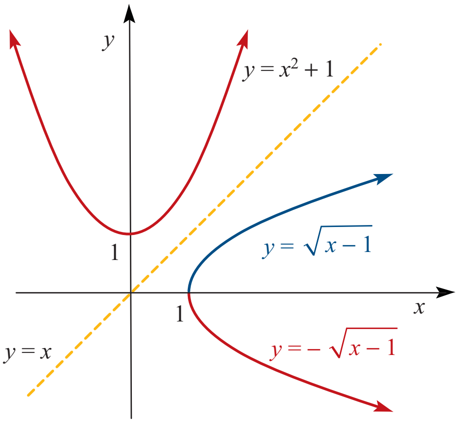
$$\begin{aligned} x &= y^2 + 1 && (-\,1) \\ x - 1 &= y^2 && (\sqrt{\ }) \\ \pm\sqrt{x - 1} &= y \end{aligned}$$

$y = \pm\sqrt{x - 1}$ is not a function.

However,

$y = \sqrt{x - 1}$ and $y = -\sqrt{x - 1}$,

are both possible inverse functions of $y = x^2 + 1$.

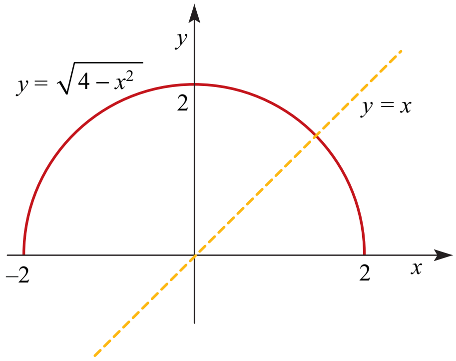


b $f(x) = \sqrt{4 - x^2}$

$y = \sqrt{4 - x^2}$

Inverse function is

$$\begin{aligned} x &= \sqrt{4 - y^2} && \text{(squared)} \\ x^2 &= 4 - y^2 && (+\,y^2) \\ x^2 + y^2 &= 4 && (-\,x^2) \\ y^2 &= 4 - x^2 && (\sqrt{\ }) \\ y &= \pm\sqrt{4 - x^2} \end{aligned}$$



However,

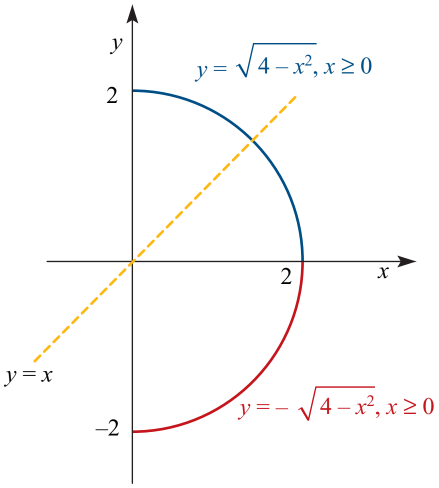
$y = \pm\sqrt{4 - x^2}$ is not a function.

Possible inverse functions are:

$f^{-1}(x) = \sqrt{4 - x^2}, \quad x \geq 0$

or

$f^{-1}(x) = -\sqrt{4 - x^2}, \quad x \geq 0$



Note: It is prudent to sketch both $y = f(x)$ and its reflection about the line $y = x$ in order to ensure the correct domain is chosen for $f^{-1}(x)$.

Exercise 4H

Example 19 **1** Find the inverse function of:

- **a** $\{(1, 3), (-2, 6), (4, 5), (7, 1)\}$
- **b** $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
- **c** $f(x) = 6 - 2x$
- **d** $f(x) = 3x + 3$
- **e** $f(x) = \frac{2}{3}x - 6$
- **f** $f(x) = 2^x$
- **g** $f(x) = 2\log_3 x$
- **h** $f(x) = \sqrt{x + 2}$
- **i** $f(x) = \sqrt{2 - x}$

Example 20 **2** Find the inverse function of each of the following and sketch both $f(x)$ and $f^{-1}(x)$ on the same Cartesian plane:

- **a** $f(x) = 2 - 8x, \quad x \geq 0$
- **b** $f(x) = \frac{1}{2}x + 3, \quad x \leq 0$
- **c** $f(x) = x + 4, \quad x \geq -4$
- **d** $f(x) = 3x - 6, \quad 0 \leq x \leq 2$
- **e** $f(x) = x^2, \quad x \leq 0$
- **f** $f(x) = \begin{cases} 2x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Example 21 **3** Find the inverse function of:

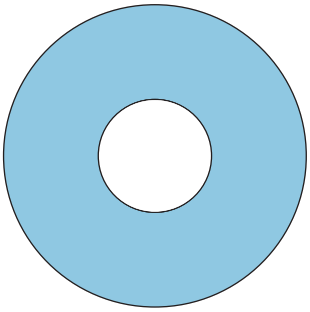
- **a** $f(x) = x^2 - 1$
- **b** $f(x) = x^3 + 1$
- **c** $f(x) = \sqrt{9 - x^2}$
- **d** $f(x) = (x - 3)^2 + 1$

4.9 Modelling life-related situations

It is necessary to restrict the domain in many life-related situations. The formula $C = 2\pi r$ is defined for all real values of r, including zero and negative values. However, in the context of finding the perimeter of a circle a more rigorous rule would be $C(r) = 2\pi r, r > 0$. Note that the domain has to be restricted because circles must have a positive radius.

Example 22

Find the area of the annulus shown as a function of the radius of the outer circle. The diameter of the inner circle is 1 cm.



Solution

For an annulus, $A = \pi R^2 - \pi r^2$.

$$r = \frac{1}{2}, \text{ as diameter} = 1 \qquad \text{(given)}$$
$$A = \pi R^2 - \pi \left(\frac{1}{2}\right)^2$$
$$\therefore A(R) = \pi \left(R^2 - \frac{1}{4}\right), \ R > \frac{1}{2}.$$

Example 23

Find the length of the base of this rectangle as a function of the height, given that the perimeter is 16 cm.

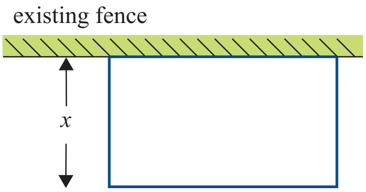
Solution

For a rectangle, $P = 2(b + h)$.
In this case, $P = 16$ and $h = x$.

$$\therefore 16 = 2(b + x)$$
$$8 = b + x$$
$$b = 8 - x$$
$$\therefore b(x) = 8 - x, \quad 0 < x < 8$$

Example 24

A farmer has 100 m of fencing with which to construct three sides of a rectangular paddock connected to an existing fence. If the width of paddock is x m and the area inside the yard is A m^2, find A as a function of x.



Solution

Let the length of the paddock be y.

For a rectangle:

$$A = \text{length} \times \text{width}$$
$$A = xy \ldots\ldots\ldots\ldots (1)$$

Also:

$$P = 2x + y$$
$$100 = 2x + y$$
$$y = 100 - 2x \ldots\ldots\ldots (2)$$

Substituting equation 2 into equation 1:

$$A = x(100 - 2x)$$

Also, x and y are both positive because they are lengths.

$$\therefore x < 50$$
$$\therefore A(x) = x(100 - 2x), \quad 0 < x < 50.$$

Note: The restricted domain is an essential part of the solution of each of Examples 22–24.

Exercise 4I

1 Find the inverse function for:

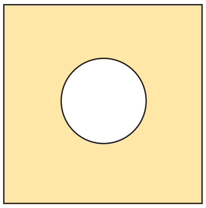
a $f(x) = \sqrt{a^2 - x^2}$ **b** $f(x) = \sqrt{x^2 - a^2}$ **c** $f(x) = \sqrt{x^2 + a^2}$

2 $g(x) = x^2 - 3x + 1$. Simplify $\dfrac{g(a + h) - g(a)}{h}$.

3 $f(x) = a - x$. Solve $f(x) = f^{-1}(x)$ for a.

4 Express the surface area of a sphere as a function of its radius.

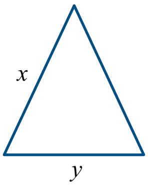
Example 22 **5** A circle is cut from a square piece of paper, with side 8 cm, as shown. Express the area of the remaining paper as a function of the radius of the circle.



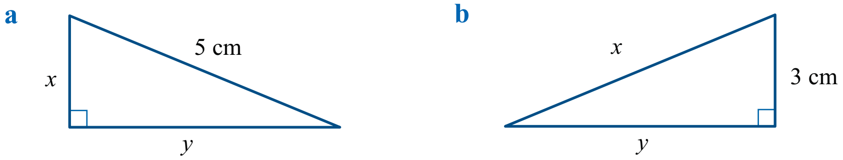
Example 23 **6** **a** Find the area of the rectangle as a function of x if the *perimeter* is 16 cm.
b Find the perimeter of the rectangle as a function of x if the *area* is 20 cm^2.

7 Find the area of a right-angled isosceles triangle as a function of the length of the hypotenuse.

8 Find y as a function of x for which the isosceles triangle shown has a total perimeter of 16 m.

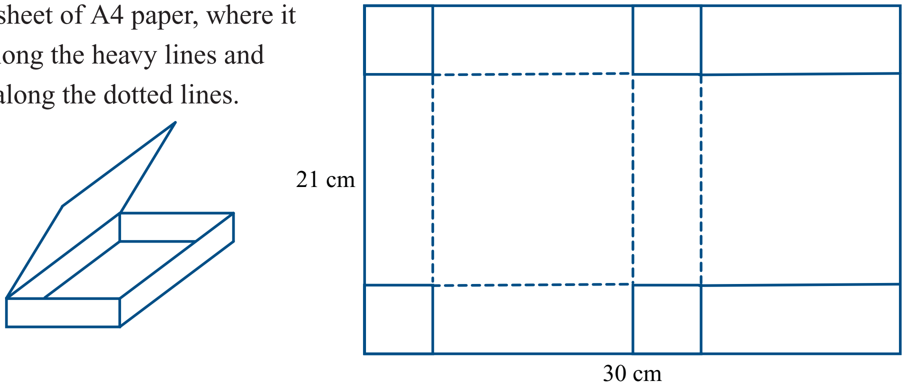


9 Find y as a function of x for the triangle shown.



Example 24 **10** Find the surface area as a function of the radius in a cylinder with a volume of 200π cm^3.

11 A lidded box is to be constructed from a sheet of A4 paper, where it is cut along the heavy lines and folded along the dotted lines.

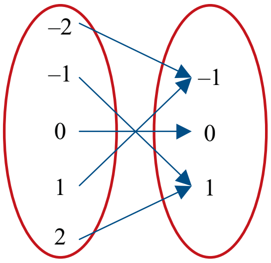


Find ‘Volume of the box’ as a function of ‘Length of each short cut’.

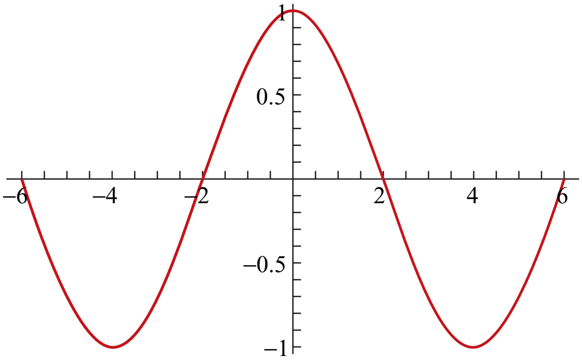
12 An ‘**even function**’ is one in which $f(-a) = f(a)$, for all possible values of a.
An ‘**odd function**’ is one in which $f(-a) = -f(a)$ for all possible values of a.
Classify each of the following functions as being either an ‘**odd function**’ or an ‘**even function**’ or as ‘**neither an odd nor even function**’. Give an appropriate explanation and justification for your classification of each one.

a $f(x) = 7 - 3x^2$

b $g(x)$



c $h(x)$



13 Telstra made the following offers:

Offer A

- Services and equipment at \$11.65 per month; and
- Local calls at 25c per call.

Offer B

- Services and equipment at \$15 per month; and
- Local calls at 20c per call.

Which is the better offer?

Chapter summary

- Draw a **neat and accurate graph** means use graph paper and as many points as necessary to get a smooth curve.
- A **sketch** should have the essential features of the graph being asked for and always enough points to reproduce a graph of that type.
- An **asymptote** or **boundary** line is a dotted line showing the edge of the graph. Although the graph approaches the asymptote, it never touches it.
- Graphing is not always a process of dot-to-dot. The dots are joined when it is possible to have the dots in between as well.
- A **boundary point** (or end point) is shown as a solid or open circle, depending on whether it is included or not.
- A **relation** is a set of ordered pairs.
- The first number in an ordered pair is called the **independent variable**.
- The second number in an ordered pair is the **dependent variable**.
- The **domain** is the set of all possible values of the independent variable.
- The **range** is the set of all possible values of the dependent variable.
- A **mapping** is a diagram showing the links between the **elements** of the domain and the elements of the range.
- A **discrete variable** is a variable for which the elements of the domain can be listed.
- A **continuous variable** is a variable for which there is an infinite number of values of the variable between any two given values.
- A **function** is a relation for which no two values of the independent variable are the same.
- The **vertical line** test is useful when determining if a relation is a function or not.
- A **continuous function** can be graphed without removing your pencil from the paper.
- A **discontinuous function** is a function that is not continuous.
- A **discrete function** is a function for which the ordered pairs can be listed in order of the x values.
- The notation $f(x)$, which is read 'f of x', is an important concept.
- The **implied (maximal)** domain of a relation consists of all real numbers for which the rule has a meaning.
- Functions that have different rules for different subsets of the domain are called **hybrid functions**.
- The **inverse** function is simply the same rule with the domain and range swapped over.
- To find the **inverse** function when the equation is given, swap x for y in the equation and make y the subject. If the resulting equation is not a function it is necessary to restrict the domain so that it is.
- The inverse function of $f(x)$ is $f^{-1}(x)$, which reads 'inverse f of x'.
- It is necessary to restrict the domain in many life-related situations.

Multiple-choice questions

1 Which of the following is **not** a function?

A $y = 9 - x$

B $y = 9 - x^2$

C $y = \sqrt{9 - x^2}$

D $x = 9 - y^2$

E $x = 9 - y$

2 Find the maximal domain and the corresponding range of $y = \sqrt{4 - x^2}$.

A $-4 \le x \le 4,\ -4 \le y \le 4$

B $-2 \le x \le 2,\ -2 \le y \le 2$

C $0 \le x \le 4,\ -4 \le y \le 4$

D $0 \le x \le 2,\ -2 \le y \le 2$

E $-2 \le x \le 2,\ 0 \le y \le 2$

3 Find the inverse function of $y = 2x - 6$.

A $y = 2x + 6$

B $y = \frac{1}{2}x + 3$

C $y = 6x - 2$

D $y = \frac{1}{2}x - 3$

E $y = 3 - \frac{1}{2}x$

4 $f(x) = 6 - 2(x + 7)$. Evaluate $f(-3)$.

A 16

B -2

C 19

D 26

E -14

5 $g(x) = x^2 - 3x + 5$. Simplify $g(a + 2)$.

A $a^2 + a + 3$

B 3

C $a^2 - 3a + 3$

D 7

E $a^2 - 3a + 7$

*Use the following material to answer questions **6** and 7.*

The Australia Post rates table for DL (220 × 110 mm) sized letters, which are between 5 and 20 mm thick, describes the cost of postage as a function of the weight of the envelope.

Postage rates: DL (220 × 110 mm)
between 5 and 20 mm thick

Weight	Cost of Postage
Up to 50 g	\$1.00
Over 50 g to 125 g	\$1.00
Over 125 g to 250 g	\$1.45
Over 250 g to 500 g	\$2.45

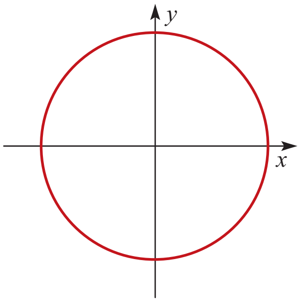
6 Choose the **false** statement from the following:

A The function is also a relation.
B The dependent variable is the cost of postage.
C The domain is $0 < x \leq 500$.
D The function is continuous.
E The range is {\$1.00, \$1.45, \$2.45}.

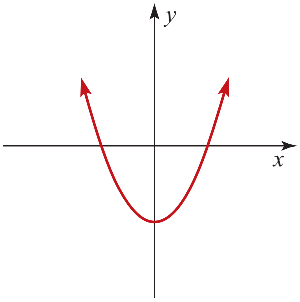
7 Choose the **true** statement from the following:

A Both the independent and dependent variables are continuous.
B The independent variable is discrete and the dependent variable is continuous.
C The dependent variable is discrete and the independent variable is continuous.
D Both the independent and dependent variables are discrete.
E None of the above is true.

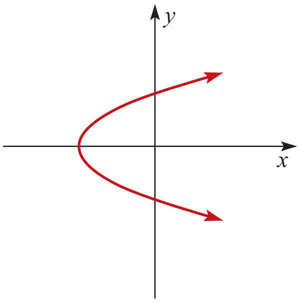
8 **A**



B



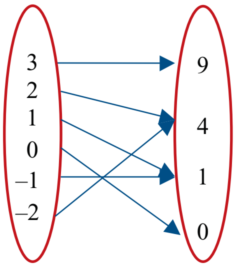
C



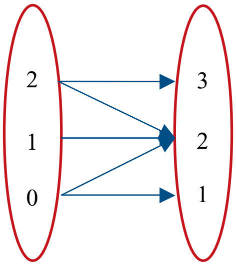
Choose the **true** statement from the following:

A B and C are functions and A is not.
B A and C are functions and B is not.
C A, B and C are all functions.
D A is a function and B and C are not.
E B is a function and A and C are not.

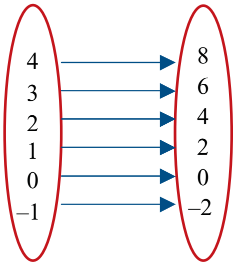
9 **A**



B



C



Choose the **true** statement from the following:

A B and C are functions and A is not.

B A and C are functions and B is not.

C A, B and C are all functions.

D C is a function and A and B are not.

E B is a function and A and C are not.

10 $f(x) = 4 - \frac{3}{5}x$, $g(x) = \sqrt{1 - x^2}$ and $h(x) = \log_3 x$. Choose the **true** statement from the following.

It is necessary to restrict the range when finding an inverse function for:

A both $g(x)$ and $h(x)$

B both $f(x)$ and $h(x)$

C all three functions

D $g(x)$ but not the other two

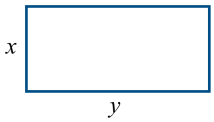
E $h(x)$ but not the other two

Short-response questions

1 Using a 1 cm grid, draw neat and accurate graphs of the following equations on the number plane:

a $y = 4 - 2^x$

b $y^2 - x^2 + 16 = 0$



2 **a** Use the Cartesian plane to show all possible ordered pairs (x, y) for which the rectangle shown has a *perimeter* of 20 cm.

b Use the Cartesian plane to show all possible ordered pairs (x, y) for which the rectangle shown has an *area* of 15 cm^2.

c Use the Cartesian plane to show all possible ordered pairs (x, y) for which the rectangle shown is covered with small 1 cm^2 tiles and has an area of 12 cm^2.

3 For the relations in Question **2**:

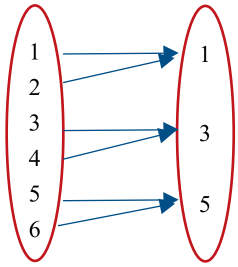
- **i** What are the independent and dependent variables used?
- **ii** State the domain and range of each relation.
- **iii** Draw a mapping diagram for the relation in Question **2c**.
- **iv** State an algebraic equation for each relation.
- **v** Classify each of the variables as being either discrete or continuous.
- **vi** Classify each of the relations as being either continuous, discontinuous or discrete.

4 Classify each of these relations as being either 'functions' or 'not'.

a $4x + 3y = 23$

b $y^2 = x^2 - 8$

c



d

x	y
2	1
2	2
4	3
4	4
6	5

5 For $f(x) = 2x + 5$ and $g(x) = x^2 + 2$

a Evaluate: **i** $f(3)$ **ii** $g(-2)$

b Simplify: **i** $f(a + 3)$ **ii** $g(b - 2)$

c Solve: **i** $f(x) = 20$ **ii** $g(x) = 24$

d Sketch $y = f(x) + g(x)$.

6 State the maximal domain and find the corresponding range of:

a $y = \sqrt{x + 2}$

b $y = \dfrac{3}{4x - 2}$

c $y = \log_4 (x - 8)$

d $y = \dfrac{3x}{4} + 1$

7 Sketch each of the following on the Cartesian plane:

a $f(x) = 3x - 1, \quad 0 \le x < 3$

b $g(x) = 4 - x^2, \quad x \ge -2$

c $$f(x) = \begin{cases} x + 1 & \text{for } x < 0 \\ 1 - 2x & \text{for } 0 \le x \le 2 \\ \frac{1}{2}x - 2 & \text{for } x > 2 \end{cases}$$

8 Find the inverse function of:

a $\{(1, 3), (3, 4), (4, 6), (7, 8)\}$

b $f(x) = 2x + 3$

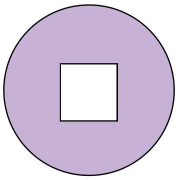
c $g(x) = 3 - 4x, \quad x \geq 0$

9 Find the inverse function of:

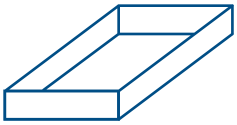
a $y = x^2 - 3$

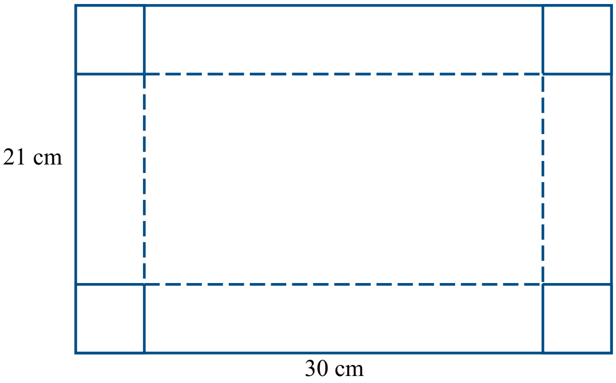
b $f(x) = \sqrt{x^2 - 9}$

10 A square is cut from a circular piece of paper, with a diameter of 10 cm, as shown. Express the area of the remaining paper as a function of the length of the side of the square.



11 A chocolate tray is to be constructed from a sheet of A4 paper, where it is cut along the heavy lines and folded along the dotted lines. Find 'Volume of the tray' as a function of 'Length of each cut'.





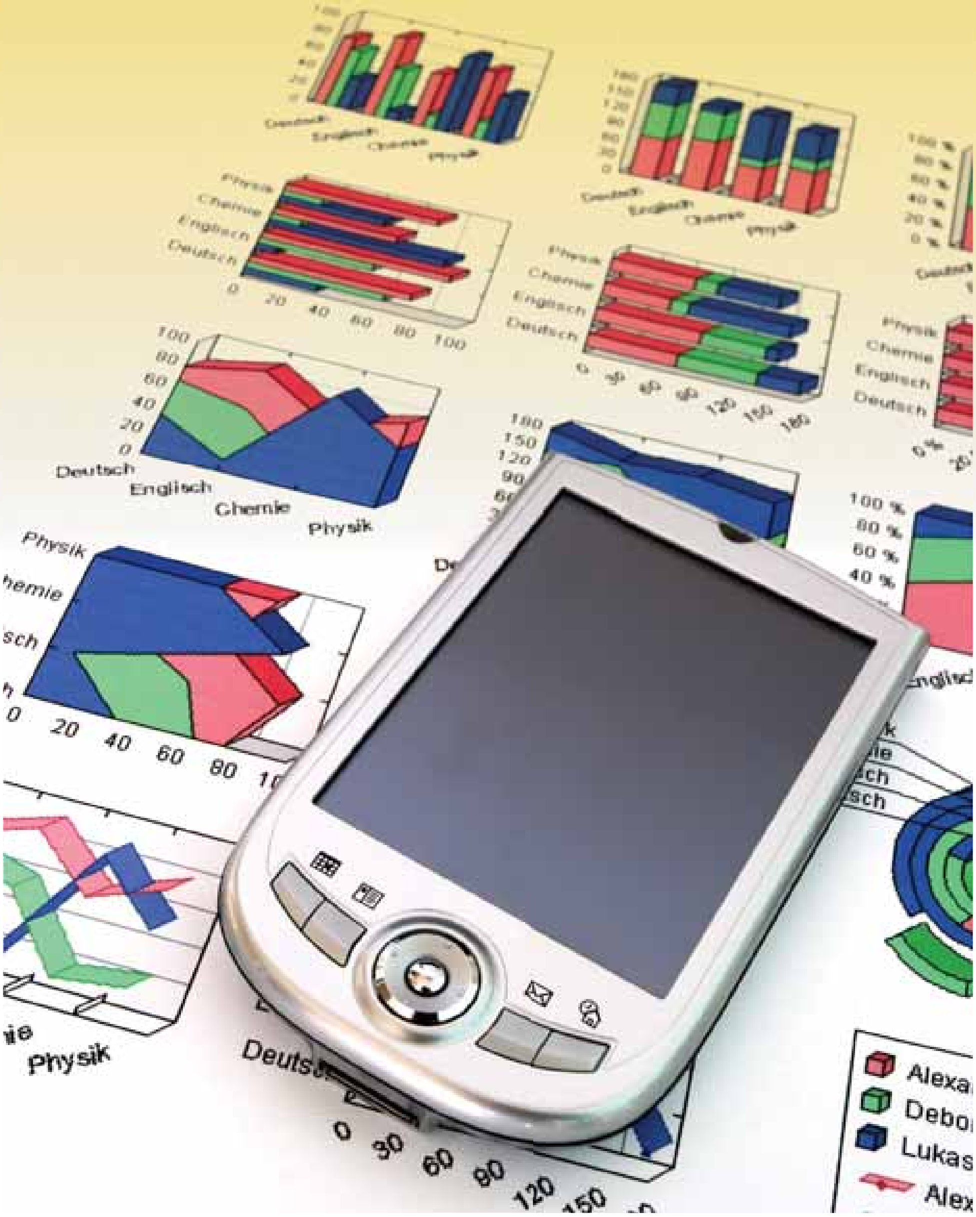
CHAPTER

5

Applied statistical analysis

Objectives

- To introduce the two main types of data – **categorical** and **numerical.**
- To use **bar charts** and **sector graphs** to display **frequency distributions** of categorical data.
- To use **histograms** and **frequency polygons** to display frequency distributions of numerical data.
- To use **cumulative frequency polygons** and **cumulative relative frequency polygons** to display cumulative frequency distributions.
- To use the **stem-and-leaf plot** to display numerical data.
- To use the **histogram** to display numerical data.
- To use these plots to describe the distribution of a numerical variable in terms of **symmetry**, **centre**, **spread** and **outliers**.
- To define and calculate the summary statistics **mean**, **median**, **range**, **interquartile range** and **standard deviation**.
- To understand the properties of these summary statistics and when each is appropriate.
- To construct and interpret **boxplots**, and use them to compare data sets.
- To use scatterplots to display bivariate (numerical) data.
- To identify patterns and features of sets of data from scatterplots.
- To identify positive, negative or no association between variables from a scatterplot.
- To introduce the ***q*-correlation coefficient** to measure the strength of the relationship between two variables.
- To introduce **Pearson's product-moment correlation coefficient *r*** to measure the strength of the linear relationship between two variables.
- To fit a straight line to data by **eye**, and using the **method of least squares.**
- To interpret the **slope** of a regression line and its intercept, if appropriate.
- To **predict** the value of the dependent (response) variable from an independent (explanatory) variable, using a linear equation.



5.1 Types of variables

A characteristic about which information is recorded is called a **variable** because its value is not always the same. Several types of variable can be identified. Consider the following situations.

- Students answer the question: What colour is your hair?
- People are asked to tick one of the following: ‘strongly agree’, ‘agree’, ‘neutral’, ‘disagree’ or ‘strongly disagree’.
- Householders are asked how many newspapers they bought last week.
- Students’ heights are measured.

These situations give rise to four different data types.

1. **Nominal** – Each response is simply a name. There is no order to the possible responses. For example, there is no logical order to the responses ‘black’, ‘blonde’, ‘brown’ etc.

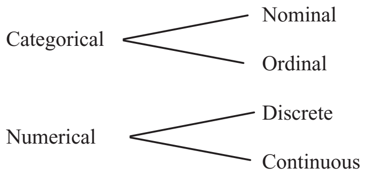
2. **Ordinal** – Each answer is still simply a name, however, the responses can be ordered. The ranking of ‘strongly agree’, ‘agree’, ‘neutral’, disagree’ and ‘strongly disagree’ reflects the change of opinion from one extreme to the other.

3. **Discrete** – Each answer is a number and only certain values are possible. In this case, the numbers 0, 1, 2, 3, . . . are possible responses. Discrete data are usually sourced from counting.

4. **Continuous** – Each answer is a number, although the values can be any number (usually within a particular range of values). Although most students would give their height as a whole number, such as 175 cm, their actual height could be 174.5 cm or 175.49859 . . . Continuous data are usually sourced from measuring.

The data types nominal and ordinal are collectively referred to as **categorical** because the data are classified into categories. The data types discrete and continuous are collectively referred to as **numerical** because the data are always numbers.

This diagram summarises the data classification described above.



Note:

- Shoe sizes come as numbers and so they are often misclassified as being discrete. They are, in fact, ordinal. T-shirts come in ‘small’, ‘medium’, ‘large’ and ‘extra large’. The words small, medium, large and extra large are used simply as a way of indicating the order of sizes. Customers know that if the medium is too tight that they should try the large or the extra large. In the same sense, when purchasing shoes, customers who find the size 7 too small would try a $7\frac{1}{2}$ or an 8. The numbering is simply a way of indicating the order of sizes.

- The amount of money in each student's pocket is discrete data. Although decimals are possible, giving rise to values such as \$3.85 and \$0.20, as well as \$5 and \$10, the amount will never have more than 2 decimal places.

Exercise 5A

1 Classify the data that arise from the following situations as nominal, ordinal, discrete or continuous:

- **a** Kindergarten pupils bring along their favourite toy, and they are grouped together under the headings: 'dolls', 'soft toys', 'games', 'cars' and 'other'.
- **b** The number of students on each of twenty school buses is counted.
- **c** A group of people each write down their favourite colour.
- **d** Each student in a class is weighed in kilograms.
- **e** Each student in a class is weighed and then classified as 'light', 'average' or 'heavy'.
- **f** People rate their enthusiasm for a certain rock group as 'low', 'medium' or 'high'.
- **g** The amount of money traded on the stock exchange each day is recorded.
- **h** The distance jumped by each competitor in the Women's long jump is recorded.

2 Classify the data that arise from the following situations as categorical or numerical:

- **a** The intelligence quotient (IQ) of a group of students is measured using a test.
- **b** A group of people are asked to indicate their attitude to capital punishment by selecting a number from 1 to 5 where $1 =$ strongly disagree, $2 =$ disagree, $3 =$ undecided, $4 =$ agree and $5 =$ strongly agree.
- **c** People are asked to write down their favourite number.
- **d** The total number of buses, 'public' and 'private', crossing the Gate Way Bridge each hour is noted.

3 Classify the following numerical data as either discrete or continuous:

- **a** The number of pages in a book.
- **b** The price paid to fill the tank of a car with petrol.
- **c** The volume of petrol used to fill the tank of a car.
- **d** The time between the arrival of successive customers at an autobank teller.
- **e** The number of tosses of a die required before a six is thrown.

5.2 Displaying categorical data: The bar chart

Nominal

Suppose a group of 130 students was asked to nominate their favourite kind of music under the categories ‘hard rock’, ‘oldies’, ‘classical’, ‘hip hop’, ‘country’ or ‘other’. The table shows the data for the first few students.

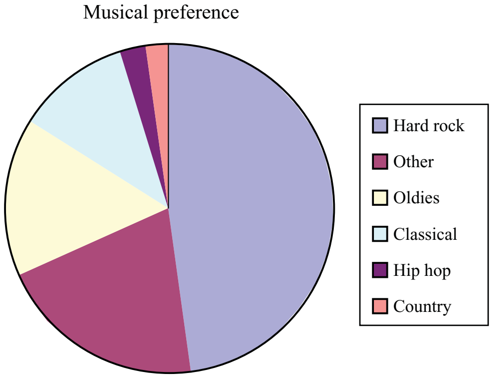
Student’s name	Favourite music
Daniel	hard rock
Karina	classical
John	country
Jodie	hard rock

The table gives data for individual students. To consider the group as a whole the data should be collected into a table called a **frequency distribution** by counting how many of each of the different values of the variable have been observed.

Counting the number of students who responded to the question on favourite kinds of music gave the following results in each category.

Hard rock	Other	Oldies	Classical	Hip hop	Country
62	27	20	15	3	3

Although a clear indication of the group’s preferences can be seen from the table, a visual display may be constructed to illustrate this. When the data are nominal, an appropriate display is a **sector** or **pie graph**.



When constructing a sector graph use the method given below to find the angles.

Hard rock	62	$\frac{62}{130} \times 360 = 172°$
Other	27	$\frac{27}{130} \times 360 = 75°$
Oldies	20	$\frac{20}{130} \times 360 = 55°$
Classical	15	$\frac{15}{130} \times 360 = 42°$
Hip hop	3	$\frac{3}{130} \times 360 = 8°$
Country	3	$\frac{3}{130} \times 360 = 8°$
Total	130	360°

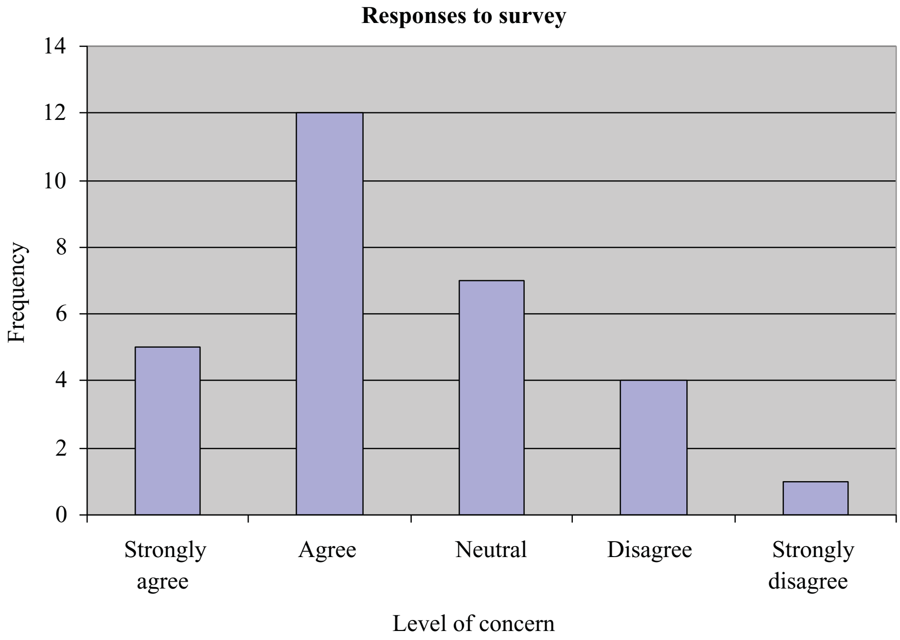
Ordinal

Suppose the responses to a questionnaire are summarised as follows:

Strongly agree	5
Agree	12
Neutral	7
Disagree	4
Strongly disagree	1

Ordinal data are best represented as a bar chart because bar charts highlight the order inherent in the responses.

The data above would appear as:



Note: In both situations described above, the most common or frequent response is clearly identifiable from the graph. 'Hard rock' is clearly the preferred type of music and 'agree' is the most common response in the opinion survey.

The most frequent or common response is called the **mode**.

Note: Although bar charts can be used for nominal data, there is no order inherent in the responses and, so, the bars can occur in an order that suits the purposes of the presenter.

Exercise 5B

1 A group of students was asked to select their favourite type of fast food, with the following results.

a Draw a sector graph for these data.

b Which is the most popular food type?

Food type	Number of students
hamburgers	23
chicken	7
fish and chips	6
Chinese	7
pizza	18
other	8

2 Presented are the responses received to a question regarding the return of capital punishment.

a Draw a bar chart for these data.

b How many respondents either agree or strongly agree?

strongly agree	21
agree	11
don't know	42
disagree	53
strongly disagree	129

3 The local basketball club is buying singlets for their U16 players. Their purchase order is shown in the table.

a Construct an appropriate graph of the data.

b What is the mode of the data?

S	2
M	5
L	12
XL	13
XXL	10

4 The results of a survey of secondary school students' preferred ways of spending their leisure time at home is given.

a Construct an appropriate graph to illustrate these data.

b What is the mode of the data?

watch TV	42%
read	13%
listen to music	23%
watch a video	12%
phone friends	4%
other	6%

5.3 Displaying numerical data: The histogram

Example 1

The numbers of siblings reported by each student in Year 11 at a local school is as follows:

2	3	4	0	3	2	3	0	4	1	0	0	1	2	3
0	2	1	1	4	5	3	2	5	6	1	1	1	0	2
2	3	4	1	1	0	9	0	1	1	1	1	1	0	1

a Construct a frequency distribution of the number of siblings.

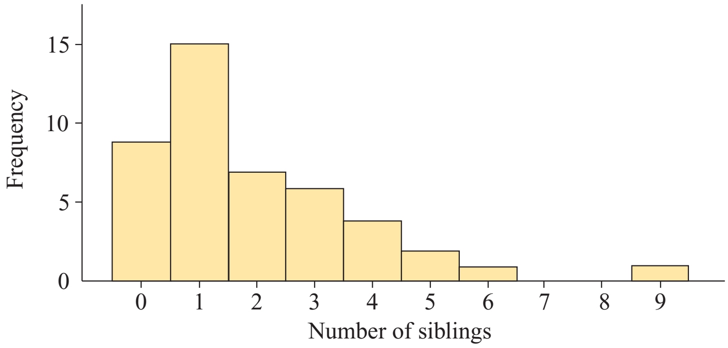
b Construct a frequency histogram of the number of siblings.

Solution

a

Number	0	1	2	3	4	5	6	7	8	9
Frequency	9	15	7	6	4	2	1	0	0	1

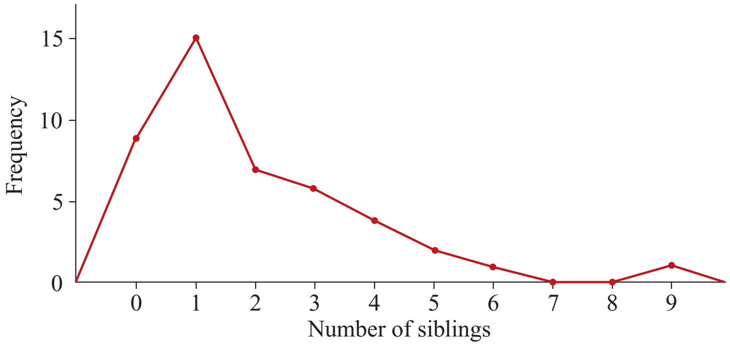
b



Note: To construct the frequency distribution, count the numbers of students corresponding to each of the numbers of siblings, as shown in part **a**.

Note: A **histogram** looks similar to a bar chart. The difference is that the columns all touch each other. For discrete data the actual data values are located at the middle of the appropriate column, as shown above.

An alternative display for a frequency distribution is a **frequency polygon**. It is formed by plotting the values in the frequency table with points, which are then joined by straight lines. A frequency polygon for the data in Example 1 is shown by the red line in this diagram.



When the range of responses is large it is usual to gather the data together into subgroups or **class intervals**. The number of data values corresponding to each class interval is called the **class frequency**.

Class intervals should be chosen according to the following principles:

- Every data value should be in an interval.
- The intervals should not overlap.
- There should be no gaps between the intervals.

The choice of intervals can vary but, generally, a division that results in about 5 to 15 groups is preferred. It is also usual to choose an interval width that is easy for the reader to interpret, such as 10 units, 100 units, 1000 units etc. (depending on the data). By convention, the beginning of the interval is given the appropriate exact value, rather than the end. For example, intervals of 0–49, 50–99, 100–149 would be preferred over the intervals 1–50, 51–100, 101–150 etc.

Example 2

A researcher asked a group of people to record how many cups of coffee they drank in a particular week. Here are the results.

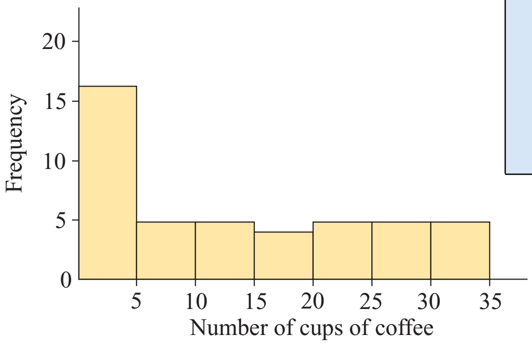
0	0	9	10	23	25	0	0	34	32	0	0	30	0	4
5	0	17	14	3	6	0	33	23	0	32	13	21	22	6
8	19	25	25	0	0	0	2	28	25	14	20	12	17	16

Construct a frequency distribution and, hence, a histogram of these data.

Solution

The range of values is 0–34. An interval width of 5 gives seven intervals.

Number of cups of coffee	Frequency
0–4	16
5–9	5
10–14	5
15–19	4
20–24	5
25–29	5
30–34	5



Example 2 was concerned with a discrete numerical variable. When constructing a frequency distribution of continuous data, the data are again grouped, as shown in Example 3.

Example 3

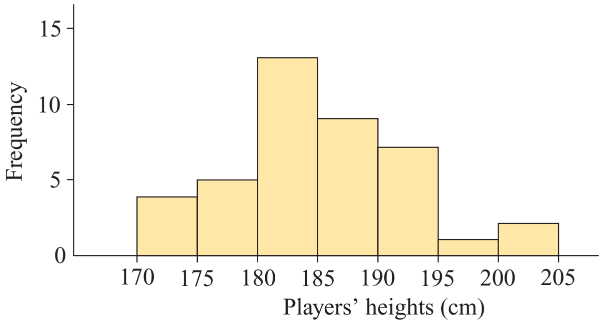
The following are the heights of the players in a basketball club, measured to the nearest millimetre.

178.1	185.6	173.3	193.4	183.1	193.0	188.3	189.5	184.6	202.4	170.9
183.3	180.3	182.0	183.6	184.5	185.8	189.1	178.6	194.7	185.3	188.7
192.4	203.7	191.1	189.7	191.1	180.4	180.0	180.1	170.5	179.3	193.8
196.3	189.6	183.9	177.7	184.1	183.8	174.7	178.9			

Construct a frequency distribution and, hence, a histogram of these data.

Solution

Players' heights (cm)	Frequency
170 –	4
175 –	5
180 –	13
185 –	9
190 –	7
195 –	1
200 –	2



Note: All values of the variable that are 170 or more, but less than 175, have been included in the first interval. The second interval includes values from 175 to less than 180, and so on for the rest of the table.

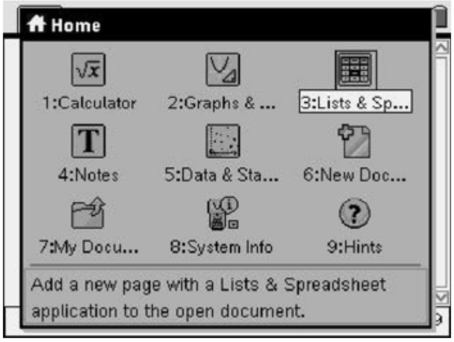
The interval in a frequency distribution that has the highest class frequency is called the **modal class**.

In Example 3, the modal class is 180.0–184.9.

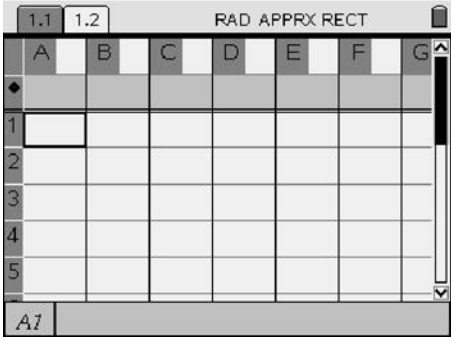
Using technology

Using the TI-Nspire:

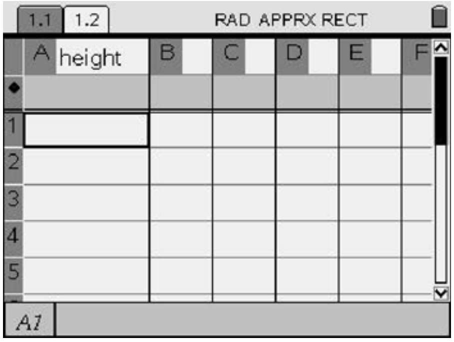
1 To access the Lists press enter on the Lists & Spreadsheet application.



Add a new page with a Lists & Spreadsheet application to the open document.

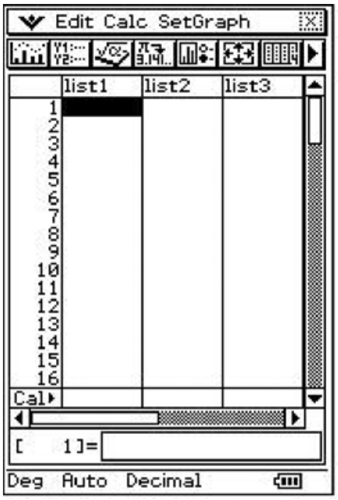


2 Move the cursor to the blank cell next to column A, type **height** then press enter. This has now given column 1 the name 'height'.

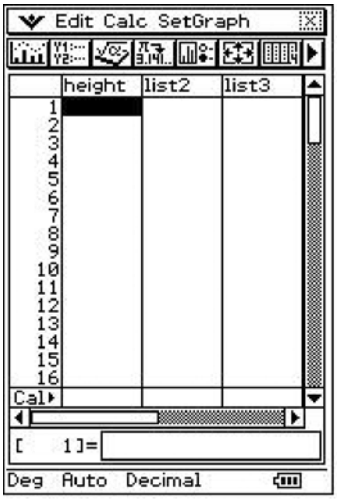


Using the ClassPad:

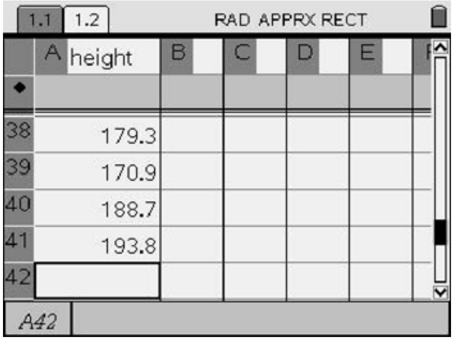
1 To access the Lists application tap Statistics.



2 Rename list 1 and call it 'height'. This is done by tapping on the word 'list1' and then typing **height**, followed by EXE.



3 Enter the data into ‘height’.

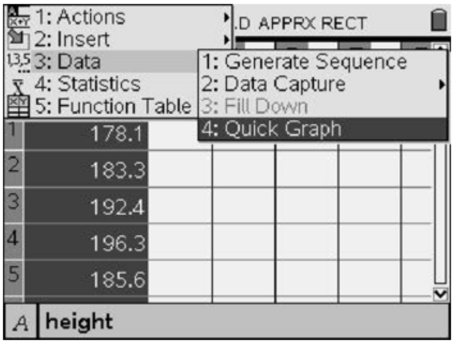


	A height	B	C	D	E	F
◆						
38	179.3					
39	170.9					
40	188.7					
41	193.8					
42						

4 Highlight the first column by pressing the up arrrow until the column is shaded.

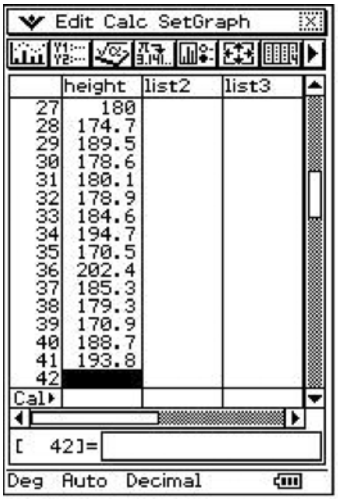
	A height	B	C	D	E	F
◆						
1	178.1					
2	183.3					
3	192.4					
4	196.3					
5	185.6					

5 To draw a histogram for these data, press menu and press enter on *Quick Graph* from the Data submenu.



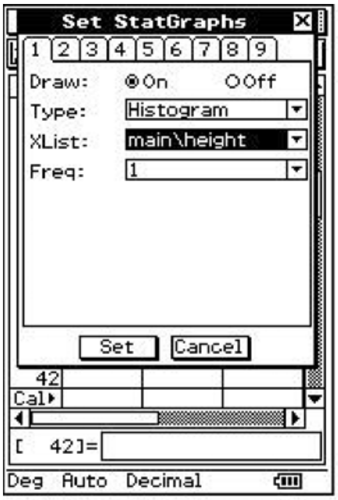
1	178.1
2	183.3
3	192.4
4	196.3
5	185.6

3 Enter the data into ‘height’.



	height	list2	list3
27	180		
28	174.7		
29	189.5		
30	178.6		
31	180.1		
32	178.9		
33	184.6		
34	194.7		
35	170.5		
36	202.4		
37	185.3		
38	179.3		
39	170.9		
40	188.7		
41	193.8		
42			

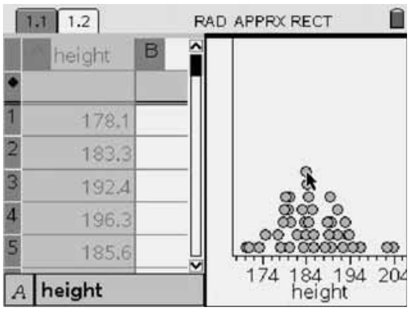
4 To ensure a histogram is drawn for height, tap SetGraph then tap on Setting. . . Change Type to Histogram and change the XList to main/height. Tap SET to save the changes.



Draw: ● On ○ Off
Type: Histogram
XList: main\height
Freq: 1

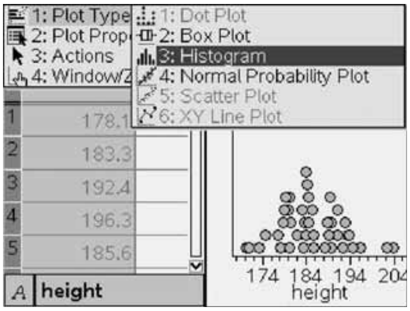
5 To see the histogram, tap 📊.

6 Put HStart: 170 and HStep: 5, and then tap OK.

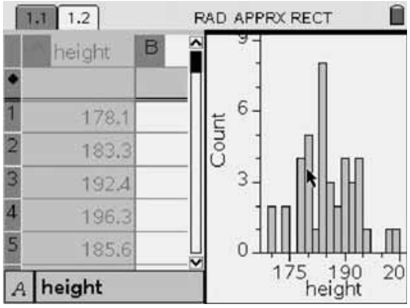


	height
1	178.1
2	183.3
3	192.4
4	196.3
5	185.6

Since the default setting is a dot plot, press menu and select *Histogram* from the Plot Type submenu.

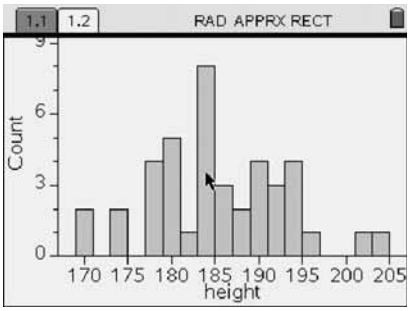


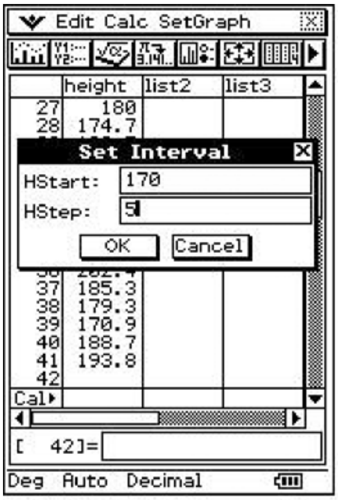
1	178.1
2	183.3
3	192.4
4	196.3
5	185.6



	height
1	178.1
2	183.3
3	192.4
4	196.3
5	185.6

6 For a full-screen view of the graph, press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, ⌂. Scroll to *Page Layout* → *Select Layout* → *1*: *Layout 1*.



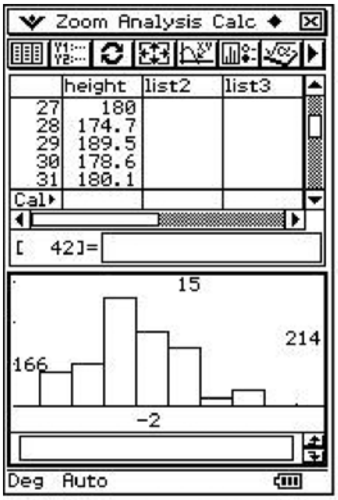


	height
27	180
28	174.7
37	185.3
38	179.3
39	170.9
40	188.7
41	193.8
42	

HStart: 170

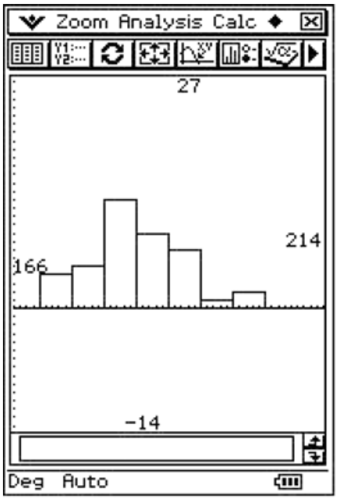
HStep: 5

[42]=

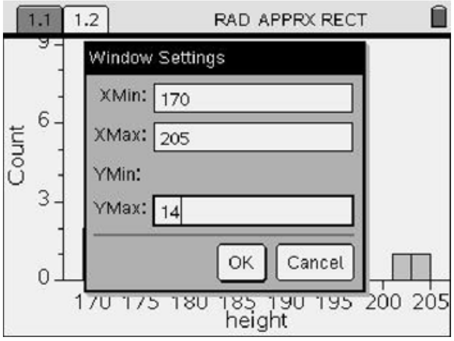


	height
27	180
28	174.7
29	189.5
30	178.6
31	180.1

7 Tap Resize for a full-screen shot of the histogram.



7 Press menu and select *Window Settings* from the Window/Zoom submenu and enter the information below.



Window Settings

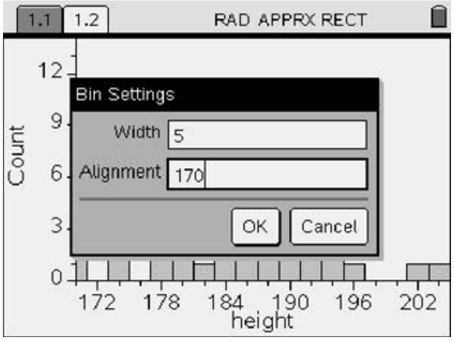
XMin: 170

XMax: 205

YMin:

YMax: 14

8 To change these data values to a more suitable interval width, press menu, then navigate to *Plot Properties* → *Histogram Properties* → *Bin Settings* and enter the information below.

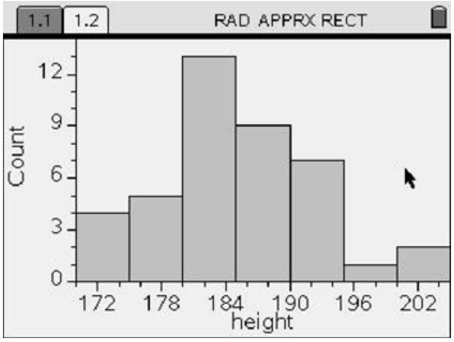


Bin Settings

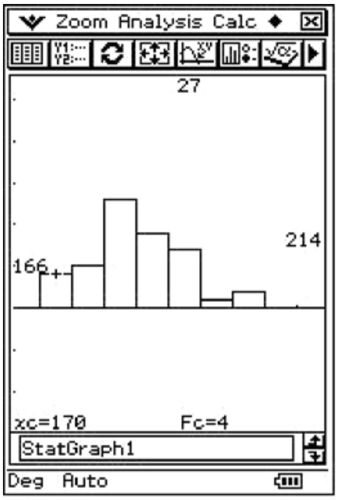
Width 5

Alignment 170

Press enter on OK.

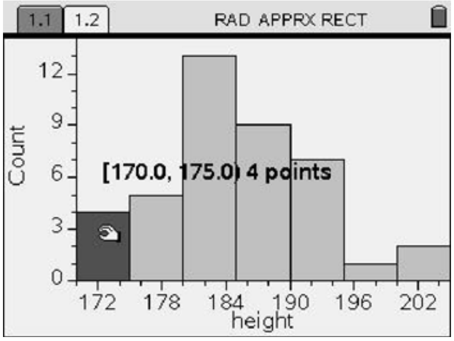


8 Select *Trace* from the Analysis menu to determine the frequencies for each class.



xc=170 Fc=4

9 To determine the frequencies of each class, put the cursor on the bar for the required interval and hold down the button for 3 seconds, then let go.



Relative and percentage frequencies

When frequencies are expressed as a proportion of the total number they are called **relative frequencies**. By expressing the frequencies as relative frequencies more information is obtained about the data set. Multiplying the relative frequencies by 100 readily converts them to **percentage frequencies**, which are easier to interpret.

An example of the calculation of relative and percentage frequencies is shown in Example 4.

Example 4

Construct a relative frequency distribution and a percentage frequency distribution for the players' heights data.

Solution

Players' heights (cm)	Frequency	Relative frequency	Percentage frequency
170 –	4	$\frac{4}{41} \approx 0.10$	10%
175 –	5	$\frac{5}{41} \approx 0.12$	12%
180 –	13	$\frac{13}{41} \approx 0.32$	32%
185 –	9	$\frac{9}{41} \approx 0.22$	22%
190 –	7	$\frac{7}{41} \approx 0.17$	17%
195 –	1	$\frac{1}{41} \approx 0.02$	2%
200 –	2	$\frac{2}{41} \approx 0.05$	5%

Note: From this table it can be seen, for example, that 9 out of 41, or 22% of players, have heights from 185 cm to less than 190 cm.

Both the relative frequency histogram and the percentage frequency histogram are identical to the frequency histogram – only the vertical scale is changed.

Cumulative frequency distribution

To answer questions concerning the number or proportion of the data values that are less than a given value, a **cumulative frequency distribution** or a **cumulative relative frequency distribution** can be constructed. For both a cumulative frequency distribution and a cumulative relative frequency distribution, the number of observations in each class are accumulated from low to high values of the variable.

Example 5

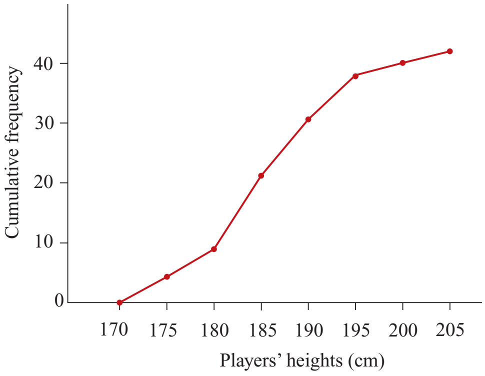
Construct a cumulative frequency distribution and a cumulative relative frequency distribution for the data in Example 4.

Solution

Players' heights (cm)	Frequency	Cumulative frequency	Cumulative relative frequency
<170	0	0	0.00
<175	4	4	0.10
<180	5	9	0.22
<185	13	22	0.54
<190	9	31	0.76
<195	7	38	0.93
<200	1	39	0.95
<205	2	41	1.00

Note: Each cumulative frequency was obtained by adding preceding values of the frequency. In the same way, the cumulative relative frequencies were obtained by adding preceding relative frequencies. Thus, it can be said that a proportion of 0.54, or 54%, of players are less than 185 cm tall.

A graphical representation of a cumulative frequency distribution is called a **cumulative frequency polygon** and has a distinctive appearance, as it always starts at zero and is non-decreasing.



This graph shows, on the vertical axis, the number of players shorter than any height given on the horizontal axis. The cumulative relative frequency distribution could also be plotted as a **cumulative relative frequency polygon**, which would differ from the cumulative frequency polygon only in the scale on the vertical axis, which would run from 0 to 1.

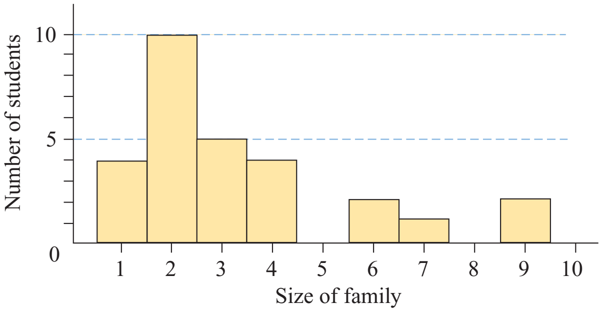
Exercise 5C

Example 1 **1** The number of pets reported by each student in a class is given in the following table:

2	3	4	0	3	2	3	0	4	1	0
0	2	1	1	4	5	3	2	5	6	1

Construct a frequency distribution of the numbers of pets reported by each student.

2 The number of children in the family for each student in a class is shown in this histogram.



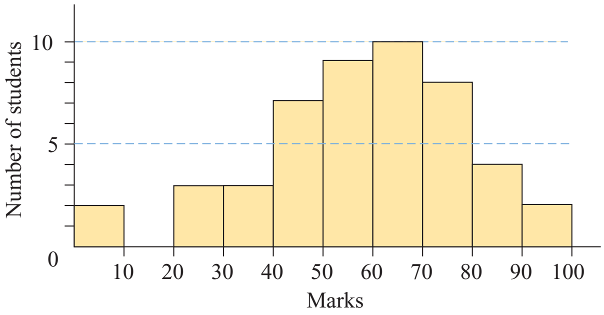
a How many students are the only child in a family?

b What is the mode of the data?

c How many students come from families with six or more children?

d How many students are there in the class?

3 This histogram gives the scores on a general knowledge quiz for a class of Year 11 students.



a How many students scored 10–19 marks?

b How many students attempted the quiz? **c** What is the modal class?

d If a mark of 50 or more is designated as a pass, how many students passed the quiz?

4 The maximum temperatures for several capital cities around the world on a particular day, in degrees Celsius, were:

17	26	36	32	17	12	32	2
16	15	18	25	30	23	33	33
17	23	28	36	45	17	19	37
31	19	25	22	24	29	32	38

Example 2 **a** Construct a frequency distribution for these data.

Example 4 **b** Construct the corresponding relative frequency distribution.

c Draw a histogram from the frequency distribution.

d What percentage of cities had a maximum temperature of less than 25°C?

5 A student purchases 21 new textbooks from a school book supplier, with the following prices (in dollars):

21.65	14.95	12.80	7.95	32.50	23.99	23.99
7.80	3.50	7.99	42.98	18.50	19.95	3.20
8.90	17.15	4.55	21.95	7.60	5.99	14.50

Example 3 **a** Draw a histogram of these data, using appropriate class intervals.

b What is the modal class?

Example 5 **c** Construct a cumulative frequency distribution for these data and draw the cumulative frequency polygon.

6 A group of students was asked to draw a line which they estimated to be the same length as a 30 cm ruler. The lines were then measured (in cm), giving the following results:

30.3	30.9	31.2	32.3	31.3	30.7	32.8	31.0	33.3	30.7
32.2	30.1	31.6	32.1	31.4	31.8	32.9	31.9	29.4	31.6
32.1	31.2	30.7	32.1	30.8	29.7	30.1	28.9		

a Construct a histogram of the frequency distribution.

b Construct a cumulative frequency distribution for these data and draw the cumulative frequency polygon.

c Write a sentence to describe the students' performance on this task.

7 Given are the marks obtained by a group of Year 11 Chemistry students on the end-of-year exam.

21	49	58	68	72	31	49	59	68	72
33	52	59	68	82	47	52	59	70	91
47	52	63	71	92	48	53	65	71	99

a Using a graphics calculator, or otherwise, construct a histogram of the frequency distribution.

b Construct a cumulative frequency distribution for these data and draw the cumulative frequency polygon.

c Write a sentence to describe the students' performance on this exam.

8 The following 50 values are the lengths (in metres) of some par 4 golf holes from Brisbane golf courses.

302	272	311	351	338	325	314	307	336	310
371	334	369	334	320	374	364	353	366	260
376	332	338	320	321	364	317	362	310	280
366	361	299	321	361	312	305	408	245	279
398	407	337	371	266	354	331	409	385	260

a Construct a histogram of the frequency distribution.

b Construct a cumulative frequency distribution for these data and draw the cumulative frequency polygon.

c Use the cumulative frequency polygon to estimate:

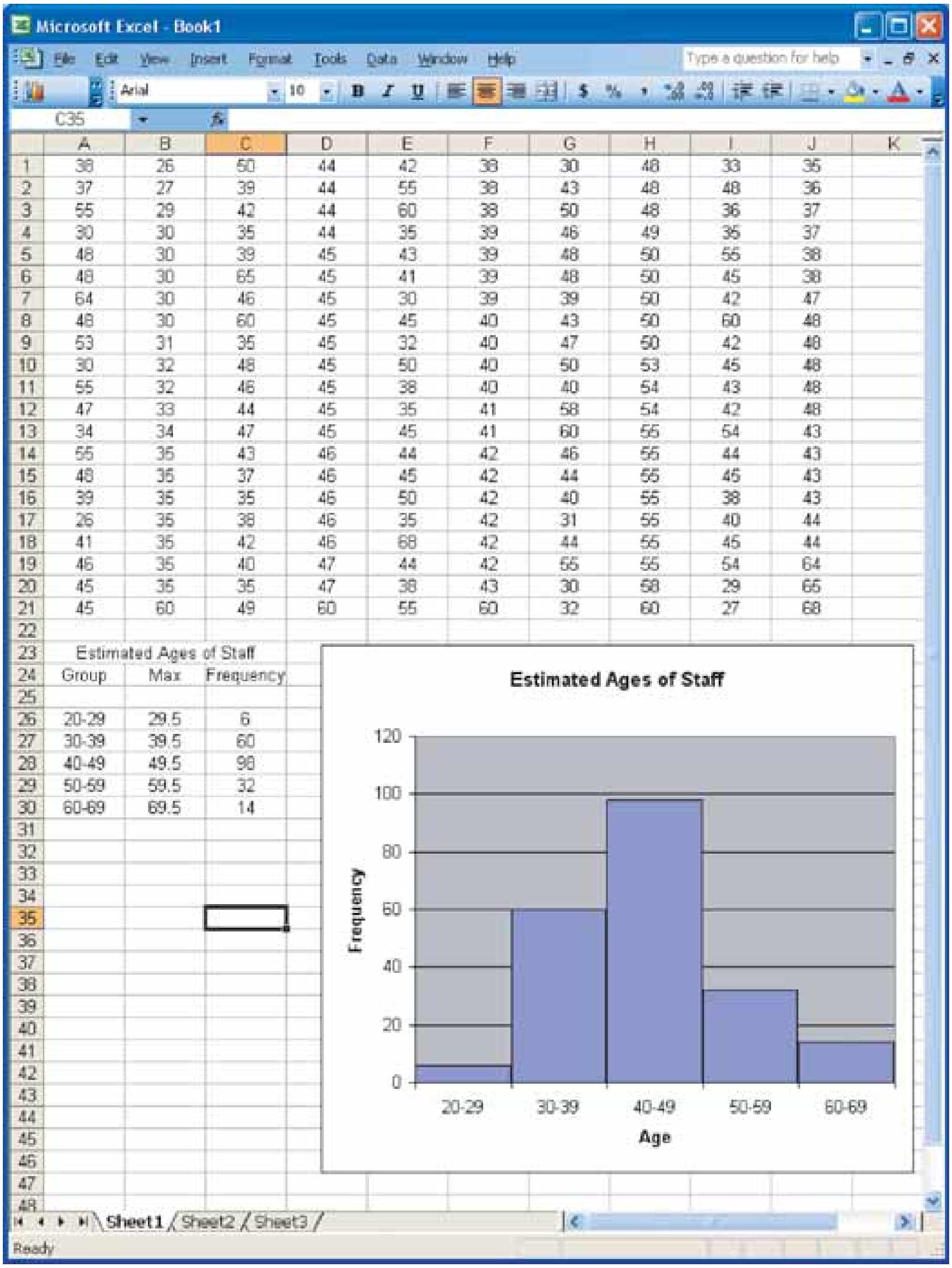
  i the proportion of par 4 holes less than 300 m in length

  ii the proportion of par 4 holes 360 m or more in length

  iii the length that is exceeded by 90% of the par 4 holes

Using Excel to draw graphs

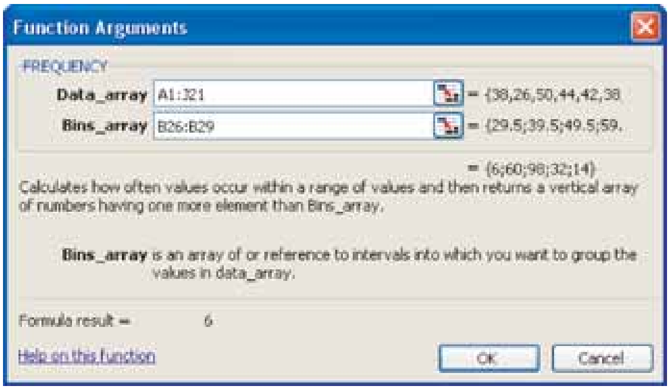
The students at a Brisbane secondary school were asked to estimate the ages (in years) of the teaching staff at the school and their results were presented as a graph, as shown below.



	A	B	C	D	E	F	G	H	I	J	K
1	38	26	50	44	42	38	30	48	33	35	
2	37	27	39	44	55	38	43	48	48	36	
3	55	29	42	44	60	38	50	48	36	37	
4	30	30	35	44	35	39	46	49	36	37	
5	48	30	39	45	43	39	48	50	55	38	
6	48	30	65	45	41	39	48	50	45	38	
7	64	30	46	45	30	39	39	50	42	47	
8	48	30	60	45	45	40	43	50	60	48	
9	53	31	35	45	32	40	47	50	42	48	
10	30	32	48	45	50	40	50	53	45	48	
11	55	32	46	45	38	40	40	54	43	48	
12	47	33	44	45	35	41	58	54	42	48	
13	34	34	47	45	45	41	60	55	54	43	
14	55	35	43	46	44	42	46	55	44	43	
15	48	35	37	46	45	42	44	55	45	43	
16	39	35	35	46	50	42	40	55	38	43	
17	26	35	38	46	35	42	31	55	40	44	
18	41	35	42	46	68	42	44	55	45	44	
19	46	35	40	47	44	42	55	55	54	64	
20	45	36	35	47	38	43	30	58	29	65	
21	45	60	49	60	55	60	32	60	27	68	

23	Estimated Ages of Staff		
24	Group	Max	Frequency
25			
26	20-29	29.5	6
27	30-39	39.5	60
28	40-49	49.5	98
29	50-59	59.5	32
30	60-69	69.5	14

To create the frequency table



Function Arguments

FREQUENCY

Data_array A1:J21 = {38,26,50,44,42,38

Bins_array B26:B29 = {29.5;39.5;49.5;59.

= {6;60;98;32;14}

Calculates how often values occur within a range of values and then returns a vertical array of numbers having one more element than Bins_array.

Bins_array is an array of or reference to intervals into which you want to group the values in data_array.

Formula result = 6

1 Highlight cells **C26:C30**.

2 Select **Function** from the **Insert** menu.

3 Select **FREQUENCY**.

4 Enter **A1:J21** in the **Data_array**.

5 Enter **B26:B29** in the **Bins_array**. (*Note*: The Bins_array is one cell shorter than the frequency column highlighted in step 1. Read about FREQUENCY in Excel Help for more information on this.)

6 Press **CTRL SHIFT ENTER** to execute. (*Note*: Enter by itself will not work. Again, read Excel Help on this.)

To create the frequency histogram

1 Select **Column Graph** from the **Chart Wizard**.

2 The **Data Range** is the Frequency column. In **Series**, the **Category (X) axis labels** is the Group column.

3 Complete **Chart Title**, **Category (X) axis** and **Value (Y) axis**.

4 Select **Finish**.

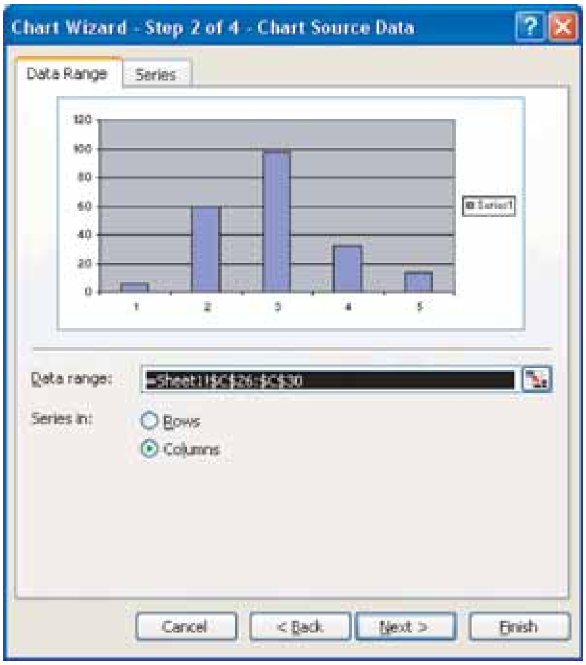


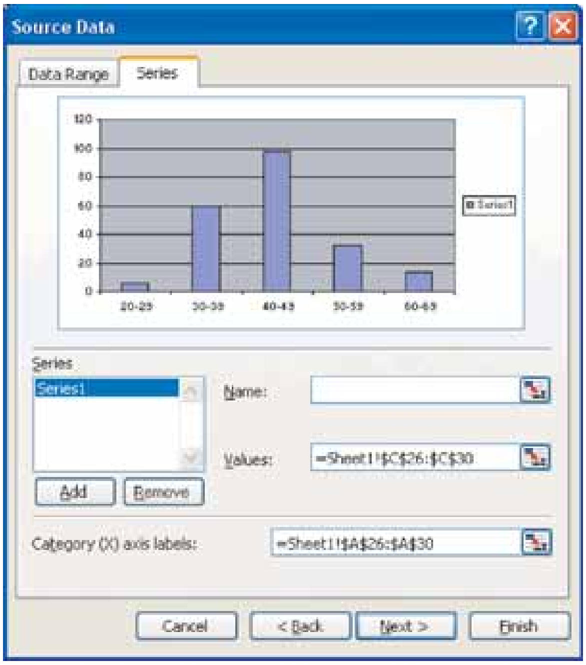
Chart Wizard - Step 2 of 4 - Chart Source Data

Data range: =Sheet1!$C$26:$C$30

Series in: ○ Rows ● Columns

To convert the column graph to a histogram

1. Right click on a column and select **Format Data Series**.
2. Set the **Gap Width** to Zero in **Options**.



Series: Series1

Name:

Values: =Sheet1!$C$26:$C$30

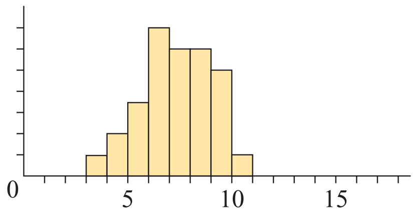
Category (X) axis labels: =Sheet1!$A$26:$A$30

5.4 Characteristics of distributions of numerical variables

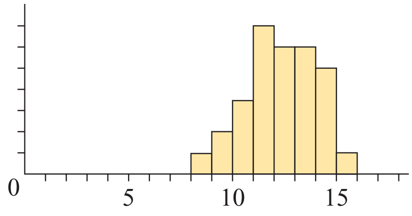
Distributions of numerical variables are characterised by their shapes and special features, such as centre and spread.

Two distributions are said to differ in **centre** if the values of the variable in one distribution are generally larger than the values of the variable in the other distribution. Consider, for example, the following histograms shown on the same scale:

a



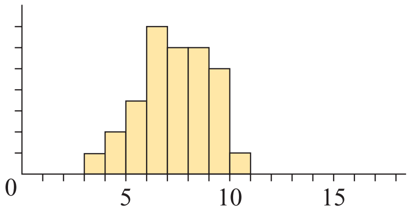
b



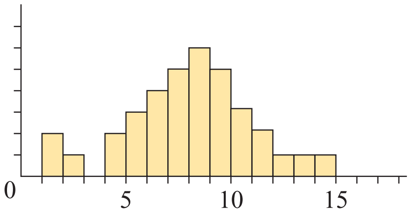
It can be seen that plot **b** is identical to plot **a** but moved horizontally several units to the right, indicating that these distributions differ in the location of their centres.

The next pair of histograms also differ, but not in the same way. Although both histograms are centred at about the same place, histogram **d** is more spread out. Two distributions are said to differ in **spread** if the values of the variable in one distribution tend to be more spread out than the values of the variable in the other distribution.

c

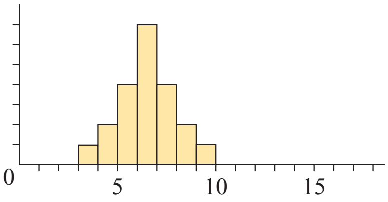


d

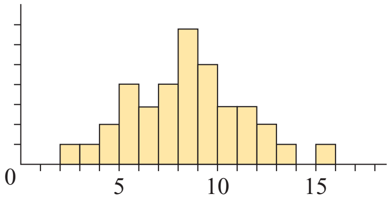


A distribution is said to be **symmetrical** if it forms a mirror image of itself when folded in the ‘middle’ along a vertical axis; otherwise it is said to be **skewed**. Histogram **e** is perfectly symmetrical, whereas **f** shows a distribution that is approximately symmetrical.

e



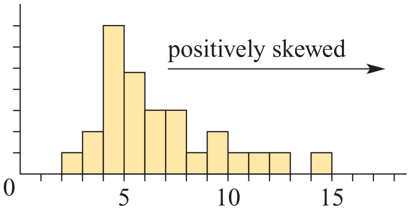
f



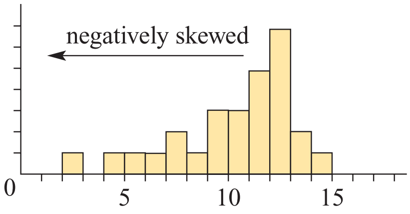
If a histogram has a short tail to the left and a long tail pointing to the right it is said to be **positively skewed** (because of the many values towards the positive end of the distribution), as shown in the histogram **g**.

If a histogram has a short tail to the right and a long tail pointing to the left it is said to be **negatively skewed** (because of the many values towards the negative end of the distribution), as shown in histogram **h**.

g



h

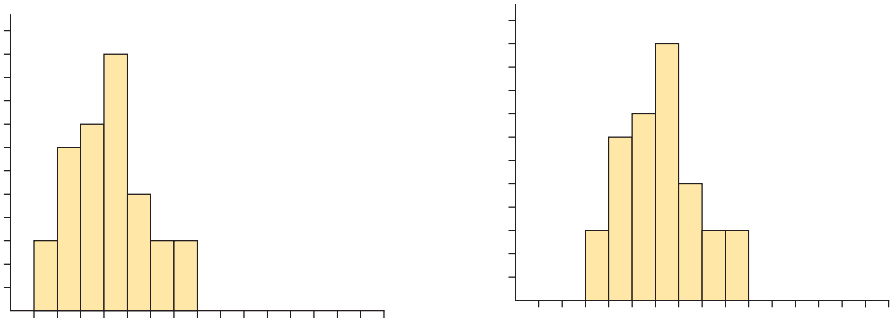


Knowing whether a distribution is skewed or symmetrical is important as this gives considerable information concerning the choice of appropriate summary statistics, as will be seen in the next section.

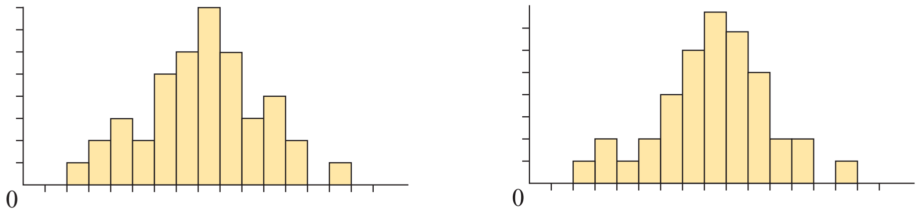
Exercise 5D

1 Do these pairs of distributions differ in centre, spread, both or neither?

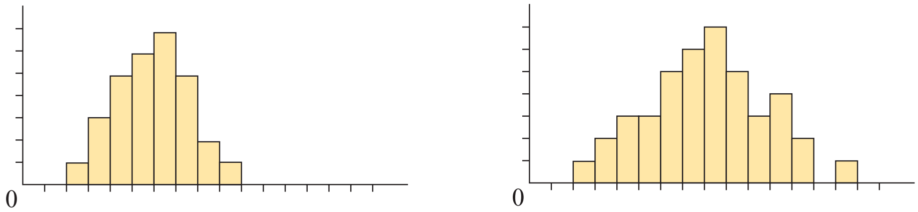
a



b

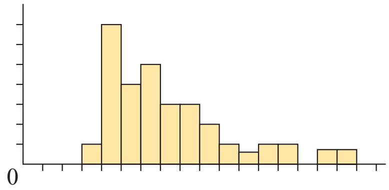


c

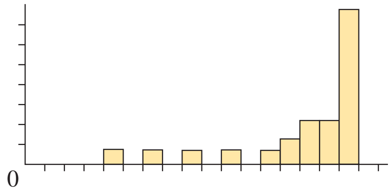


2 Describe the shape of each of the following histograms:

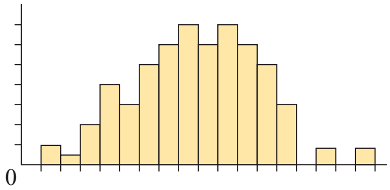
a



b



c



3 What is the shape of the histogram drawn in Question **6**, Exercise **5C**?

4 What is the shape of the histogram drawn in Question **7**, Exercise **5C**?

5 What is the shape of the histogram drawn in Question **8**, Exercise **5C**?

5.5 Stem-and-leaf plots

An informative data display for a small (fewer than 50 values) numerical data set is the **stem-and-leaf plot**. The construction of the stem-and-leaf plot is illustrated in Example 6.

Example 6

Early in 2008 the number of test matches played, as captain, by each of the Australian cricket captains was:

3	16	2	1	8	3	6	4	8	21	2	15	10	6
10	11	2	5	25	5	24	1	24	2	17	1	5	28
1	39	2	25	1	30	48	7	28	93	50	57	6	37

Construct a stem-and-leaf plot of these data.

Solution

0	3 2 1 8 3 6 4 8 2 6 2 5 5 1 2 1 5 1 2 1 7 6
1	6 5 0 0 1 7
2	1 5 4 4 8 5 8
3	9 0 7
4	8
5	0 7
6	
7	
8	
9	3

Number of test matches as captain

0	1 1 1 1 1 2 2 2 2 2 3 3 4 5 5 5 6 6 6 7 8 8
1	0 0 1 5 6 7
2	1 4 4 5 5 8 8
3	0 7 9
4	8
5	0 7
6	
7	
8	
9	3

3 | 9 indicates 39 matches

Note: Students are advised to make an unordered plot followed by an ordered plot, as shown in the example above. The unordered plot is made by writing the numbers in the order they appear in the original list and allows the student to roughly sort the data into groups first. Inclusion of the heading and key with the ordered plot is essential to a complete solution.

It can be seen from this plot that one captain has led Australia in many more test matches than any other (Allan Border, who captained Australia in 93 test matches). When a value sits away from the main body of the data it is called an **outlier**.

Stem-and-leaf plots have the advantage of retaining all the information in the data set while achieving a display not unlike that of a histogram (turned on its side). In addition, a stem-and-leaf plot clearly shows:

- the range of values
- where the values are concentrated
- the shape of the data set
- whether there are any gaps in which no values are observed
- any unusual values (outliers).

Grouping the leaves in tens is simplest – other convenient groupings are in fives or twos, as shown in Example 7.

Example 7

The birth weights, in kilograms, of the first 30 babies born at a hospital in a selected month are as follows:

2.9	2.7	3.5	3.6	2.8	3.6	3.7	3.6	3.6	2.9
3.7	3.6	3.2	2.9	3.2	2.5	2.6	3.8	3.0	4.2
2.8	3.5	3.3	3.1	3.0	4.2	3.2	2.4	4.3	3.2

Construct a stem-and-leaf plot of these data.

Solution 1

Birth weights of babies

2	9 7 8 9 9 5 6 8 4
3	5 6 6 7 6 6 7 6 2 2 8 0 5 3 1 0 2 2
4	2 2 3

3 | 0 indicates 3.0 kilograms

Birth weights of babies

2	4 5 6 7 8 8 9 9 9
3	0 0 1 2 2 2 2 3 5 5 6 6 6 6 6 7 7 8
4	2 2 3

3 | 0 indicates 3.0 kilograms

Solution 2

Birth weights of babies

2	4
2	5 6 7 8 8 9 9 9
3	0 0 1 2 2 2 2 3
3	5 5 6 6 6 6 6 7 7 8
4	2 2 3

3 | 0 indicates 3.0 kilograms

Solution 3

Birth weights of babies

2	4	5					
2	6	7					
2	8	8	9	9	9		
3	0	0	1				
3	2	2	2	2	3		
3	5	5					
3	6	6	6	6	6	7	7
3	8						
4							
4	2	2	3				

3 | 0 indicates 3.0 kilograms

Note: The stem-and-leaf plot in Solution 2 can be described as having an **interval** of 5 because there are only five possible numbers next to each stem: either 0–4 or 5–9. It could also be described as having two **rows per stem**. Similarly, the stem-and-leaf plot in Solution 3 can be described as having an interval of 2 because there are only two possible numbers next to each stem: either 0–1, 2–3, 4–5, 6–7 or 8–9. It could also be described as having five rows per stem.

None of the stem-and-leaf displays shown are correct or incorrect. A stem-and-leaf plot is used to explore data and more than one may need to be constructed before the most informative one is obtained. Again, between 5 and 15 rows is generally the most helpful, but this may vary in individual cases.

When the data have too many digits for a convenient stem-and-leaf plot they should be rounded or truncated. Truncating a number means simply dropping off the unwanted digits. So, for example, a value of 149.99 would become 149 if truncated to three digits, but 150 if rounded to three digits. Since the object of a stem-and-leaf display is to give a feeling for the shape and patterns in the data set, the decision on whether to round or truncate is not very important; however, generally when constructing a stem-and-leaf display the data are truncated, as this is what commonly used data analysis computer packages will do.

Some of the most interesting investigations in statistics involve comparing two or more data sets. Stem-and-leaf plots are useful displays for the comparison of two data sets, as shown in the following example.

Example 8

This table gives the number of points scored by Queensland and New South Wales in the Rugby League State of Origin Series each year since 1994.

Queensland

43	38	26	38	53	27	42	72	48	52	44	56	62	39

New South Wales

38	36	47	35	53	30	104	56	68	58	63	84	37	42

Construct a back-to-back stem-and-leaf plot of the data and use it to compare Queensland and New South Wales in the Rugby League State of Origin Series.

Solution

Points scored in the State of Origin Series

Queensland		New South Wales
7 6	2	
9 8 8	3	0 5 6 7 8
8 4 3 2	4	2 7
6 3 2	5	3 6 8
2	6	3 8
2	7	
	8	4
	9	
	10	4
2 | 4 represents 42 points		4 | 2 represents 42 points

The back-to-back stem-and-leaf plot suggests that New South Wales scored more points in a State of Origin Series than Queensland. This is shown in two ways. The last two entries on the right-hand side are lower than any entry on the right; that is, the highest two scores went to New South Wales. Also, the New South Wales entries are centred lower in the table than the Queensland entries, which indicates that, on average, New South Wales scores more points than Queensland.

Note: The text written below the graph is crucial to a complete response to the question. The question asks that the plots of data be used to compare the two States. Therefore, the response must make the comparison and refer to the graph in doing so.

Exercise 5E

Example 6 **1** The mean number of days of rain of ≥ 1 mm each month in Brisbane are shown in the table below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Days of rain	8.4	8.8	9.4	6.8	5.8	4.5	4.2	4.0	4.4	5.9	6.5	7.7

a Construct a stem-and-leaf plot of the mean number of days of rain each month in Brisbane.

b In how many months is the mean number of days of rain in Brisbane 7 or more?

Example 7 **2** An investigator recorded the amount of time 24 similar batteries lasted in a toy. The results in hours were:

25.5 39.7 29.9 23.6 26.9 31.3 21.4 27.4 19.5 29.8 33.4 21.8
4.2 25.6 16.9 18.9 46.0 33.8 36.8 27.5 25.1 31.3 41.2 32.9

a Make a stem-and-leaf plot of these times with two rows per stem.

b How many of the batteries lasted for more than 30 hours?

3 The amount of time (in minutes) that a class of students spent on homework on one particular night was:

10 27 46 63 20 33 15 21 16 14 15
39 70 19 37 67 20 28 23 0 29 10

a Make a stem-and-leaf plot of these times.

b How many students spent more than 60 minutes on homework?

c What is the shape of the distribution?

4 The cost of various brands of athletic shoes at a retail outlet are as follows:

\$49.99 \$75.49 \$68.99 \$164.99 \$75.99 \$39.99 \$35.99 \$52.99
\$210.00 \$84.99 \$36.98 \$95.49 \$28.99 \$25.49 \$78.99 \$45.99
\$46.99 \$76.99 \$82.99 \$79.99 \$149.99

a Construct a stem-and-leaf plot of these data.

b What is the shape of the distribution?

Example 8 **5** The students in a class were asked to write down the ages (in years) of their mothers and fathers.

Mother's age

49 50 43 50 47 50 40 46 49 49 42 44 38
43 44 40 39 40 41 43 45 48 38 43 37 43

Father's age

50	51	41	55	51	48	47	47	52	54	41	44	40
43	46	44	44	48	43	48	43	46	48	49	45	46

a Construct a back-to-back stem-and-leaf plot of these data sets.

b How do the ages of the students' mothers and fathers compare in terms of shape, centre and spread?

6 The results of a mathematics test for two different classes of students are given in the table.

Class A

22	19	48	39	68	47	58	77	76	89	85	82
85	79	45	82	81	80	91	99	55	65	79	71

Class B

12	13	80	81	83	98	70	70	71	72	72	73
74	76	80	81	82	84	84	88	69	73	88	91

a Construct back-to-back stem-and-leaf plots to compare the data sets.

b How many students in each class scored less than 50%?

c Which class do you think performed better overall on the test? Give reasons for your answer.

5.6 Interpretation of graphs

Exercise 5F

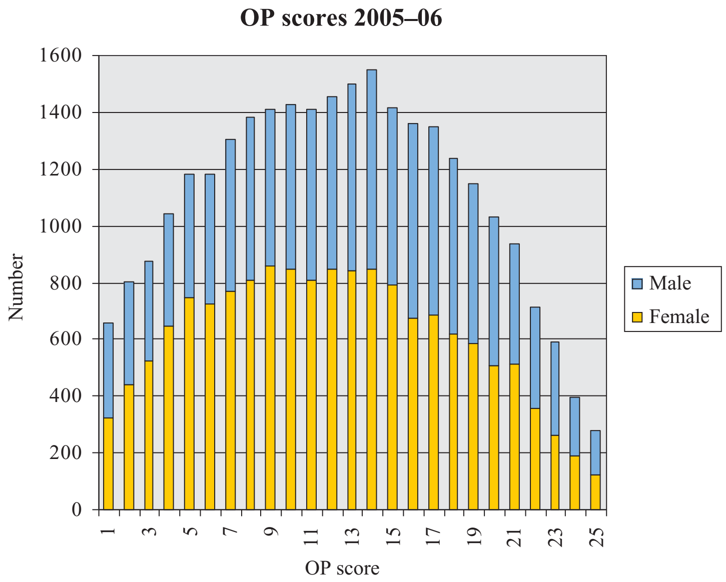
1 Refer to the graph to answer these questions.

a How many females got an OP15?

b How many males got an OP4?

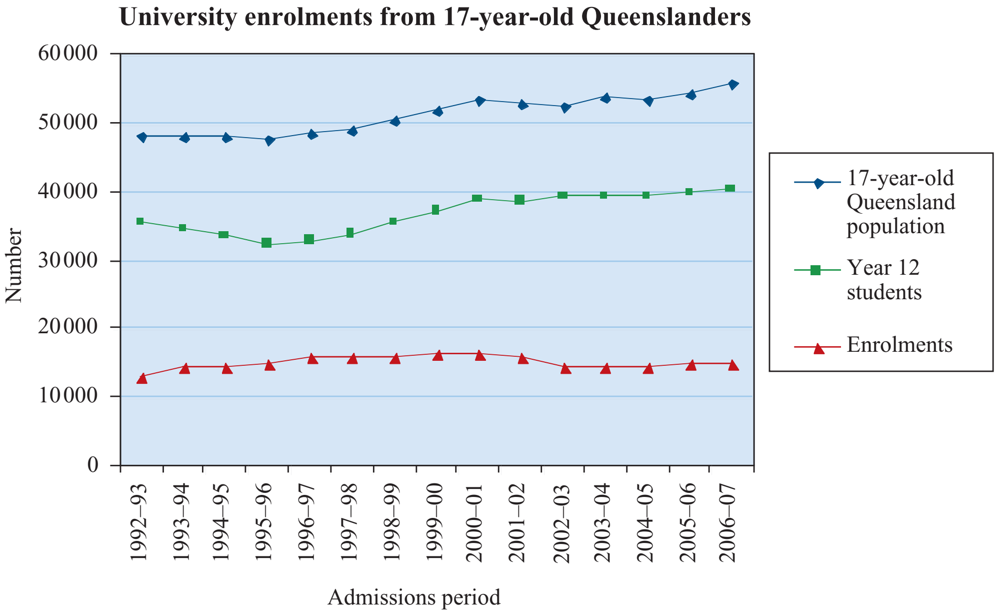
c What was the modal OP score?

d What was the modal OP score for females?



Source: www.qtac.edu.au

2 Refer to the graph to answer these questions.



Source: www.qtac.edu.au

a How many 17-year-old students were there in Queensland in the admission period 2006–07?

b When did the 17-year-old population of Queensland rise above 50 000?

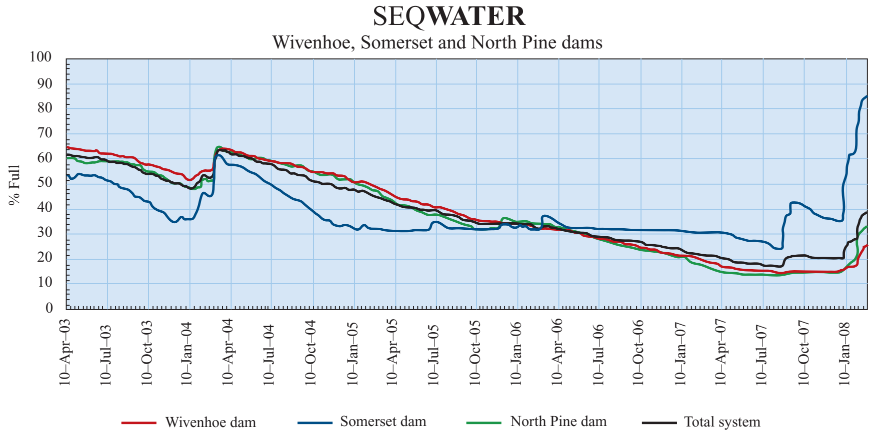
c In which admission period did the lowest number of 17-year-old Queenslanders complete Year 12?

d What proportion of 17-year-old Queenslanders completed Year 12?

e In which admission period did the highest proportion of Year 12 students enrol into university?

3 Brisbane's main water supply is managed by SEQWater and comes from the three dams shown in this graph and table.

Item	Wivenhoe	Somerset	North Pine	SEQWater totals
Storage volume FSL (ML)	1 165 240	379 850	214 960	1 760 050



Source: www.seqwater.com.au

a What percentage of total capacity was the total system holding on 10th October, 2004?

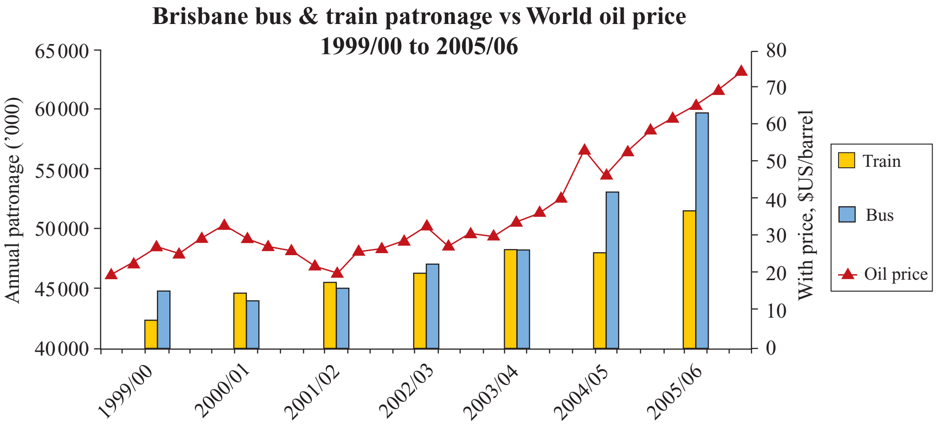
b When did the amount of water in Wivenhoe Dam first drop below 50% over the period April 2003–February 2008?

c When was the Somerset Dam at its lowest over the period April 2003–February 2008?

d How many megalitres of water were in the North Pine Dam on 10th July, 2003?

e Which dam held the most water on 10th April, 2007?

4 In the graph below, the world oil price is shown at the end of each quarter and the bus and train patronage is shown at the end of each financial year.



a How many people caught the bus in the financial year 2002/03?

b How many people caught the bus or train in the financial year 2004/05?

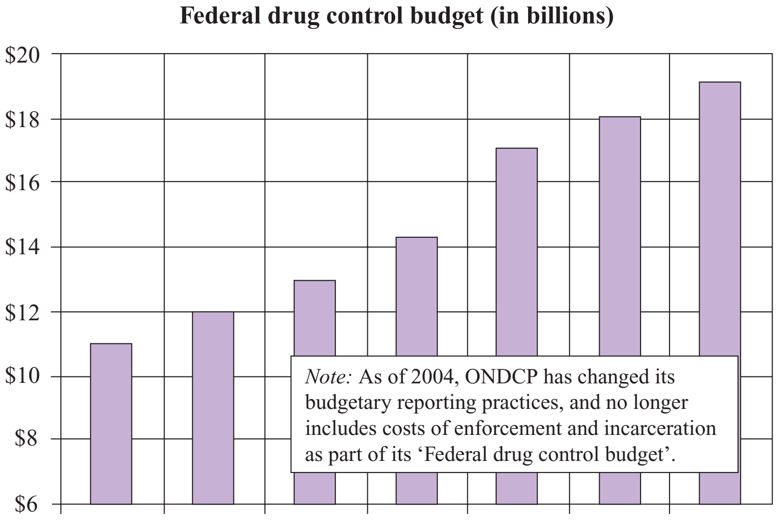
c When did the world oil price first rise above US$50/barrel?

d Find the percentage increase in the use of trains over the period shown in the graph.

e A researcher suggests that the use of public transport increases as the world oil price increases. Discuss.

5 Refer to the graph to answer these questions.

- **a** By how much did the US Federal drug control budget increase over the period 1991–2003?
- **b** Find the average annual growth rate as a percentage over the period 1991–2003.

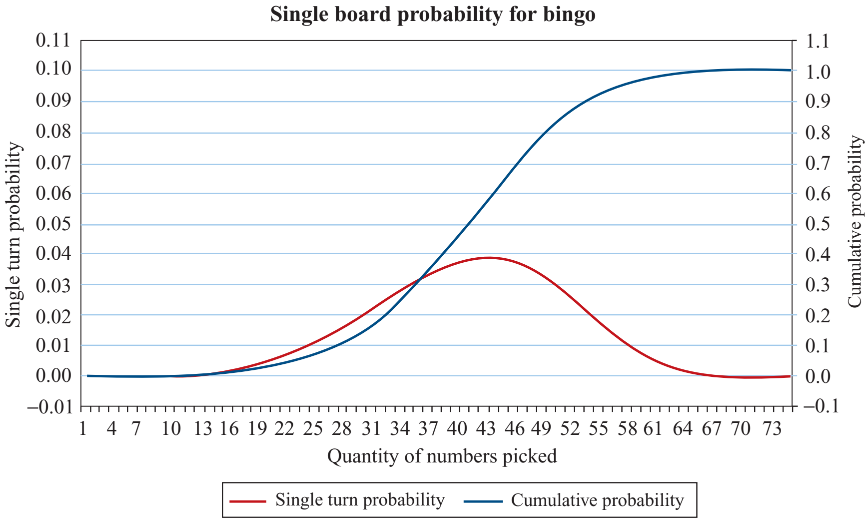


Note: As of 2004, ONDCP has changed its budgetary reporting practices, and no longer includes costs of enforcement and incarceration as part of its 'Federal drug control budget'.

Source: US Bureau of Justice Statistics: ONDCP, FY 2003 National Drug Control Budget, February 2002.

6 The red line in the graph below shows the probability that a single board will score a bingo exactly when the *n*th number is called. The blue line shows the cumulative single-board probability that a bingo will hit on or before the *n*th number is called.

- **a** What is the probability that bingo will be called when the 34th number is called?
- **b** What is the probability that bingo will be called on or before the 34th number is called?
- **c** By which number will 90% of bingo games be over?



5.7 Summarising data

The purpose of collecting data and either presenting it in a graph, finding the average of the data or making some other calculation is often to make some prediction or statement about the general underlying population.

A **statistic** is a value that is calculated from a **sample of data**. It is often used to estimate or compare with some **parameter** of the underlying **population** from which the sample was taken.

Consider the exercise: Measure the heights of all of the students in your Maths class, then add up the heights and divide by the number of students. The **data** are the heights of the students; the **statistic** is the mean height of the students in your Maths class; and the list of heights can be considered to be a **sample** of the heights of the general **population** of Year 11 students across Queensland.

If your class can be considered to be a **random sample** of Year 11 students across Queensland then the mean height of your class can be used to estimate the mean height (a **parameter**) of the underlying population. The sample mean is said to be an **unbiased estimate** of the population mean.

Summary statistics are generally either **measures of centre** or **measures of spread**. There are many different examples for each of these measures and there are situations when one of the measures is more appropriate than another.

Measures of centre

Mean

The most commonly used measure of centre of a distribution of a numerical variable is the **mean**. This is calculated by summing all the data values and dividing by the number of values in the data set.

Example 9

The number of points scored each season by the Queensland Reds in the Super 12 competition is shown below.

320 263 273 233 317 300 336 281 217 185

Find the mean number of points scored per season.

Solution

$$\begin{aligned}\text{Mean} &= \frac{320 + 263 + 273 + \cdots + 185}{10}\\ &= 272.5\end{aligned}$$

The mean of a sample is always denoted by the symbol $\bar{x}$, which is called ‘x bar’.

In general, if n observations are denoted by $x_1, x_2, \ldots, x_n$ the mean is

$$\bar{x} = \frac{x_1 + x_2 + \cdots\cdots + x_n}{n}$$

or, in a more compact version

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where the symbol $\sum$ is the upper case Greek sigma, which in mathematics means ‘the sum of the terms’.

Note: The subscripts on the x pronumerals are used to identify all of the n different values of x. They do not mean that the x pronumerals have to be written in any special order.

Median

Another useful measure of the centre of a distribution of a numerical variable is the middle value or **median**. To find the value of the median, all the observations are *listed in order* and the middle one is the median.

The median of

$$2 \quad 3 \quad 4 \quad 5 \quad 5 \quad \overset{\text{median}}{\boxed{6}} \quad 7 \quad 7 \quad 8 \quad 8 \quad 11$$

is 6, as there are five observations on either side of this value when the data are listed in order.

Example 10

Find the median number of points scored in a season by the Queensland Reds in the Super 12 competition. (Use the data from Example 9.)

Solution

$$n = 10$$

$$\frac{n+1}{2} = 5.5$$

$$185 \quad 217 \quad 233 \quad 263 \quad \underbrace{273 \quad 281} \quad 300 \quad 317 \quad 320 \quad 336$$

$$\begin{aligned}\text{Median} &= \frac{273 + 281}{2}\\ &= 277\end{aligned}$$

In general, to compute the **median** of a distribution:

- Arrange all the observations in ascending order, according to size.
- Calculate $\frac{n+1}{2}$.
- Count along the ordered row to the $\frac{n+1}{2}$ th place. If $\frac{n+1}{2}$ is a whole number, then you have the median. If $\frac{n+1}{2}$ is not a whole number, then the counting will finish between *two* numbers. *Average* them to find the median. (*Note*: This was done in Example 10.)

Note: The median value is easily determined from a stem-and-leaf plot by counting to the required observation or observations from either end.

Mode

The mode is the observation that occurs most often. It is a useful summary statistic, particularly for categorical data that do not lend themselves to some of the other numerical summary methods. Many texts state that the mode is a third option for a measure of centre but this is generally not true. Sometimes data sets do not have a mode, or they have several modes, or they have a mode that is at one or other end of the range of values.

Measures of spread

Range

A measure of spread is calculated in order to judge the **variability** of a data set. That is, are most of the values clustered together or are they rather spread out? The simplest measure of spread can be determined by considering the difference between the smallest and the largest observations. This is called the **range**.

Example 11

Consider the marks, for two different tasks, awarded to a group of students.

Task A

2	6	9	10	11	12	13	22	23	24	26	26	27	33	34
35	38	38	39	42	46	47	47	52	52	56	56	59	91	94

Task B

11	16	19	21	23	28	31	31	33	38	41	49	52	53	54
56	59	63	65	68	71	72	73	75	78	78	78	86	88	91

Find the range of each of these data sets.

Solution

For Task A, the minimum mark is 2 and the maximum mark is 94.

Range for Task A = 94 − 2 = 92

For Task B, the minimum mark is 11 and the maximum mark is 91.

Range for Task B = 91 − 11 = 80

The range for Task A is greater than the range for Task B. Is the range a useful summary statistic for comparing the spread of the two distributions? To help make this decision, consider the stem-and-leaf plots of the data sets.

				Task A				Task B					
			9	6	2	0							
		3	2	1	0	1	1	6	9				
7	6	6	4	3	2	2	1	3	8				
9	8	8	5	4	3	3	1	1	3	8			
		7	7	6	2	4	1	9					
	9	6	6	2	2	5	2	3	4	6	9		
						6	3	5	8				
						7	1	2	3	5	8	8	8
						8	6	8					
				4	1	9	1						

From the stem-and-leaf plots of the data it appears that the spread of marks for the two tasks is not described well by the range. The marks for Task A are more concentrated than the marks for Task B, except for the two unusual values for Task A. Another measure of spread is needed, one that is not so influenced by these extreme values. For this the **interquartile range** is used.

Interquartile range

To find the **interquartile range** of a distribution:

- Arrange all observations in order according to size.
- Divide the observations into two equal-sized groups. If n, the number of observations, is odd, then the median is omitted from both groups.
- Locate Q_1, the first quartile, which is the median of the lower half of the observations, and Q_3, the third quartile, which is the median of the upper half of the observations.
- The interquartile range (i.e. IQR) is defined as the difference between the quartiles. i.e.

$$\text{IQR} = Q_3 - Q_1$$

Definitions of the quartiles of a distribution sometimes differ slightly from the one given here. Using different definitions may result in slight differences in the values obtained, but these will be minimal and should not be considered a difficulty.

Example 12

Find the interquartile ranges for Task A and Task B data given in Example 11.

Solution

For Task A the marks listed in order are:

2 6 9 10 11 12 13 22 23 24 26 26 27 33 34
35 38 38 39 42 46 47 47 52 52 56 56 59 91 94

The median of the lower group is the eighth observation, 22, so $Q_1 = 22$.
The median of the upper group is 47, so $Q_3 = 47$.

$$\begin{aligned}\text{Thus, the interquartile range, IQR} &= 47 - 22\\ &= 25\end{aligned}$$

Similarly, for Task B data,

$$\begin{aligned}\text{the lower quartile} &= 31 \text{ and}\\ \text{the upper quartile} &= 73,\end{aligned}$$

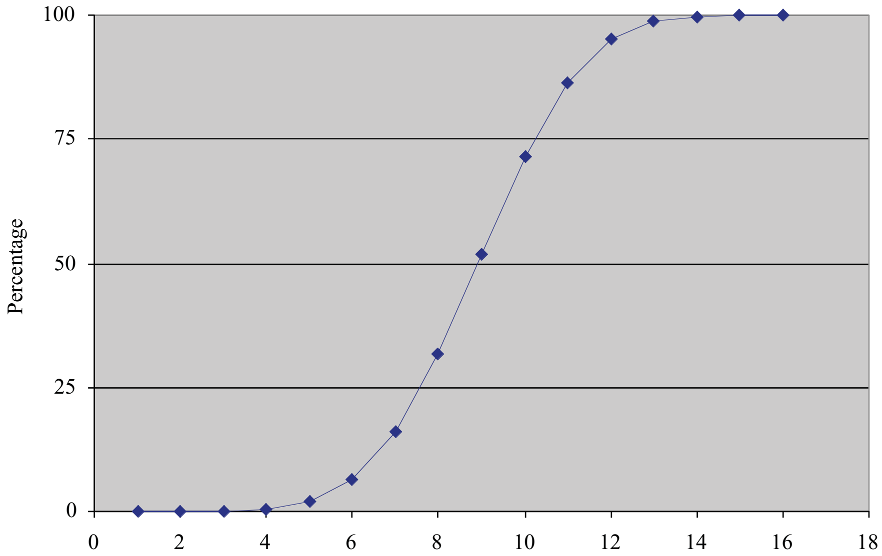
giving an interquartile range for this data set of 42.
Comparing the two values of interquartile range shows the spread of Task A marks to be much smaller than the spread of Task B marks, which seems consistent with the display.

The interquartile range is a measure of spread of a distribution that describes the range of the middle 50% of the observations. Since the upper 25% and the lower 25% of the observations are discarded, the interquartile range is generally not affected by the presence of outliers in the data set, which makes it a reliable measure of spread.

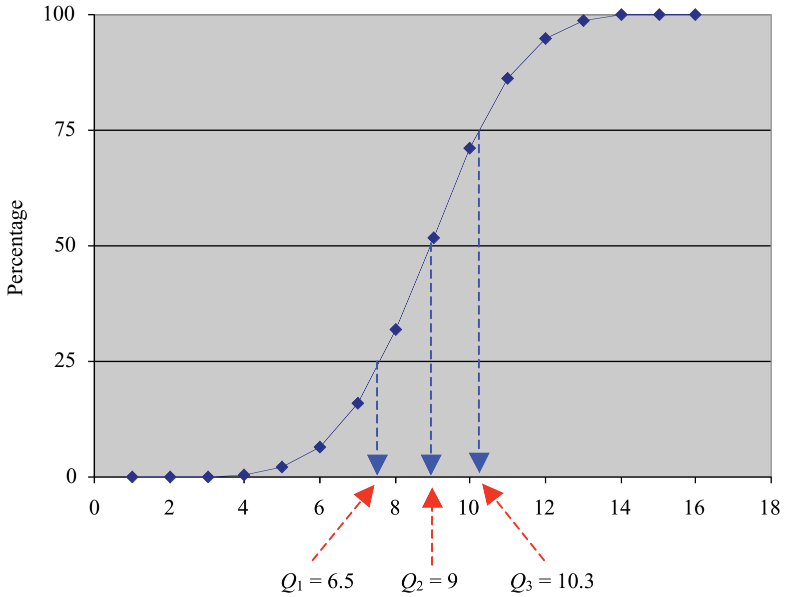
The median and quartiles of a distribution may also be determined from a cumulative relative frequency polygon. Since the median is the observation that divides the data set in half, this is the data value that corresponds to a cumulative relative frequency of 0.5 or 50%. Similarly, the first quartile corresponds to a cumulative relative frequency of 0.25 or 25%, and the third quartile corresponds to a cumulative relative frequency of 0.75 or 75%.

Example 13

Use the cumulative relative frequency polygon to find the median and interquartile range for the data shown in the graph.



Solution



$$
\begin{aligned}
\text{Median} &= 9 \\
\text{IQR} &= Q_3 - Q_1 \\
&= 10.3 - 6.5 \\
&= 3.8
\end{aligned}
$$

Note: The median is also referred to as Q_2 or the 50% point because it is *two* quarters or 50% of the way along the list of ranked data.

Standard deviation

Another extremely useful measure of spread is the **standard deviation**. It is calculated using the formula below.

If a data set consists of n observations, denoted by $x_1,\ x_2,\ x_3, \ldots x_n,$ the standard deviationof the data set is

$$\sigma_n = \sqrt{\tfrac{1}{n}\left((x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2\right)}$$

or, in more compact notation,

$$\sigma_n = \sqrt{\tfrac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

where σ_n is referred to as the **population standard deviation**.

Although the σ_n is the standard deviation of the data set, it cannot be used to estimate the standard deviation of the underlying population. An unbiased estimate of the standard deviation of the wider population, from which the sample was taken, is given by σ_{n-1}. Its formula is shown below.

$$\sigma_{n-1} = \sqrt{\tfrac{1}{n-1}\left((x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2\right)}$$

or, in more compact notation,

$$\sigma_{n-1} = \sqrt{\tfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

σ_{n-1} is referred to as the **sample standard deviation**.

Note: The population standard deviation σ_n is used when data of the entire population is known. The sample standard deviation σ_{n-1} is used when the data is a sample taken from a wider population.

Example 14

a Calculate the standard deviation of the following data set:

13 12 14 6 15 12 7 6 7 8

b Calculate an unbiased estimate of the standard deviation of the population.

Solution

a

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
13	3	9
12	2	4
14	4	16
6	−4	16
15	5	25
12	2	4
7	−3	9
6	−4	16
7	−3	9
8	−2	4
$\Sigma x_i = 100$		$\Sigma(x_i - \bar{x})^2 = 112$

$$\begin{aligned}\sigma_n &= \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}\\ &= \sqrt{\frac{1}{10} \times 112}\\ &\approx 3.347\end{aligned}$$

b

$$\begin{aligned}\sigma_{n-1} &= \sqrt{\frac{1}{n-1}\Sigma(x_i - \bar{x})^2}\\ &= \sqrt{\frac{1}{10-1} \times 112}\\ &\approx 3.528\end{aligned}$$

Interpreting the standard deviation

The standard deviation can be made more meaningful by interpreting it in relation to the data set. The interquartile range gives the spread of the middle 50% of the data. It can be shown that, for most data sets, about 95% of the observations lie within two standard deviations of the mean.

Example 15

The cost of a lettuce at a number of different shops on a particular day is given in the table.

\$3.85	\$2.65	\$1.90	\$2.95	\$2.40	\$2.42	\$2.63	\$3.20	\$4.20	\$2.33	\$0.85
\$3.81	\$1.69	\$3.66	\$2.60	\$2.70	\$3.10	\$2.80	\$1.80	\$2.88	\$1.40	

Calculate a 95% confidence interval for the cost of lettuce on that day.

Solution

From the calculator, $\overline{x} \approx 2.66$ and $\sigma_{n-1} \approx 0.84$.

$$\begin{aligned}\overline{x} - 2\sigma_{n-1} &\approx 2.66 - 2 \times 0.84\\ &= 0.98\end{aligned}$$

$$\begin{aligned}\overline{x} + 2\sigma_{n-1} &\approx 2.66 + 2 \times 0.84\\ &= 4.34\end{aligned}$$

It is estimated that 95% of shops were charging between \$0.98 and \$4.34 for a lettuce on the given day.

Note: The sample standard deviation is being used because the interval must relate to the general population, and so an estimate of the underlying standard deviation is required.

Example 16

The prices of 40 second-hand motorbikes listed in a newspaper are as follows:

\$5442	\$5439	\$2523	\$2358	\$2363	\$2244	\$1963	\$2142
\$2220	\$1356	\$738	\$656	\$715	\$1000	\$1214	\$1788
\$3457	\$4689	\$8218	\$11 091	\$11 778	\$11 637	\$8770	\$8450
\$6469	\$7148	\$10 884	\$14 450	\$15 731	\$13 153	\$10 067	\$9878
\$5294	\$3847	\$4219	\$4786	\$2280	\$3019	\$7645	\$8079

Calculate a 95% confidence interval for the listing prices of second-hand motorbikes.

Solution

From the calculator, $\overline{x} \approx 5730$ and $\sigma_{n-1} \approx 4233$.

$$\begin{aligned}\overline{x} - 2\sigma_{n-1} &\approx 5730 - 2 \times 4233\\ &= -2636 \qquad \text{(round up to zero)}\\ &= 0\end{aligned}$$

$$\begin{aligned}\overline{x} + 2\sigma_{n-1} &\approx 5730 + 2 \times 4233\\ &= 14\,196\end{aligned}$$

It is estimated that 95% of motorbikes were listed below \$14 196 on that day.

Note: The lower limit was rounded to zero because the listed price is always positive.

The exact percentage of observations that lie within two standard deviations of the mean varies from data set to data set but, in general, it will be around 95%, particularly for symmetrical data sets.

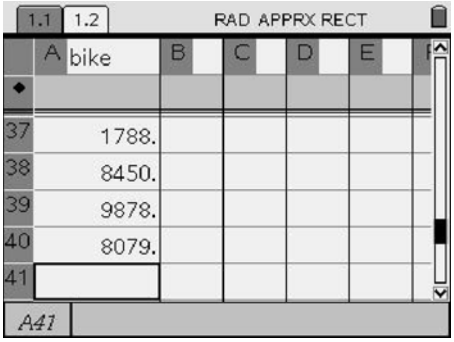
It was noted earlier that even a single outlier can have a very marked effect on the value of the mean of a data set, while leaving the median unchanged. The same is true when the effect of an outlier on the standard deviation is considered, in comparison to the interquartile range. The median and interquartile range are called **resistant** measures, whereas the mean and standard deviation are not resistant measures. When considering a data set it is necessary to do more than just compute the mean and standard variation. First, it is necessary to examine the data using a histogram or stem-and-leaf plot to determine which set of summary statistics is more suitable.

Using technology

Using the TI-Nspire:

1 Enter into the Lists & Spreadsheet application and give column A the name **bike**.

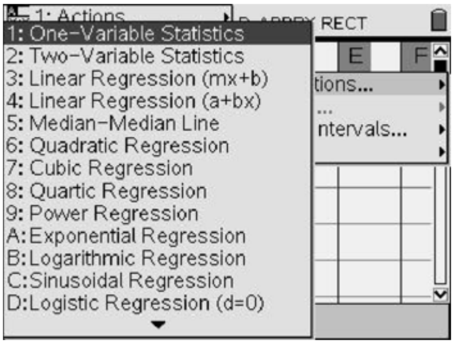
2 Enter the given data into 'bike'.



	A bike
37	1788.
38	8450.
39	9878.
40	8079.
41	

3 Highlight the column by pressing the up arrow until the column becomes shaded.

4 For a list of summary statistics, press menu and navigate to *Statistics* ⟶ *Stat Calculations* ⟶ *One-Variable Statistics*.

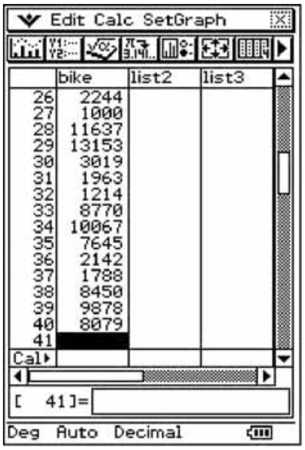


5 When prompted with 'Num of Lists:' type **1**, then press enter on OK.

Using the ClassPad:

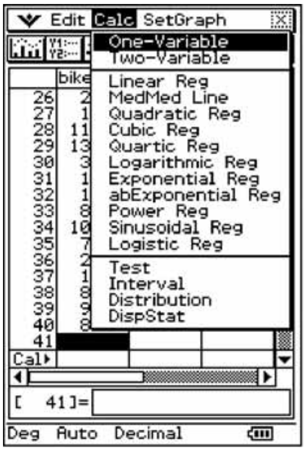
1 Enter into the Statistics application and rename list1 to **bike**.

2 Enter the given data into 'bike'.

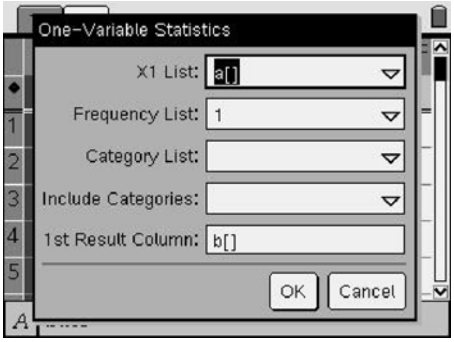


	bike	list2	list3
26	2244		
27	1000		
28	11637		
29	13153		
30	3019		
31	1963		
32	1214		
33	8770		
34	10067		
35	7645		
36	2142		
37	1788		
38	8450		
39	9878		
40	8079		
41			

3 For a list of summary statistics, select *One-Variable* from the Calc menu.



6 Enter the information below then press enter on OK.



One-Variable Statistics

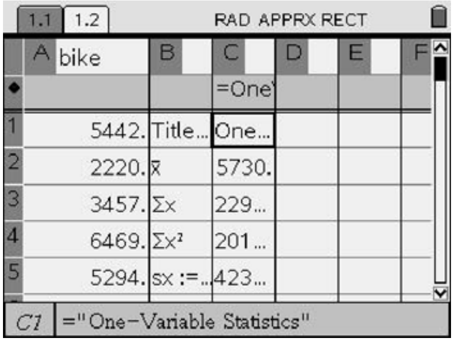
X1 List: a[]

Frequency List: 1

Category List:

Include Categories:

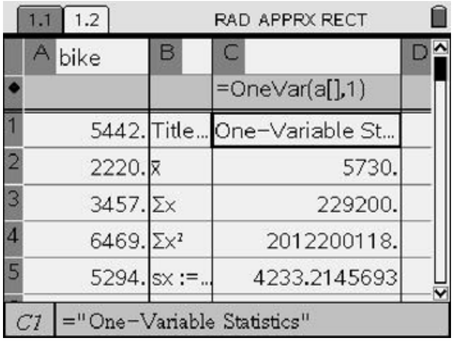
1st Result Column: b[]



	A bike	B	C	D	E	F
◆			=One'			
1	5442.	Title...	One...			
2	2220.	$\overline{x}$	5730.			
3	3457.	Σx	229...			
4	6469.	Σx^2	201 ...			
5	5294.	sx :=...	423...			

C1 ="One–Variable Statistics"

7 Resize the width of column C to view the entire numbers.

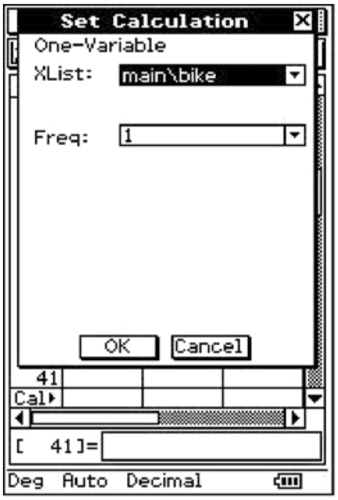


	A bike	B	C	D
◆			=OneVar(a[],1)	
1	5442.	Title...	One–Variable St...	
2	2220.	$\overline{x}$	5730.	
3	3457.	Σx	229200.	
4	6469.	Σx^2	2012200118.	
5	5294.	sx :=...	4233.2145693	

C1 ="One–Variable Statistics"

Use the down arrow key to view further summary statistics.

4 Change XList to main\bike then tap OK.

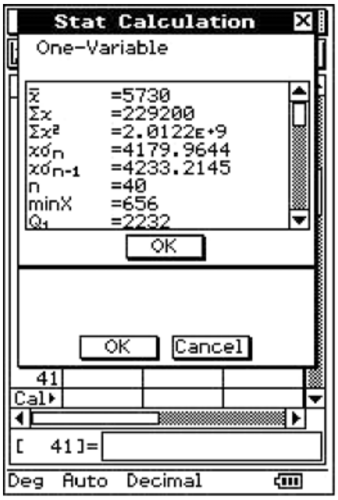


Set Calculation

One-Variable

XList: main\bike

Freq: 1

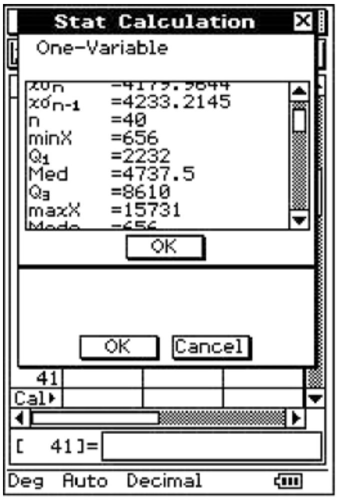


Stat Calculation

One-Variable

```
x̄      =5730
Σx     =229200
Σx²    =2.0122E+9
xσn    =4179.9644
xσn-1  =4233.2145
n      =40
minX   =656
Q1     =2232
```

Use the scroll bar on the right-hand side to view further summary statistics.



Stat Calculation

One-Variable

```
xσn-1  =4233.2145
n      =40
minX   =656
Q1     =2232
Med    =4737.5
Q3     =8610
maxX   =15731
```

Either graphics calculator can be used to determine the summary statistics when the data are given in a frequency table, such as:

x	1	2	3	4
Frequency	5	8	7	2

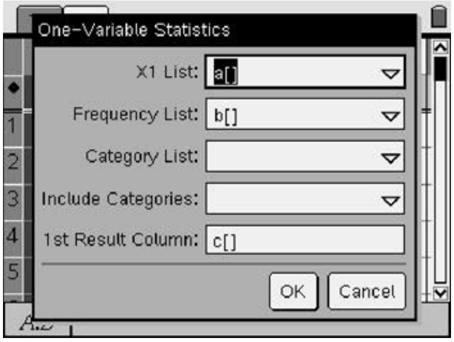
Using the TI-Nspire:

1 Enter the x values into column A and the frequencies into column B.

2 Highlight both columns then navigate to *One-Variable Statistics* and press enter.

3 When prompted with 'Num of Lists:' type **1** and press enter on OK.

4 Enter the information below then press enter on OK for a list of summary statistics.



One-Variable Statistics

X1 List: a[]

Frequency List: b[]

Category List:

Include Categories:

1st Result Column: c[]

Using the ClassPad:

1 Enter the x values into list1 and the frequencies into list2.

2 Select *One-Variable* from the Calc menu and enter the following:

XList: list1

Freq: list2

Tap OK to view the summary statistics.

Set Calculation

One-Variable

XList: list1

Freq: list2

Exercise 5G

Examples 9,10

1 Find the mean and the median of the following data sets:

a 29 14 11 24 14 14 28 14 18 22 14

b 5 9 11 3 12 13 12 6 13 7 3 15 12 15 5 6

c 8.3 5.6 8.2 6.5 8.2 7.0 7.9 7.1 7.8 7.5

d 1.5 0.2 0.7 0.7 0.2 0.2 0.1 1.7 0.5 1.2 2.0 1.7
1.0 3.4 1.3 0.9 1.1 5.8 2.7 3.2 0.6 4.6 0.5 3.1

2 Find the mean and the median of the following data sets:

a

x	1	2	3	4	5
Frequency	6	3	10	7	8

b

x	−2	−1	0	1	2
Frequency	5	8	11	3	2

3 The price, in dollars, of houses sold in a particular suburb during a one-week period are given in the following list:

$187 500 $129 500 $93 400 $400 000 $118 000 $168 000 $550 000
$133 500 $135 500 $140 000 $186 000 $140 000 $204 000 $122 000

Find the mean and the median of the prices. Which do you think is a better measure of centre of the data set? Explain your answer.

4 Concerned with the level of absence from his classes, a teacher decided to investigate the number of days each student had been absent from the classes for the year to date. These are the results.

No. of days missed	0	1	2	3	4	5	6	9	21
No. of students	4	2	14	10	16	18	10	2	1

Find the mean and the median number of days each student had been absent so far that year. Which is the better measure of centre in this case?

Examples 11, 12

5 Find the range and the interquartile range for each of the following data sets:

a 718 630 1002 560 715 1085 750 510 1112 1093

b 0.7 −1.6 0.2 −1.2 −1.0 3.4 3.7 0.8

c 8.56 8.51 8.96 8.39 8.62 8.51 8.58 8.82 8.54

d 20 19 18 16 16 18 21 20 17 15 22 19

6 The serum cholesterol levels for a sample of 20 people are:

231 159 203 304 248 238 209 193 225 244
190 192 209 161 206 224 276 196 189 199

a Find the range of the serum cholesterol levels.
b Find the interquartile range of the serum cholesterol levels.

7 Twenty babies were born at a local hospital on one weekend. Their birth weights, in kg, are given in the stem-and-leaf plot below.

2	1				
2	5	7	9	9	
3	1	3	3	4	4
3	5	6	7	7	9
4	1	2	2	3	
4	5				

3|6 represent 3.6 kg

a Find the range of the birth weights.
b Find the interquartile range of the birth weights.

Example 13 **8** A randomly chosen group of university students was asked to write down their ages, giving the following results:

17 17 17 17 17 17 17 18 18 18 18 18 18 18 18 18 18 18
18 18 18 18 18 19 19 19 20 20 20 21 24 25 31 41 44 45

a Construct a cumulative relative frequency polygon and use it to find the median and the interquartile range of this data set.
b Estimate the mean and standard deviation of the ages of students at the university.
c Find the percentage of students whose ages fall within two standard deviations of the mean.

Example 14 **9** Find the standard deviation for the following data sets:

a 30 16 22 23 18 18 14 56 13 26 9 31

b \$2.52 \$4.38 \$3.60 \$2.30 \$3.45 \$5.40 \$4.43 \$2.27 \$4.50
\$4.32 \$5.65 \$6.89 \$1.98 \$4.60 \$5.12 \$3.79 \$4.99 \$3.02

c 200 300 950 200 200 300 840 350 200 200

d 86 74 75 77 79 82 81 75 78 79 80 75 78 78 81 80 76 77 82

10 For each of the following data sets:

a Estimate the mean and the standard deviation of the underlying population.
Example 15 **b** Determine the percentage of observations falling within two standard deviations of the mean for the underlying population.

i 41 16 6 21 1 21 5 31 20 27 17 10 3 32 2 48 8 12
21 44 1 56 5 12 3 1 13 11 15 14 10 12 18 64 3 10

ii 141 260 164 235 167 266 150 255 168 245 258 239
152 141 239 145 134 150 237 254 150 265 140 132

11 The results of a student's chemistry experiment are as follows:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 6.0

a **i** Find the mean and the median of the results.

ii Find the interquartile range and the standard deviation of the results.

b Unfortunately, when the student was transcribing his results into his chemistry book he made a small error, and wrote:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 60

i Find the mean and the median of these results.

ii Find the interquartile range and the standard deviation of these results.

c Describe the effect the error had on the summary statistics calculated in parts **a** and **b**.

Example 16 **12** A selection of shares traded on the stock exchange had a mean price of \$50 with a standard deviation of \$3. Determine an interval that would include approximately 95% of share prices.

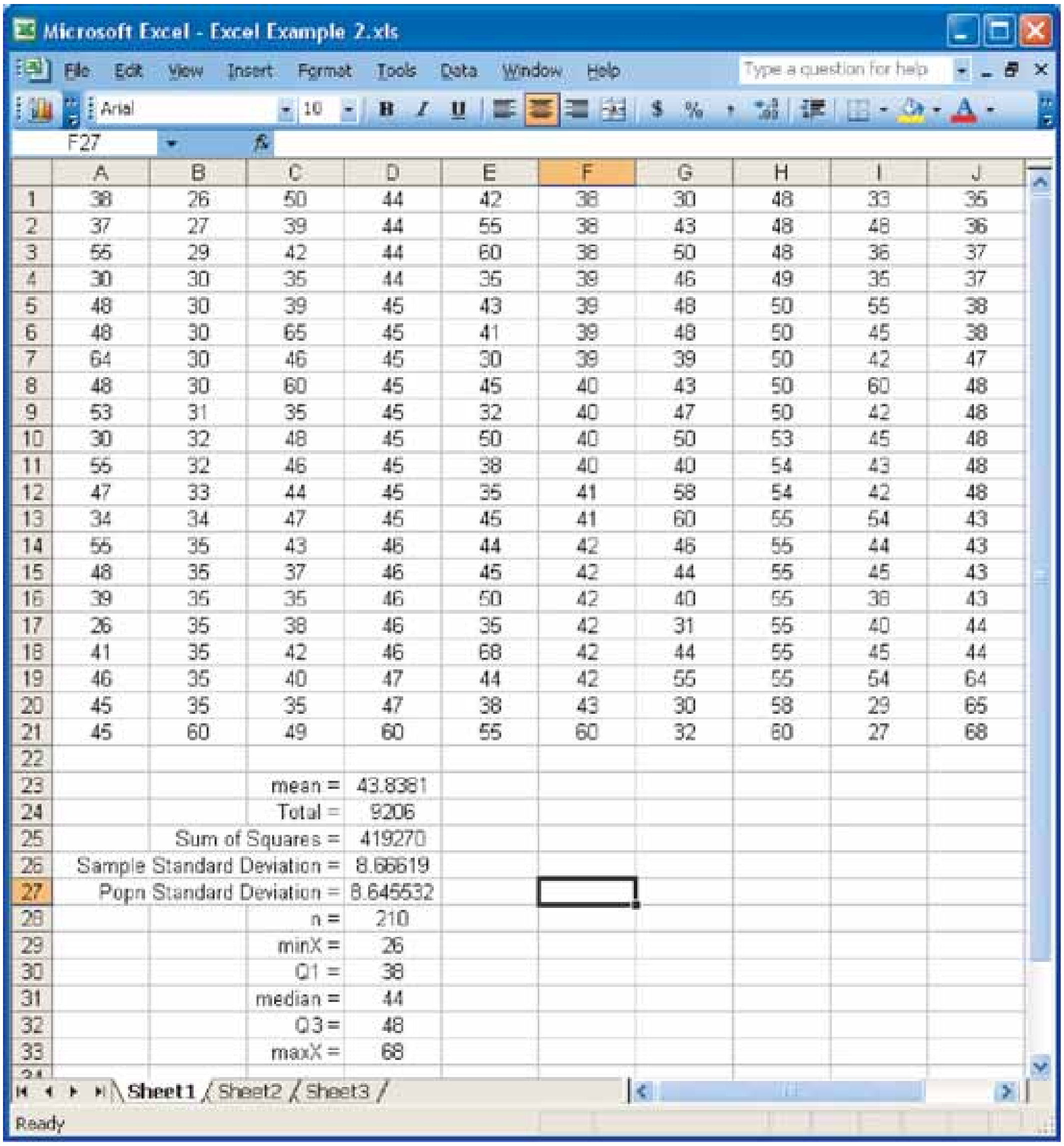
13 A store manager determined the store's mean daily receipts as \$550, with a standard deviation of \$200. On what proportion of days were the daily receipts between \$150 and \$950?

Using Excel to calculate statistics

The students at a Brisbane secondary school were asked to estimate the ages of the teaching staff at the school and to use Excel to calculate the same statistics as appeared on their graphics calculator.

The formulae to be entered in cells **D23:D33** are:

```
=AVERAGE(A1:J21)
=SUM(A1:J21)
=SUMSQ(A1:J21)
=STDEV(A1:J21)
=STDEVP(A1:J21)
=COUNT(A1:J21)
=MIN(A1:J21)
=QUARTILE(A1:J21,1)
=MEDIAN(A1:J21)
=QUARTILE(A1:J21,3)
=MAX(A1:J21)
```



	A	B	C	D	E	F	G	H	I	J
1	38	26	50	44	42	38	30	48	33	35
2	37	27	39	44	55	38	43	48	48	36
3	56	29	42	44	60	38	50	48	36	37
4	30	30	35	44	35	39	46	49	35	37
5	48	30	39	45	43	39	48	50	55	38
6	48	30	65	45	41	39	48	50	45	38
7	64	30	46	45	30	39	39	50	42	47
8	48	30	60	45	45	40	43	50	60	48
9	53	31	35	45	32	40	47	50	42	48
10	30	32	48	45	50	40	50	53	45	48
11	55	32	46	45	38	40	40	54	43	48
12	47	33	44	45	35	41	58	54	42	48
13	34	34	47	45	45	41	60	55	54	43
14	56	35	43	46	44	42	46	55	44	43
15	48	35	37	46	45	42	44	55	45	43
16	39	35	35	46	50	42	40	55	38	43
17	26	35	38	46	35	42	31	55	40	44
18	41	35	42	46	68	42	44	55	45	44
19	46	35	40	47	44	42	55	55	54	64
20	45	35	35	47	38	43	30	58	29	65
21	45	60	49	60	55	60	32	60	27	68
22										
23			mean =	43.8381						
24			Total =	9206						
25			Sum of Squares =	419270						
26			Sample Standard Deviation =	8.66619						
27			Popn Standard Deviation =	8.645532						
28			n =	210						
29			minX =	26						
30			Q1 =	38						
31			median =	44						
32			Q3 =	48						
33			maxX =	68						

5.8 The boxplot

Knowing the median and quartiles of a distribution means that quite a lot is known about the central region of the data set. If something is known about the tails of the distribution then a good picture of the whole data set can be obtained. This can be achieved by knowing the maximum and minimum values of the data. These five important statistics can be derived from a data set: the median, the two quartiles and the two extremes.

These values are called the **five-figure summary** and can be used to provide a succinct pictorial representation of a data set called the **box and whisker plot** or **boxplot**.

For this visual display, a box is drawn with the ends at the first and third quartiles. Lines are drawn that join the ends of the box to the minimum and maximum observations. The median is indicated by a vertical line in the box.

Example 17

Draw a boxplot to show the number of hours spent on a project by individual students in a particular school.

24	4	166	147	97	90	36	92	226	37	111
59	102	13	108	2	71	102	147	56	181	35
9	3	48	27	264	86	9	40	146	19	76

Solution

2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

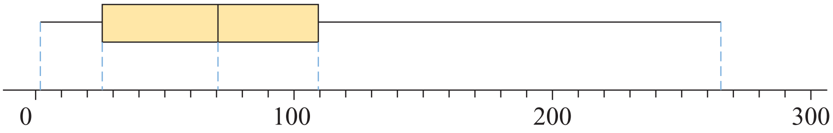
$$\text{Median, } m = 71$$

$$\text{First quartile, } Q_1 = \frac{24 + 27}{2} = 25.5$$

$$\text{Third quartile, } Q_3 = \frac{108 + 111}{2} = 109.5$$

$$\text{Minimum} = 2$$

$$\text{Maximum} = 264$$



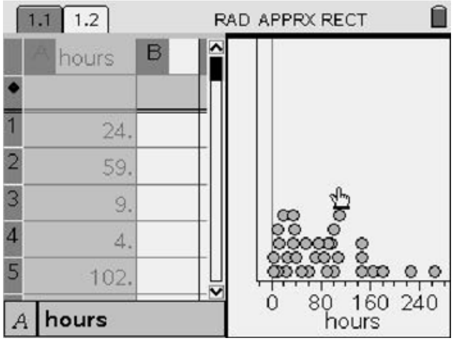
In general, to draw a boxplot:

- Arrange all the observations in order, according to size.
- Determine the minimum value, the first quartile, the median, the third quartile, and the maximum value for the data set.
- Draw a horizontal box with the ends at the first and third quartiles. The height of the box is not important.
- Join the minimum value to the lower end of the box with a horizontal line.
- Join the maximum value to the upper end of the box with a horizontal line.
- Indicate the location of the median with a vertical line.

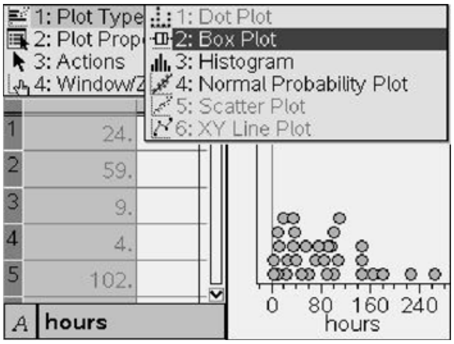
Using technology

Using the TI-Nspire:

1. Enter the data from Example 17 into a list named **hours**.
2. Highlight the column, press menu, then select *Quick Graph* from the Data submenu.

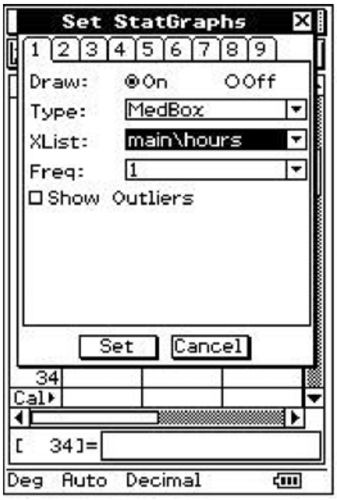


3. To draw a boxplot for the data, press menu and select *Box Plot* from the Plot Type submenu.



Using the ClassPad:

1. Enter the data from Example 17 into a list named **hours**.
2. To ensure a boxplot is drawn for ‘hours’, tap SetGraph then tap on Setting. . . Change Type to MedBox and change the XList to main\hours. Tap SET to save the changes.



Set StatGraphs

Draw: ● On ○ Off

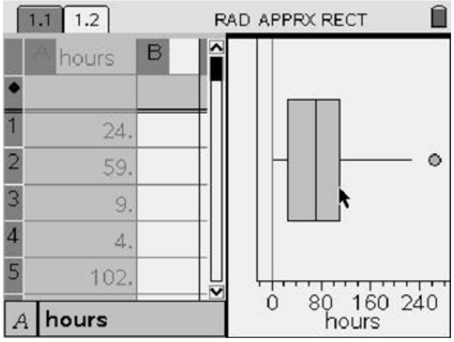
Type: MedBox

XList: main\hours

Freq: 1

☐ Show Outliers

3. To see the boxplot, tap .

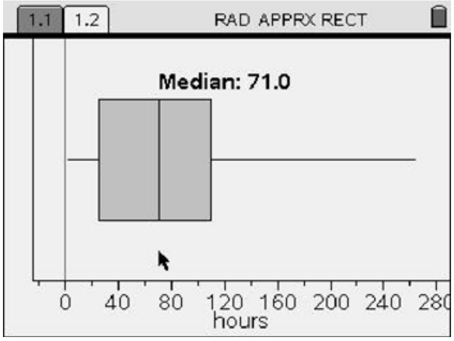


A hours	B
◆	
1 24.	
2 59.	
3 9.	
4 4.	
5 102.	

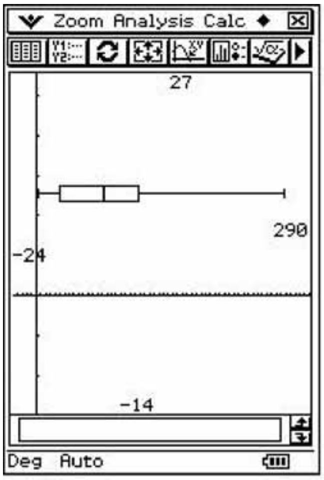
A **hours**

4 To view a boxplot with **no** outliers, press menu and select *Extended Box Plot Whiskers* from the Plot Properties submenu.

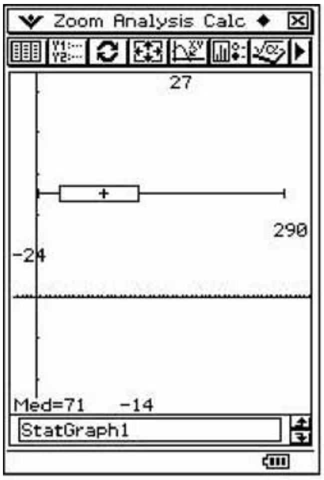
5 For a full-screen view of the boxplot, press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, ⌂. Scroll to *Page Layout ⟶ Select Layout ⟶ 1: Layout 1*.



4 Tap Resize for a full-screen view of the boxplot.



5 Select Trace from the Analysis menu to move through the summary statistics on the boxplot.

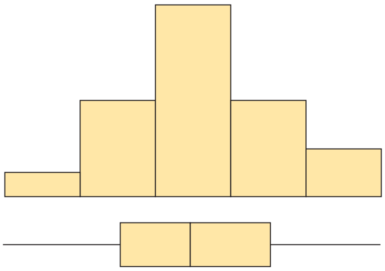


Med=71

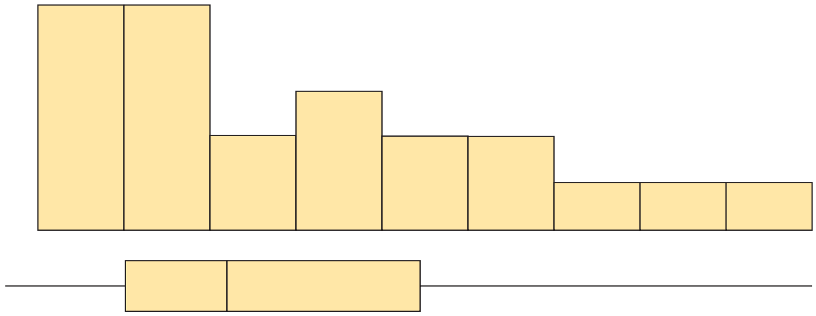
StatGraph1

Symmetry of a data set

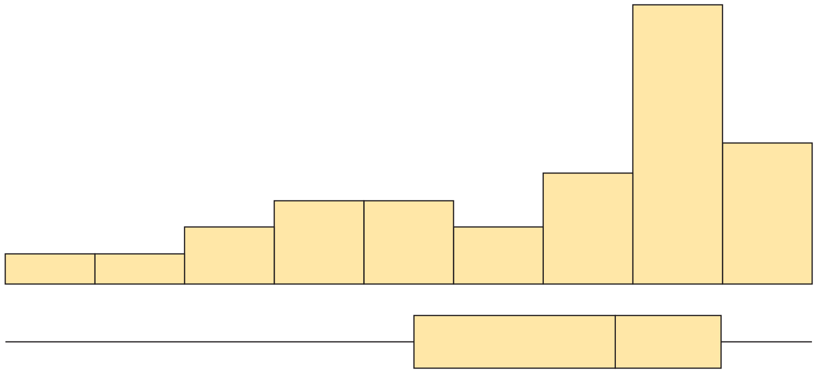
The symmetry of a data set can be determined from a boxplot. If a data set is symmetrical, then the median will be located approximately in the centre of the box, and the tails will be of similar length. This is illustrated in the following diagram, which shows the same data set displayed as a histogram and a boxplot.



A median placed towards the left of the box, and/or a long tail to the right indicates a **positively skewed** distribution, as shown in this plot.



A median placed towards the right of the box, and/or a long tail to the left indicates a **negatively skewed** distribution, as illustrated here.



A more sophisticated version of a boxplot can be drawn with the outliers in the data set identified. This is very informative, as one cannot tell from the previous boxplot if an extremely long tail is caused by many observations in that region or just one.

Before drawing this boxplot the outliers in the data set must be identified. The term **outlier** is used to indicate an observation that is rather different from other observations. Sometimes it is difficult to decide whether or not an observation should be designated as an outlier. The interquartile range can be used to give a very useful definition of an outlier.

An **outlier** is any number that is more than 1.5 interquartile ranges above the upper quartile, or more than 1.5 interquartile ranges below the lower quartile.

When drawing a boxplot, any observation identified as an outlier is indicated by an asterisk, and the whiskers are joined to the smallest and largest values that are not outliers.

Example 18

Use the data from Example 17 to draw a boxplot with outliers.

Solution

$$\text{Median} = 71$$

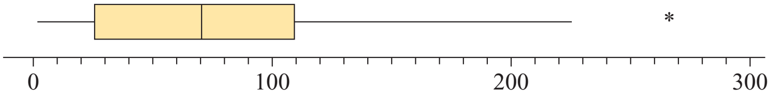
$$\begin{aligned}\text{Interquartile range} &= Q_3 - Q_1\\ &= 109.5 - 25.5\\ &= 84\end{aligned}$$

$$\begin{aligned}\text{Lower limit} &= Q_1 - 1.5 \times \text{IQR}\\ &= 25.5 - 1.5 \times 84\\ &= -100.5\end{aligned} \qquad \begin{aligned}\text{Upper limit} &= Q_3 + 1.5 \times \text{IQR}\\ &= 109.5 + 1.5 \times 84\\ &= 235.5\end{aligned}$$

264 > upper limit (235.5) and is therefore an outlier.

Minimum = 2

Maximum = 226 (ignoring the outlier)



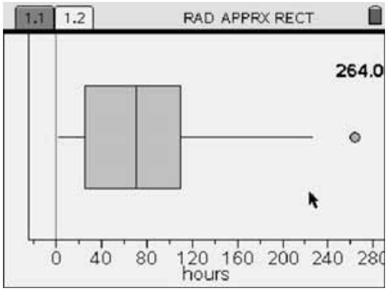
Note: The outlier is still plotted, it is just not included in the calculation of the maximum or minimum value.

Using technology

Either graphics calculator can also construct a boxplot with outliers. Consider the data from Example 17.

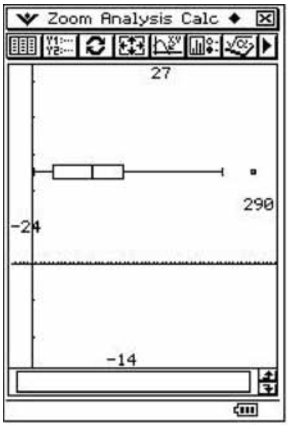
Using the TI-Nspire:

1. Highlight the data and select Quick Graph from the Data submenu.
2. Change the Plot Type to Box Plot.
3. Press menu and select *Show Box Plot Outliers* from the Plot Properties submenu.



Using the ClassPad:

1. With the data in a list called **hours**, tap SetGraph then tap on Setting. . .
   Change Type to MedBox and change the XList to main\hours. Place a tick in the box next to Show Outliers by tapping it with the stylus. Tap SET to save the changes.



Exercise 5H

Example 17 **1** The heights (in centimetres) of a class of girls are:

160 165 123 143 154 180 133 123 157 157 135 140 140 150
154 159 149 167 176 163 154 167 168 132 145 143 157 156

a Determine the five-figure summary for this data set.
b Draw a boxplot of the data.
c Describe the pattern of heights in the class in terms of shape, centre and spread.

Example 18 **2** A researcher is interested in the number of books people borrow from a library. She decides to select a sample of 38 cards and record the number of books each person has borrowed in the previous year. Here are her results.

7 28 0 2 38 18 0 0 4 0 0 2 13
1 1 14 1 8 27 0 52 4 0 12 28 15
10 1 0 2 0 1 11 5 11 0 13 0

a Determine the five-figure summary for this data set.
b Determine if there are any outliers.
c Draw a boxplot of the data, showing any outliers.
d Describe the number of books borrowed in terms of shape, centre and spread.

3 The winnings of the top 25 male tennis players in 2008 are given in the table below.

Player	Winnings	Player	Winnings
Roger Federer	$39 012 348	David Ferrer	$4 894 568
Lleyton Hewitt	$17 368 039	Guillermo Canas	$4 726 635
Rafael Nadal	$14 327 494	Mikhail Youzhny	$4 516 698
Andy Roddick	$13 337 041	Juan Ignacio Chela	$4 341 261
Carlos Moya	$12 913 650	Tomas Berdych	$3 533 072
Juan Carlos Ferrero	$11 620 750	Richard Gasquet	$3 149 224
David Nalbandian	$8 446 124	Paul-Henri Mathieu	$2 584 896
Nikolay Davydenko	$7 390 873	Marcos Baghdatis	$2 450 046
Ivan Ljubicic	$6 739 032	Ivo Karlovic	$2 084 169
Fernando Gonzalez	$6 225 685	Andy Murray	$1 983 077
Novak Djokovic	$6 039 631	Juan Monaco	$1 548 419
Tommy Robredo	$5 948 991	Jo-Wilfried Tsonga	$1 102 944
James Blake	$5 340 285		

a Draw a boxplot of the data, indicating any outliers.
b Describe the data in terms of shape, centre, spread and outliers.

4 The hourly rate of pay for a group of students engaged in part-time work was found to be:

\$4.75	\$8.50	\$17.23	\$9.00	\$12.00	\$11.69	\$6.25
\$7.50	\$8.89	\$6.75	\$7.90	\$12.46	\$10.80	\$8.40
\$12.34	\$10.90	\$11.65	\$10.00	\$10.00	\$13.00	

a Draw a boxplot of the data, indicating any outliers.

b Describe the hourly pay rate for the students in terms of shape, centre, spread and outliers.

5 The daily circulation of several newspapers in Australia is:

570 000	327 654	299 797	273 248	258 700	230 487
217 284	214 000	212 770	171 568	170 000	125 778
98 158	77 500	56 000	43 330	17 398	

a Draw a boxplot of the data, indicating any outliers.

b Describe the daily newspaper circulation in terms of shape, centre, spread and outliers.

5.9 Using boxplots to compare distributions

Boxplots are extremely useful for comparing two or more sets of data collected on the same variable, such as marks on the same assignment for two different groups of students. By drawing boxplots on the same axis, both the centre and spread for the distributions are readily identified and can be compared visually.

Example 19

The number of hours spent by individual students on the project referred to in Example 17 at another school were:

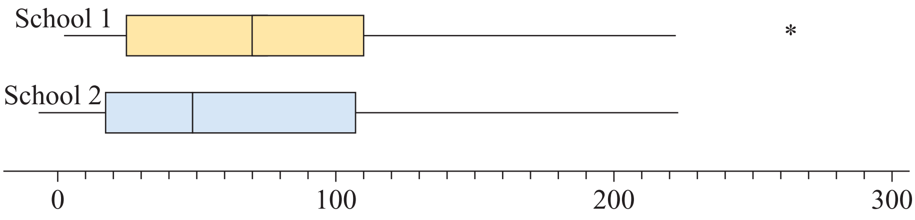
53	152	82	30	16	136	21	11	1	55	128
57	106	14	18	173	102	86	227	48	12	45
136	226	17	9	156	19	107	24	42	21	176
24	80	54	16	106	6	38	3			

Use boxplots to compare the time spent on the project by students at this school with those in Example 17.

Solution

The five-figure summary for this data set is:

Median, $m = 48$; first quartile, $Q_1 = 17.5$; third quartile, $Q_3 = 106.5$; minimum $= 1$; maximum $= 227$



From the boxplots the distributions of time for the two schools can be compared in terms of shape, centre, spread and outliers. Clearly the two distributions for both schools are skewed positively, indicating a larger range of values in the upper half of the distributions. The centre for School 1 is higher than the centre for School 2 (71 hours compared to 48 hours). As can be seen by comparing the box widths, which indicate the IQR, the spread of the data is comparable for both distributions. There is one outlier, a student who attended School 1 and spent 264 hours on the project!

Note: The boxplot is useful for summarising large data sets and for comparing several sets of data. It focuses attention on important features of the data and gives a picture of the data that is easy to interpret. When a single data set is being investigated a stem-and-leaf plot is sometimes better, as a boxplot may hide the local detail of the data set.

Exercise 5I

Example 19

1 To test the effect of a physical fitness course, the number of sit-ups that a person could do in 1 minute, both before and after the course, were recorded. Twenty randomly selected participants scored as follows:

Before	29	22	25	29	26	24	31	46	34	28
	23	22	26	26	30	12	17	21	20	30
After	28	26	25	35	33	36	32	54	50	43
	25	24	30	34	30	15	29	21	19	34

a Construct boxplots of these two sets of data on the same axis.

b Describe the effect of the physical fitness course on the number of sit-ups achieved in terms of shape, centre, spread and outliers.

2 The number of hours spent on homework per week by a group of students in Year 8 and a group of students in Year 12 are shown in the tables.

Year 8	1	2	4	2	4	4	5	3	7	7	2	4	3	3
	1	3	4	3	3	1	7	2	1	3	1	4	1	0
Year 12	1	2	3	5	6	7	7	6	7	8	7	5	4	1
	2	3	1	1	4	7	8	9	6	7	8	7	2	3

Draw boxplots of these two sets of data on the same axis and use them to answer the following questions:

a Which group does the most homework?

b Which group varies more in the number of hours of homework they do?

3 The ages of mothers at the birth of their first child were noted, for the first forty such births, at a particular hospital in 1970 and again in 1990.

1970	21	29	25	32	37	30	24	36	23	19
	37	22	26	31	26	27	19	21	33	17
	24	21	22	36	22	25	31	20	18	20
	16	21	25	26	34	27	18	39	24	21
1990	24	22	35	32	17	28	38	20	30	39
	19	33	44	24	18	27	24	33	29	23
	26	18	28	32	43	28	26	28	41	28
	25	35	31	23	19	46	29	23	34	29

a Construct boxplots of these two sets of data on the same axis.

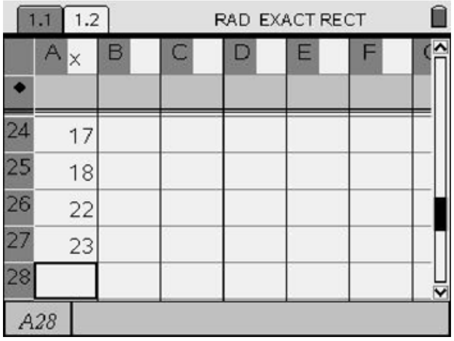
b Compare the ages of the mothers in 1970 and 1990 in terms of shape, centre, spread and outliers.

Using technology

How to construct a histogram

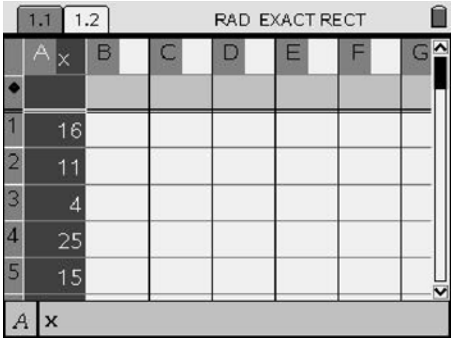
Using the TI-Nspire:

1 Enter the data into a column called **x**.



	A x
24	17
25	18
26	22
27	23
28	

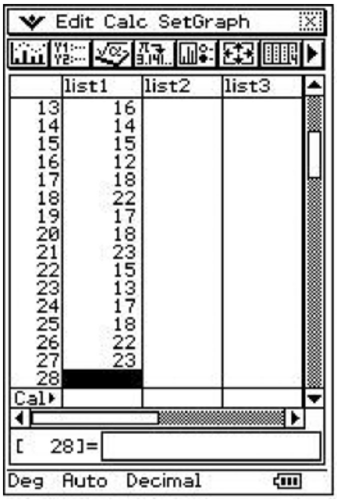
2 Highlight the column by moving the cursor to the extreme top of the column.



	A x
1	16
2	11
3	4
4	25
5	15

Using the ClassPad:

1 Enter the data into list1.

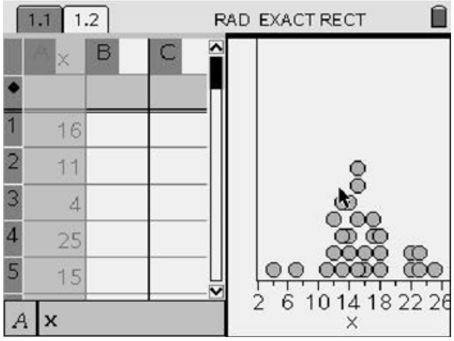


	list1	list2	list3
13	16		
14	14		
15	15		
16	12		
17	18		
18	22		
19	17		
20	18		
21	23		
22	15		
23	13		
24	17		
25	18		
26	22		
27	23		
28			

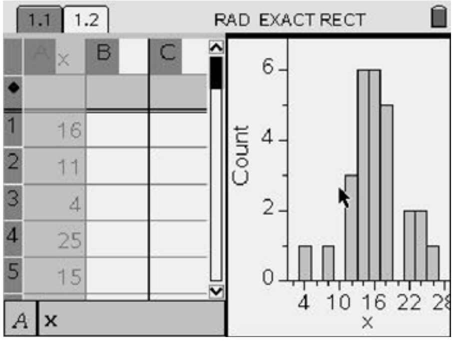
[28]=

2 To ensure a histogram is drawn for list1, tap SetGraph then tap on Setting. . . Change Type to Histogram and change the XList to list1. Tap SET to save the changes.

3 Press menu and select *Quick Graph* from the Data submenu.

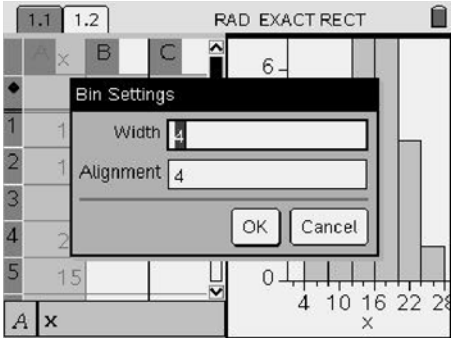


4 Press menu and select *Histogram* from the Plot Type submenu.



5 Press menu and enter into the Plot Properties menu. Select *Bin Settings* from the Histogram Properties submenu.

6 Enter the information below.



Width 4

Alignment 4

Draw: ●On ○Off

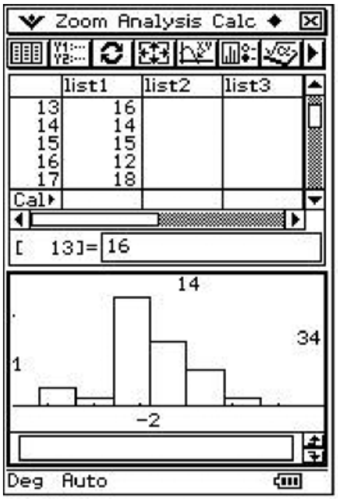
Type: Histogram

XList: list1

Freq: 1

[28]=

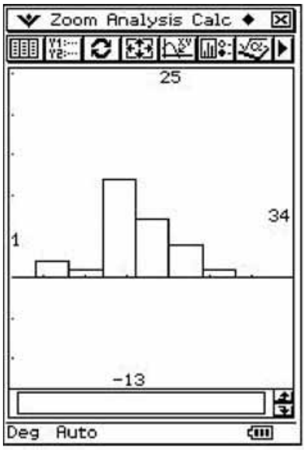
3 To see the histogram, tap .
Enter the following:
HStart: 4 and HStep: 4 and tap OK.



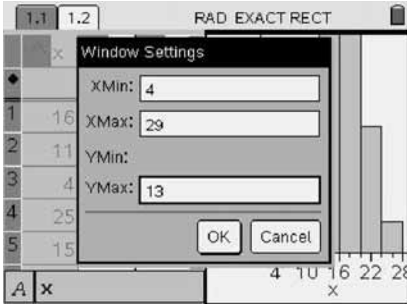
	list1	list2	list3
13	16		
14	14		
15	15		
16	12		
17	18		

[13]=16

4 Tap Resize for a full-screen view of the histogram.



7 Press menu, enter into *Window Settings* then type the information below.



Window Settings

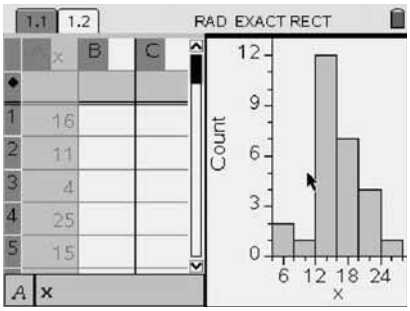
XMin: 4

XMax: 29

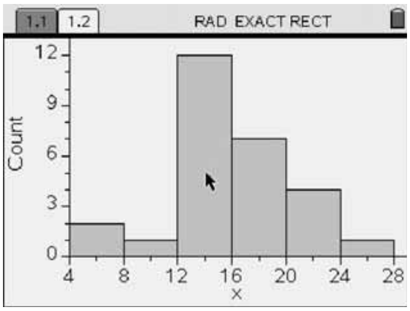
YMin:

YMax: 13

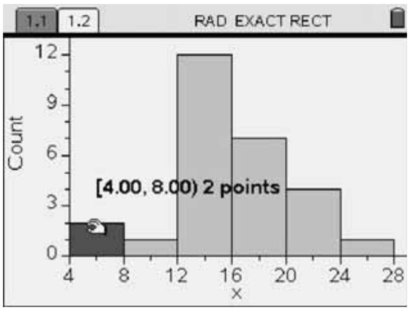
8 Press enter on OK.



9 For a full-screen view of the graph, press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, ⌂. Scroll to *Page Layout* ⟶ *Select Layout* ⟶ *1: Layout 1*.



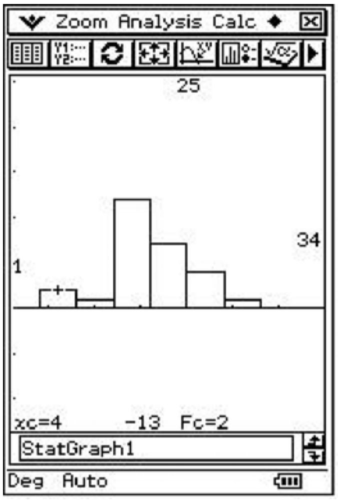
10 To view the frequency of the first class interval, move the cursor to the first bar then hold down the ✋ key.



[4.00, 8.00) 2 points

Thus, the count is 2 for the first interval.

5 Select *Trace* from the Analysis menu to determine the frequencies for each class.



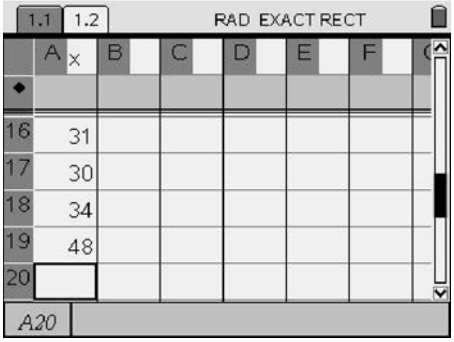
xc=4 -13 Fc=2

Thus, the count is 2 for the first interval.

How to construct a boxplot with outliers

Using the TI-Nspire:

1 Enter the data into a column called **x**.

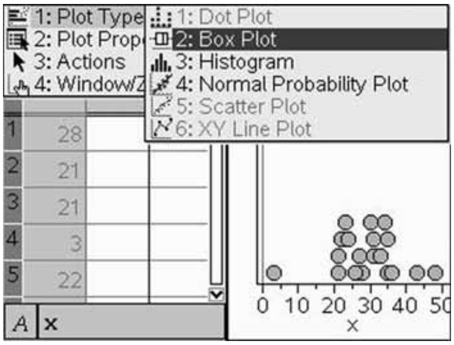


	A x
16	31
17	30
18	34
19	48
20	

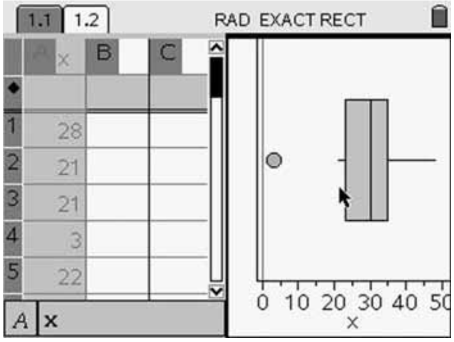
2 Highlight the column then press menu.

3 Select *Quick Graph* from the Data submenu.

4 To change the dot plot to a boxplot with outliers, press menu and select *Box Plot* from the Plot Type submenu.



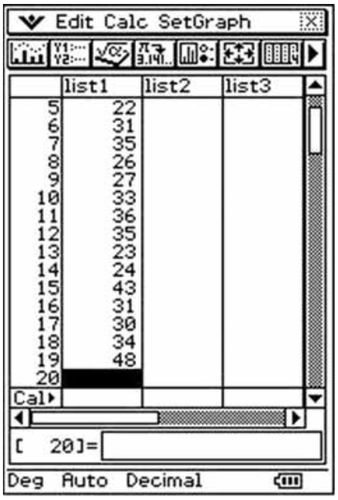
1	28
2	21
3	21
4	3
5	22



1	28
2	21
3	21
4	3
5	22

Using the ClassPad:

1 Enter the data into list1.



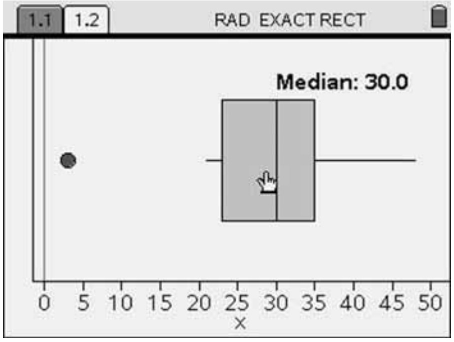
	list1	list2	list3
5	22		
6	31		
7	35		
8	26		
9	27		
10	33		
11	36		
12	35		
13	23		
14	24		
15	43		
16	31		
17	30		
18	34		
19	48		
20			

2 To ensure a boxplot is drawn for list1, tap SetGraph then tap on Setting. . . Change Type to MedBox and change the XList to list1. Ensure the box next to Show Outliers has a tick in it. Tap SET to save the changes.

Draw: ●On ○Off
Type: MedBox
XList: list1
Freq: 1
☑Show Outliers

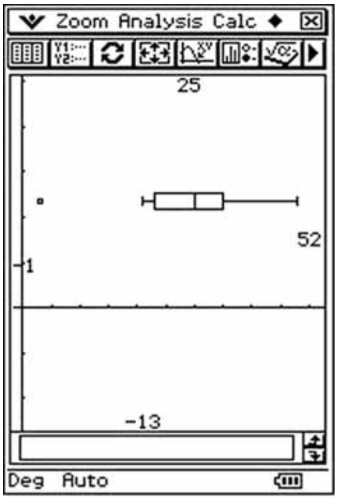
3 To see the boxplot, tap 📊.

5 For a full-screen view of the boxplot, press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, ⌂. Scroll to *Page Layout* ⟶ *Select Layout* ⟶ *1: Layout 1*.



6 Move the cursor left and right to view the statistics.

4 Tap Resize for a full-screen view of the boxplot.

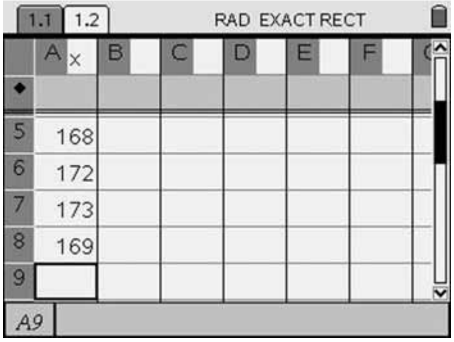


5 Select *Trace* from the Analysis menu to determine the statistics.

How to calculate the mean and standard deviation

Using the TI-Nspire:

1 Enter the data into a column called **x**.



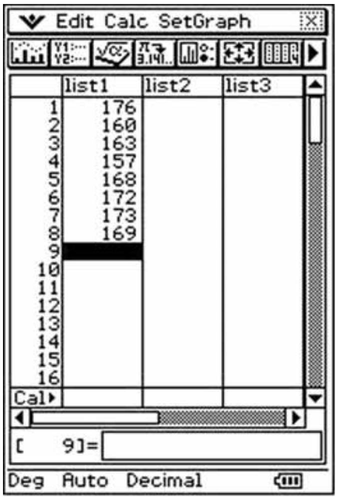
	A x	B	C	D	E	F
◆						
5	168					
6	172					
7	173					
8	169					
9						

2 Highlight the column and press menu.

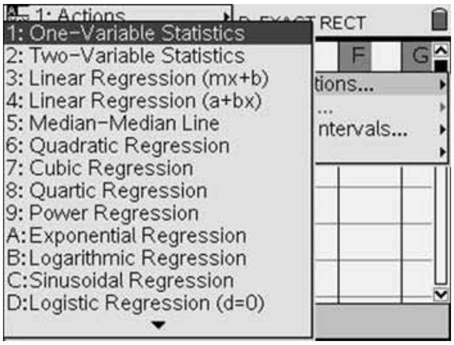
3 Navigate as follows:
Statistics ⟶ *Stat Calculations* ⟶ *One-Variable Statistics*.

Using the ClassPad:

1 Enter the data into list1.



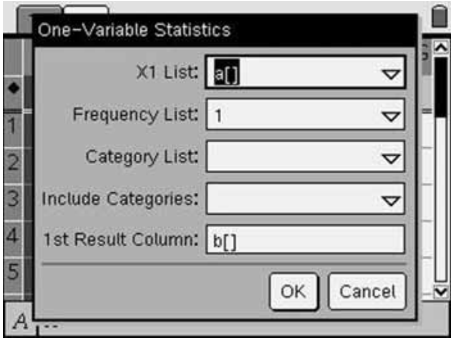
	list1	list2	list3
1	176		
2	160		
3	163		
4	157		
5	168		
6	172		
7	173		
8	169		
9			
10			
11			
12			
13			
14			
15			
16			



4 Press (enter).

5 When prompted with 'Num of Lists:' type **1** then press OK.

6 Set the following and then press OK.


One−Variable Statistics

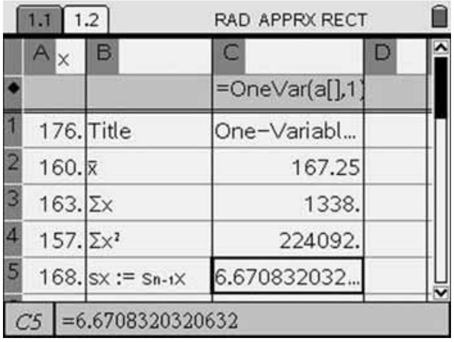
X1 List: a[]

Frequency List: 1

Category List:

Include Categories:

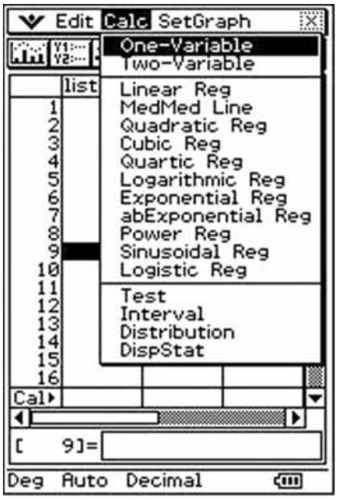
1st Result Column: b[]



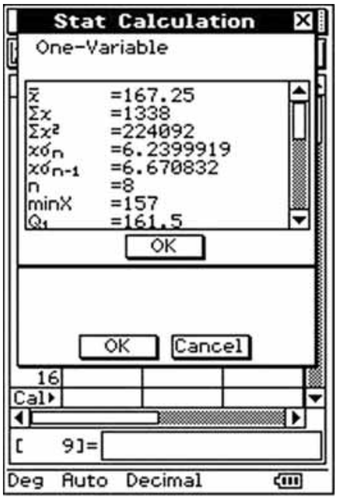
	A x	B	C	D
◆			=OneVar(a[],1	
1	176.	Title	One−Variabl...	
2	160.	x̄	167.25	
3	163.	Σx	1338.	
4	157.	Σx²	224092.	
5	168.	sx := sn-1x	6.670832032...	

C5 =6.6708320320632

2 In the Calc menu, select *One-Variable*.



3 Set XList: list1 and Freq: 1 and then tap OK.


Stat Calculation

One−Variable

```
x̄      =167.25
Σx     =1338
Σx²    =224092
xσn    =6.2399919
xσn-1  =6.670832
n      =8
minX   =157
Q1     =161.5
```

Thus, the mean is 167.25 and the standard deviation is 6.67.

5.10 Extension: Bivariate data

When two variables are observed for each subject, **bivariate** data are obtained. For example, it might be interesting to record the number of hours spent studying for an exam by each student in a class and the mark they achieved in the exam. If each of these variables were considered separately the methods discussed earlier would be used. It may be of more interest to examine the relationship between the two variables, in which case new bivariate techniques are required. When exploring bivariate data, questions arise such as: 'Is there a relationship between two variables?' or 'Does knowing the value of one of the variables tell us anything about the value of the other variable?'

Consider the relationship between the number of cigarettes smoked per day and blood pressure. Since one opinion might be that varying the number of cigarettes smoked may affect blood pressure, it is necessary to distinguish between blood pressure, which is called the **dependent** or **response** variable, and the number of cigarettes, which is called the **independent** or **explanatory** variable.

Displaying bivariate data

As with data concerning one variable, the most important first step in analysing bivariate data is the construction of a visual display. When both of the variables of interest are numerical then a scatterplot (or bivariate plot) may be constructed. This is the single most important tool in the analysis of such bivariate data, and should always be examined before further analysis is undertaken. The pairs of data points are plotted on the Cartesian plane, with each pair contributing one point to the plot. Using the normal convention, the variable plotted horizontally is denoted as x, and the variable plotted vertically as y. The next example examines the features of the scatterplot in more detail.

Example 20

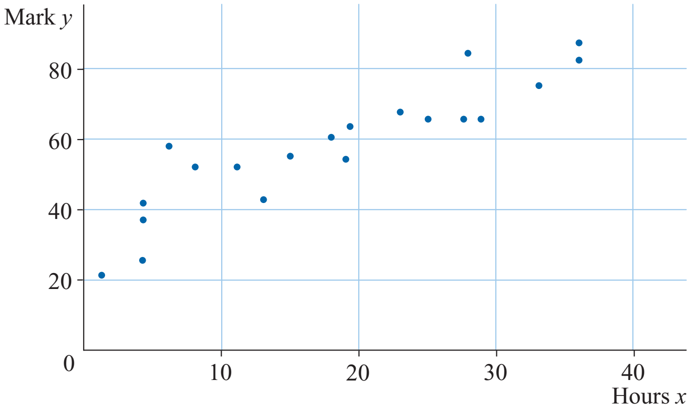
The number of hours spent studying for an examination by each member of a class, and the marks they were awarded, are given in the table.

Student	1	2	3	4	5	6	7	8	9	10
Hours	4	36	23	28	25	11	18	13	4	8
Mark	27	87	67	84	66	52	61	43	38	52

Student	11	12	13	14	15	16	17	18	19	20
Hours	4	19	6	19	1	29	33	36	28	15
Mark	41	54	57	62	23	65	75	83	65	55

Construct a scatterplot of these data.

Solution



Note: 'Hours' is treated as the independent variable and 'Marks' as the dependent variable because it is assumed that the mark achieved will depend on the number of hours worked.

From this scatterplot, a general trend can be seen of increasing marks with increasing hours of study. There is said to be a positive association between the variables.

> Two variables are positively associated when larger values of y are associated with larger values of x, as shown in the previous scatterplot.

Examples of variables that exhibit positive association are height and weight, foot size and hand size, and number of people in the family and household expenditure on food.

Example 21

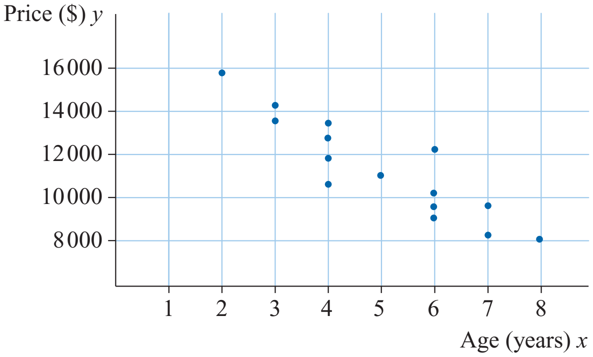
The age, in years, of several cars and their advertised price in a newspaper are given in the following table:

Age (years)	4	6	5	7	4	2	3	3
Price ($)	13 000	9 800	11 000	8 300	10 500	15 800	14 300	13 800

Age (years)	7	6	4	6	4	8	6
Price ($)	9 700	9 500	13 200	10 000	11 800	8 000	12 200

Construct a scatterplot to display these data.

Solution



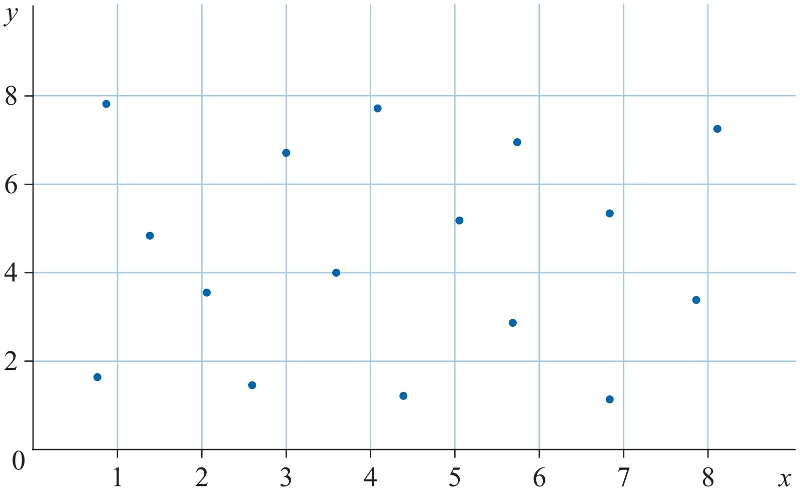
Note: In this case, the independent variable is the age of the car. The dependent variable, price, is plotted on the vertical axis.

From the scatterplot a general trend of decreasing price with increasing age of car can be seen. There is said to be a negative association between the variables.

> Two variables are negatively associated when larger values of y are associated with smaller values of x, as shown in the previous scatterplot.

Examples of other variables that exhibit negative association are weight and number of weeks spent on a healthy eating program, hearing ability and age, and number of cold rainy days per week and sales of ice-creams.

The third alternative is that a scatterplot shows no particular pattern, indicating no association between the variables.

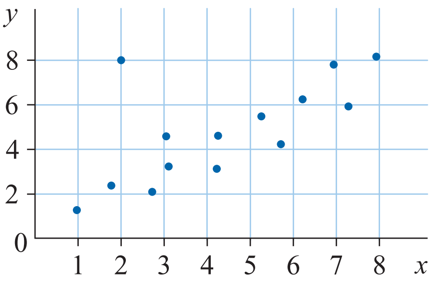


> There is no association between two variables when the values of y are not related to the values of x, as shown in the preceding scatterplot.

Examples of variables that show no association are height and IQ for adults, car price and fuel consumption, and size of family and number of pets.

When one point, or a few points, do not seem to fit with the rest of the data they are called **outliers**. Sometimes a point is an outlier, not because its x value or its y value is in itself unusual, but rather because this particular combination of values is atypical. Consequently such an outlier cannot always be detected from single variable displays, such as stem-and-leaf plots.

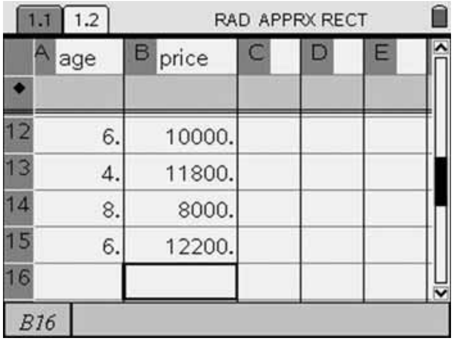
For example, consider this scatterplot. Although the variable plotted on the horizontal axis takes values from 1 to 8 and the variable plotted on the vertical axis takes values from 1 to 8, the combination (2, 8) is clearly an outlier.



Using technology

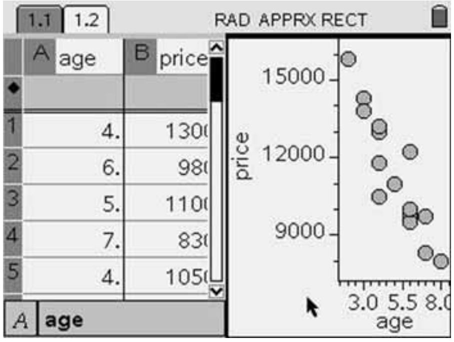
Using the TI-Nspire:

1. Enter the age data into a column called **age**.
2. Enter the price data into a column called **price**.



	A age	B price	C	D	E
◆					
12	6.	10000.			
13	4.	11800.			
14	8.	8000.			
15	6.	12200.			
16					

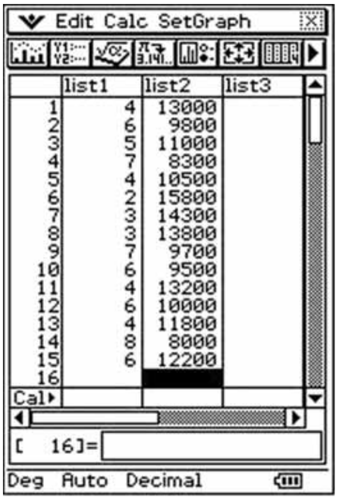
3. Highlight both columns, press (menu) and select *Quick Graph* from the Data submenu.



	A age	B price
◆		
1	4.	130
2	6.	98
3	5.	110
4	7.	83
5	4.	105

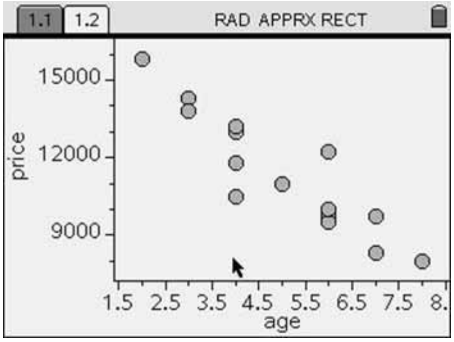
Using the ClassPad:

1. Enter the age data into list1.
2. Enter the price data into list2.

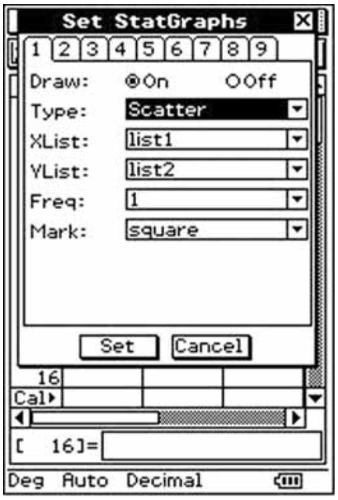


	list1	list2	list3
1	4	13000	
2	6	9800	
3	5	11000	
4	7	8300	
5	4	10500	
6	2	15800	
7	3	14300	
8	3	13800	
9	7	9700	
10	6	9500	
11	4	13200	
12	6	10000	
13	4	11800	
14	8	8000	
15	6	12200	
16			

4 Change to a full-screen view.



3 To ensure a scatterplot is drawn, tap SetGraph then tap on Setting. . . Change Type to Scatter and change the XList to list1 and YList to list2. Tap SET to save the changes.



Draw: ◉On ○Off

Type: Scatter

XList: list1

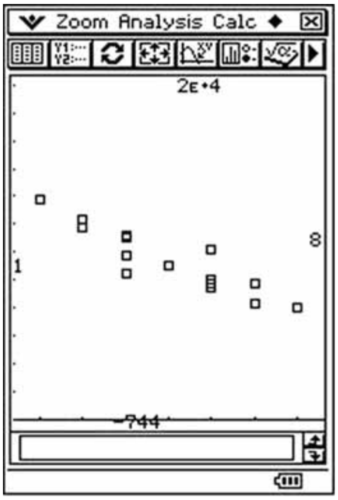
YList: list2

Freq: 1

Mark: square

4 To see the scatterplot, tap 📊.

5 Tap Resize for a full-screen view of the scatterplot.



Exercise 5J

Note: *Save your data for questions **1–4** in named lists as they will be needed for later exercises.*

1 The amount of a particular pain relief drug given to each patient and the time taken for the patient to experience pain relief are shown.

Patient	1	2	3	4	5	6	7	8	9	10
Drug dose (mg)	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	4.0
Response time (min)	65	35	15	10	22	16	10	18	70	20

a Plot the response time against drug dose.

b From the scatterplot, describe any association between the two variables.

c Identify outliers, if any, and interpret.

2 The proprietor of a hairdressing salon recorded the amount spent advertising in the local paper and the business income for each month of a year, with the following results:

Month	Advertising ($)	Business ($)
1	350	9 450
2	450	10 070
3	400	9 380
4	500	9 110
5	250	5 220
6	150	3 100

Month	Advertising ($)	Business ($)
7	350	8 060
8	300	7 030
9	550	11 500
10	600	12 870
11	550	10 560
12	450	9 850

a Plot the business income against the advertising expenditure.

b From the scatterplot, describe any association between the two variables.

c Identify outliers, if any, and interpret.

3 The number of passenger seats on the most commonly used commercial aircraft, and the airspeeds of these aircraft, in km/h, are shown in the following table:

Number of seats	405	296	288	258	240	230	193	188
Airspeed (km/h)	830	797	774	736	757	765	760	718

Number of seats	148	142	131	122	115	112	103	102
Airspeed (km/h)	683	666	661	378	605	620	576	603

a Plot the airspeed against the number of seats.

b From the scatterplot, describe any association between the two variables.

c Identify outliers, if any, and interpret.

4 The price and age of several second-hand caravans are listed in the table.

Age (years)	Price ($)
7	4 800
7	3 900
8	4 275
9	3 900
4	6 900
8	6 500
1	11 400
10	8 700
9	1 950
9	3 300
11	1 650
3	9 600
4	8 400
7	6 600

a Plot the price of the caravans against their age.

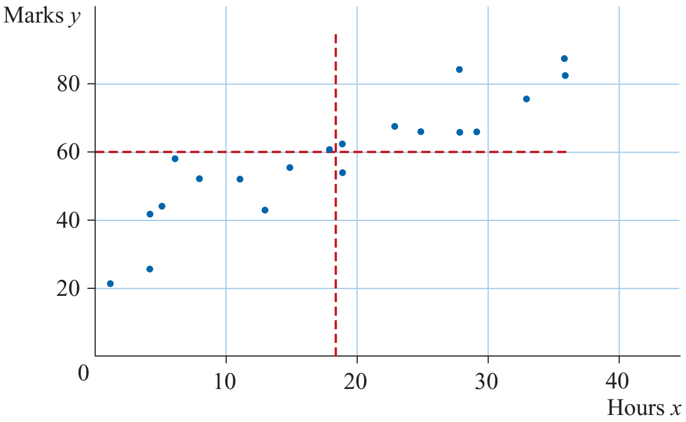
b From the scatterplot, describe any association between the two variables.

c Identify outliers, if any, and interpret.

5.11 Extension: The *q*-correlation coefficient

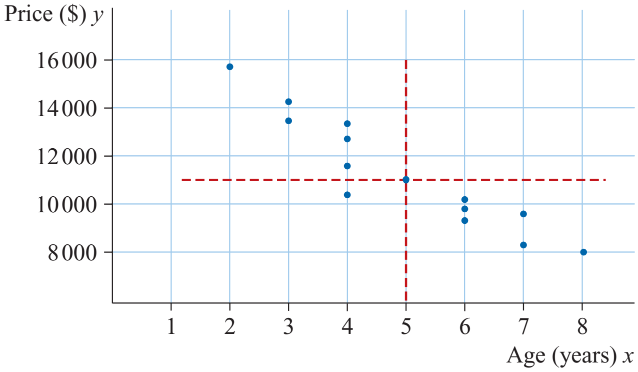
If the plot of a bivariate data set shows a basic trend, apart from some randomness, then it is useful to provide a numerical measure of the strength of the relationship between the two variables. **Correlation** is a measure of strength of a relationship that applies only to numerical variables. Thus it is sensible, for example, to calculate the correlation between the heights and weights for a group of students, but not between height and gender, as gender is not a numerical variable. There are many different numerical measures of correlation, and each has different properties. In this section the ***q*-correlation coefficient** will be introduced.

Consider the scatterplot of the number of hours spent by each member of a class when studying for an examination, and the mark they were awarded, from Example 20. This shows a positive association. To calculate the q-correlation coefficient, first find the median value for each of the variables separately. This can be done from the data, but it is usually simpler to calculate directly from the plot. There are 20 data points, and the median values are halfway between the 10th and 11th points, both vertically and horizontally. A vertical line is then drawn through the median x value, and a horizontal line through the median y value. The effect of this is to divide the plot into four regions, as shown.



Each of the four regions that have been created in this way is called a quadrant, and it can be noticed immediately that most of the points in this plot are in the upper-right and lower-left quadrants. In fact, wherever there is a positive association between variables this will be the case.

Consider the scatterplot of the age of cars and the advertised price from Example 21, which shows a negative association. Again the median value for each of the variables is found separately. There are 15 data points, giving the median values at the 8th points, both vertically and horizontally. A vertical line is then drawn through the median x value, and a horizontal line through the median y value. In this particular case, they are coordinates of the same point, but this need not be so.

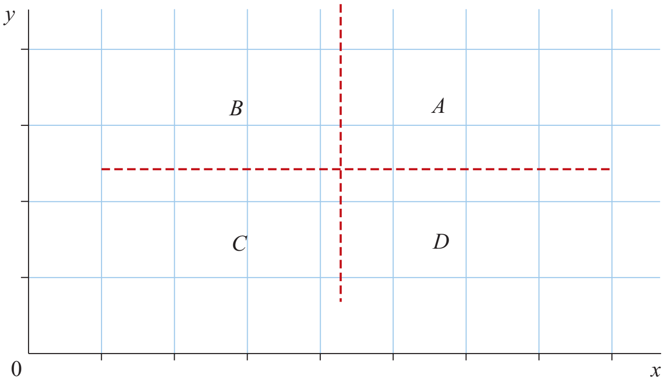


It can be seen in this example that most points are in the upper-left and the lower-right quadrants, and this is true whenever there is a negative association between variables.

These observations lead to a definition of the q-correlation coefficient.

The q-correlation coefficient can be determined from the scatterplot as follows:

- Find the median of all the x values in the data set, and draw a vertical line through this value.
- Find the median of all the y values in the data set, and draw a horizontal line through this value.



- The plane is now divided into four quadrants. Label the quadrants A, B, C and D, as shown in the diagram.
- Count the number of points in each of the quadrants A, B, C and D. Any points that lie on the median lines are omitted.
- Let a, b, c, d represent the number of points in each of the quadrants A, B, C and D, respectively. Then the q-correlation coefficient is given by

$$q = \frac{(a + c) - (b + d)}{a + b + c + d}$$

Example 22

Use the scatterplot from Example 20 to determine the q-correlation coefficient for the number of hours each member of a class spent studying for an examination and the mark they were awarded.

Solution

$$\begin{aligned} q &= \frac{(a+c)-(b+d)}{a+b+c+d} \\ &= \frac{(9+9)-(1+1)}{9+1+9+1} \\ &= \frac{18-2}{20} \\ &= \frac{16}{20} \\ &= 0.8 \end{aligned}$$

Example 23

Use the scatterplot from Example 21 to determine the q-correlation coefficient for the age of cars and their advertised price.

Solution

$$\begin{aligned} q &= \frac{(a+c)-(b+d)}{a+b+c+d} \\ &= \frac{(1+1)-(6+6)}{1+6+1+6} \\ &= \frac{2-12}{14} \\ &= \frac{-10}{14} \\ &= -0.71 \end{aligned}$$

From Examples 22 and 23 it can be seen that q-correlation coefficients may take both positive and negative values. Consider the situation when **all** the points are in the quadrants A and C.

$$\begin{aligned} q &= \frac{(a+c)-(b+d)}{a+b+c+d} \\ &= \frac{a+c}{a+c} \qquad \text{(since } b \text{ and } d \text{ are both equal to zero)} \\ &= 1 \end{aligned}$$

Thus the **maximum** value the q-correlation coefficient may take is 1, and this indicates a measure of strong positive association.

Suppose all the points are in the quadrants B and D.

$$\begin{aligned} q &= \frac{(a+c)-(b+d)}{a+b+c+d} \\ &= \frac{-(b+d)}{b+d} \qquad \text{(since } a \text{ and } c \text{ are both equal to zero)} \\ &= -1 \end{aligned}$$

Thus the **minimum** value the q-correlation coefficient may take is -1, and this indicates a measure of strong negative association.

When the same number of points are in each of the quadrants A, B, C and D then:

$$\begin{aligned} q &= \frac{(a+c)-(b+d)}{a+b+c+d} \\ &= \frac{0}{a+b+c+d} \qquad \text{(since } a = b = c = d\text{)} \\ &= 0 \end{aligned}$$

This value of the q-correlation coefficient clearly indicates that no association exists.

q-correlation coefficients can be classified as follows:

$-1 \leq q \leq -0.75$	strong negative relationship
$-0.75 < q \leq -0.50$	moderate negative relationship
$-0.50 < q \leq -0.25$	weak negative relationship
$-0.25 < q < 0.25$	no relationship
$0.25 \leq q < 0.50$	weak positive relationship
$0.50 \leq q < 0.75$	moderate positive relationship
$0.75 \leq q \leq 1$	strong positive relationship

Exercise 5K

1 Use the table of q-correlation coefficients to classify the following:

a $q = 0.20$ **b** $q = -0.30$ **c** $q = -0.85$ **d** $q = 0.33$

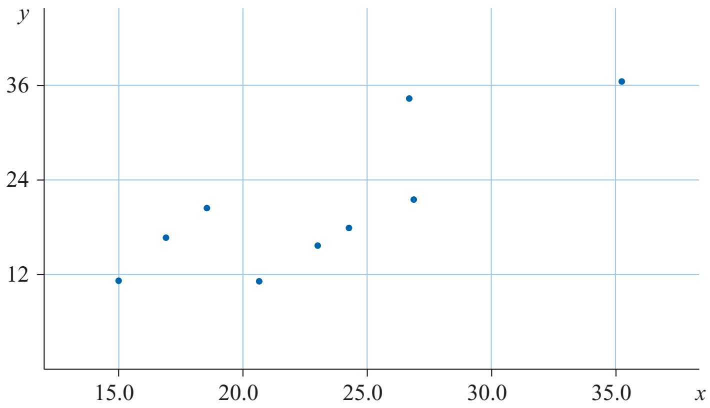
e $q = 0.95$ **f** $q = -0.75$ **g** $q = 0.75$ **h** $q = -0.24$

i $q = -1$ **j** $q = 0.25$ **k** $q = 1$ **l** $q = -0.50$

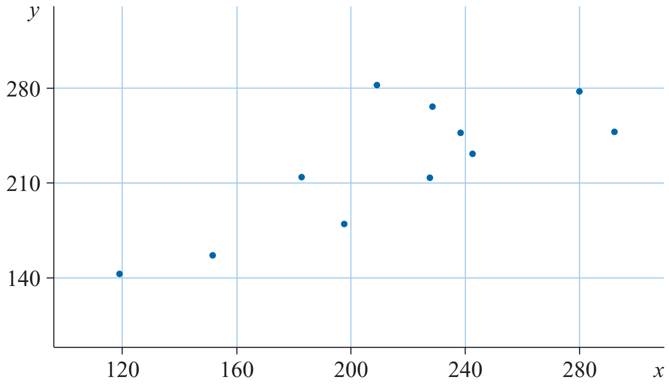
Examples 22, 23

2 Calculate the q-correlation coefficient for each pair of variables shown in the following scatterplots:

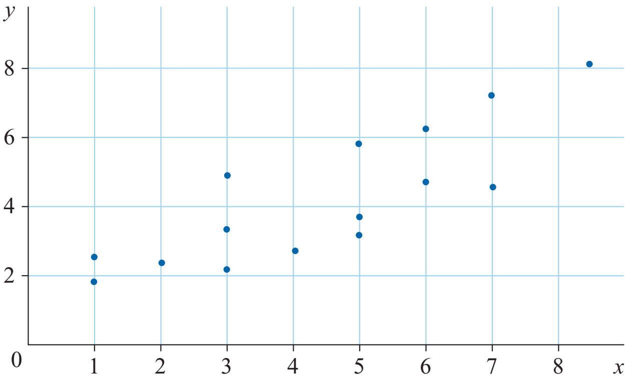
a


y
36
24
12
15.0
20.0
25.0
30.0
35.0
x

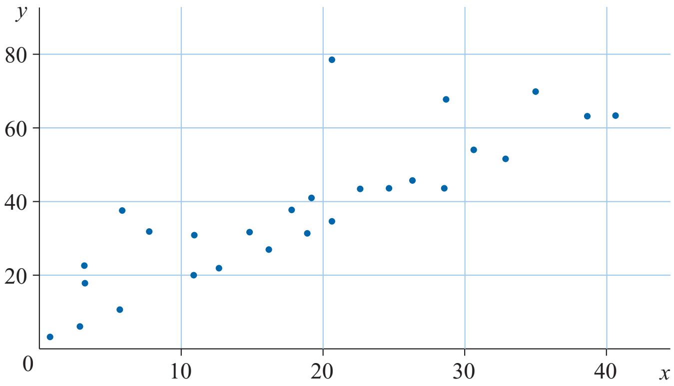
b


y
280
210
140
120
160
200
240
280
x

c


y
8
6
4
2
0
1
2
3
4
5
6
7
8
x

d


y
80
60
40
20
0
10
20
30
40
x

3 The amount of a particular pain relief drug given to each patient and the time taken for the patient to experience pain relief are shown.

Patient	1	2	3	4	5	6	7	8	9	10
Drug dose (mg)	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	4.0
Response time (min)	65	35	15	10	22	16	10	18	70	20

a Use your scatterplot from Question **1**, Exercise **5J** to find the *q*-correlation coefficient for response time and drug dosage.

b Classify the strength and direction of the relationship between response time and drug dosage according to the table given.

4 The proprietor of a hairdressing salon recorded the amount spent advertising in the local paper and the business income for each month of a year, with the following results:

Month	Advertising ($)	Business ($)
1	350	9 450
2	450	10 070
3	400	9 380
4	500	9 110
5	250	5 220
6	150	3 100

Month	Advertising ($)	Business ($)
7	350	8 060
8	300	7 030
9	550	11 500
10	600	12 870
11	550	10 560
12	450	9 850

a Use your scatterplot from Question **2**, Exercise **5J** to find the *q*-correlation coefficient for advertising expenditure and total business conducted.

b Classify the strength and direction of the relationship between advertising expenditure and business income according to the table given.

5 The number of passenger seats on the most commonly used commercial aircraft, and the airspeeds of these aircraft, in km/h, are shown in the following table:

Number of seats	405	296	288	258	240	230	193	188
Airspeed (km/h)	830	797	774	736	757	765	760	718

Number of seats	148	142	131	122	115	112	103	102
Airspeed (km/h)	683	666	661	378	605	620	576	603

a Use your scatterplot from Question **3**, Exercise **5J** to find the *q*-correlation coefficient for the number of seats on an airline and the air speed.

b Classify the strength and direction of the relationship between the number of seats on an airline and the air speed according to the table given.

6 The price and age of several second-hand caravans are listed in the table.

Age (years)	Price ($)
7	4 800
7	3 900
8	4 275
9	3 900
4	6 900
8	6 500
1	11 400

Age (years)	Price ($)
10	8 700
9	1 950
9	3 300
11	1 650
3	9 600
4	8 400
7	6 600

- **a** Use your scatterplot from Question **4**, Exercise **5J** to find the q-correlation coefficient for price and age of second-hand caravans.
- **b** Classify the strength and direction of the relationship between price and age of second-hand caravans according to the table given.

5.12 Extension: The correlation coefficient

When a relationship is linear the most commonly used measure of strength of the relationship is Pearson's product-moment correlation coefficient, r. It gives a numerical measure of the degree to which the points in the scatterplot tend to cluster around a straight line.

Pearson's product-moment correlation is defined to be

$$r = \frac{\text{Degree that the variables vary together}}{\text{Degree that the two variables vary separately}}$$

Formally, if we call the two variables x and y and we have n observations, then Pearson's product-moment correlation for this set of observations is

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$$

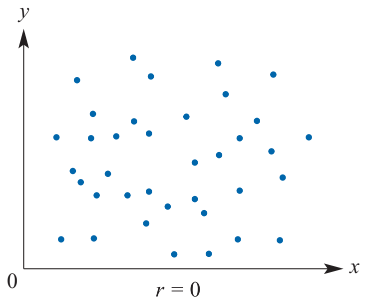
where $\bar{x}$ and s_x are the mean and standard deviation of the x scores, and $\bar{y}$ and s_y are the mean and standard deviation of the y scores.

There are two key assumptions made in calculating Pearson's correlation coefficient, r. They are:

- the data are **numerical**
- the relationship being described is **linear**.

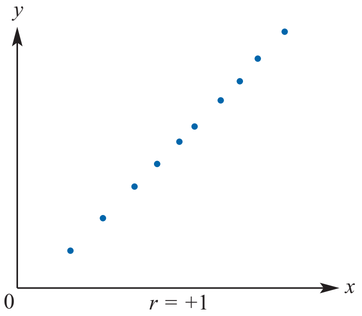
Pearson's correlation coefficient, r, has the following properties:

If there is **no linear** relationship, $r = 0$.

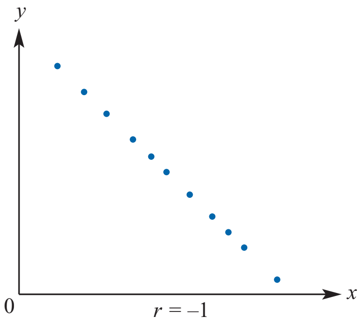


$r = 0$

For a **perfect positive linear** relationship, $r = +1$.



For a **perfect negative linear** relationship, $r = -1$.

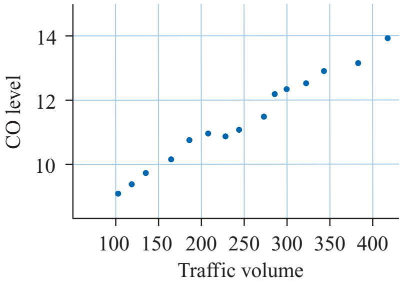


Otherwise, $-1 \leq r \leq +1$.

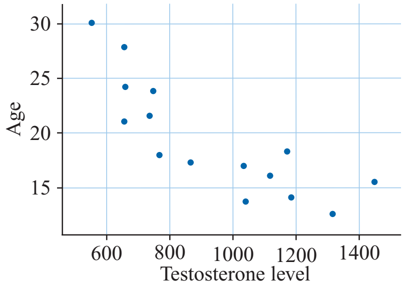
Pearson's correlation coefficient, *r*, can be classified as follows:

$-1 \leq r \leq -0.75$	strong negative linear relationship
$-0.75 \leq r \leq -0.50$	moderate negative linear relationship
$-0.50 \leq r \leq -0.25$	weak negative linear relationship
$-0.25 < r < 0.25$	no linear relationship
$0.25 \leq r < 0.50$	weak positive linear relationship
$0.50 \leq r < 0.75$	moderate positive linear relationship
$0.75 \leq r \leq 1$	strong positive linear relationship

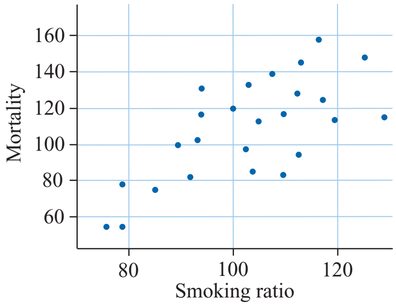
These scatterplots show linear relationships of various strengths together with the corresponding value of Pearson's product-moment correlation coefficient.



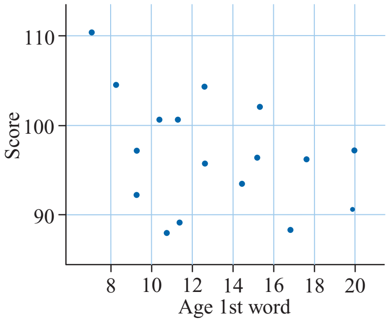
Carbon monoxide level in the atmosphere and traffic volume: $r = 0.985$



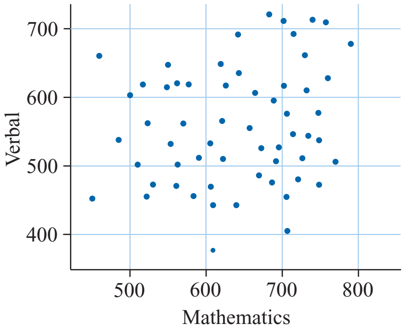
Age first convicted and testosterone (a male hormone) level of a group of prisoners: $r = -0.814$



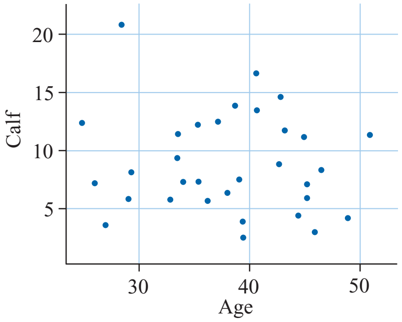
Mortality rate due to lung cancer and smoking ratio (100 average): $r = 0.716$



Score on aptitude test (taken later in life) and age (in months) when first word spoken: $r = -0.445$



Scores on standardised tests of verbal and mathematical ability: $r = 0.275$

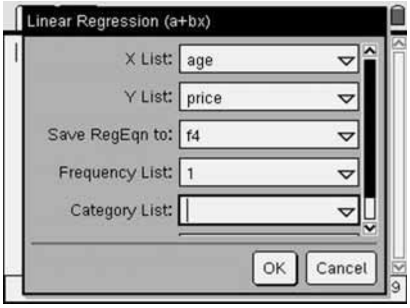


Calf measurement and age of adult males: $r = -0.005$

Using technology

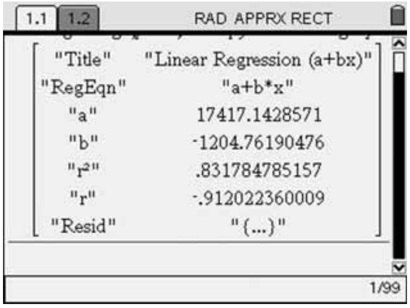
Using the TI-Nspire:

1. Ensure the data from Example 23 has been entered into two columns named **age** and **price**.
2. In the Calculator Application, press (menu) and enter into the Statistics menu. Now select *Linear Regression* $(a + bx)$ from the Stat Calculations submenu.
3. Set the following:



Linear Regression (a+bx)
X List: age
Y List: price
Save RegEqn to: f4
Frequency List: 1
Category List:

and press OK.



```
"Title"    "Linear Regression (a+bx)"
"RegEqn"   "a+b*x"
"a"        17417.1428571
"b"        -1204.76190476
"r²"       .831784785157
"r"        -.912022360009
"Resid"    "{...}"
```

Using the ClassPad:

1. Ensure the data from Example 23 has been entered into list1 and list2.
2. Tap Calc and select *Linear Reg*.
3. Set the following:

Set Calculation
Linear Reg
XList: list1
YList: list2
Freq: 1
Copy Formula: Off
Copy Residual: Off

and tap OK.

```
Stat Calculation
Linear Reg
y=a·x+b
a    =-1204.761
b    =17417.142
r    =-0.912022
r²   =0.8317847
MSe  =948336.99
```

After the mean and standard deviation, Pearson's product-moment correlation is one of the most frequently computed descriptive statistics. It is a powerful tool but it is also easily misused. The presence of a linear relationship should always be confirmed with a scatterplot before Pearson's product-moment correlation is calculated. And, like the mean and the standard deviation, Pearson's correlation coefficient, r, is very sensitive to the presence of outliers in the sample.

Exercise 5L

1 Use the table of Pearson's correlation coefficients, r, to classify the following:

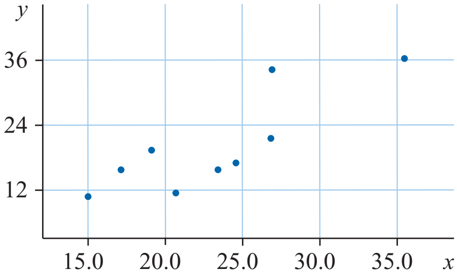
a $r = 0.20$ **b** $r = -0.30$ **c** $r = -0.85$ **d** $r = 0.33$

e $r = 0.95$ **f** $r = -0.75$ **g** $r = 0.75$ **h** $r = -0.24$

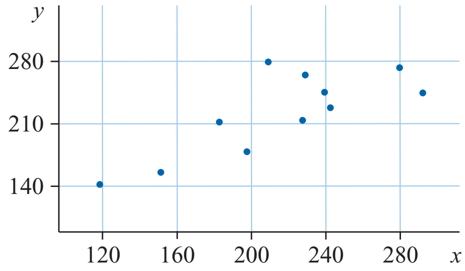
i $r = -0.50$ **j** $r = 0.25$ **k** $r = 1$ **l** $r = -1$

2 By comparing the plots given to those on pages 268–269 estimate the value of Pearson's correlation coefficient, r.

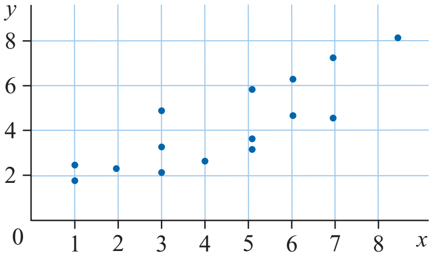
a



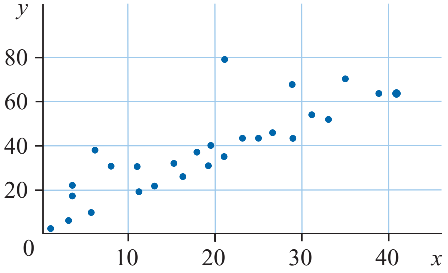
b



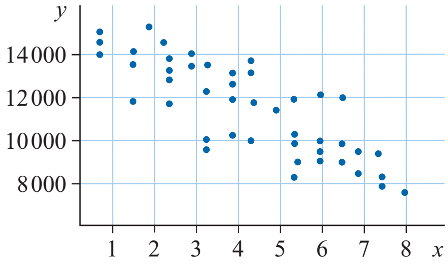
c



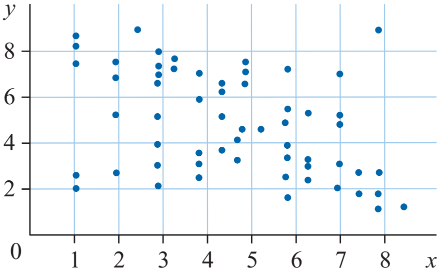
d



e



f



3 The amount of a particular pain relief drug given to each patient and the time taken for the patient to experience relief are shown.

Patient	1	2	3	4	5	6	7	8	9	10
Drug dose (mg)	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	4.0
Response time (min)	65	35	15	10	22	16	10	18	70	20

a Determine the value of Pearson's correlation coefficient.

b Classify the relationship between drug dose and response time according to the table given.

4 The proprietor of a hairdressing salon recorded the amount spent on advertising in the local paper and the business income for each month for a year, with the following results:

Month	Advertising ($)	Business ($)
1	350	9 450
2	450	10 070
3	400	9 380
4	500	9 110
5	250	5 220
6	150	3 100

Month	Advertising ($)	Business ($)
7	350	8 060
8	300	7 030
9	550	11 500
10	600	12 870
11	550	10 560
12	450	9 850

a Determine the value of Pearson's correlation coefficient.

b Classify the relationship between the amount spent on advertising and business income according to the table given.

5 The number of passenger seats on the most commonly used commercial aircraft, and the airspeeds of these aircraft, in km/h, are shown in the following table:

Number of seats	405	296	288	258	240	230	193	188
Airspeed (km/h)	830	797	774	736	757	765	760	718

Number of seats	148	142	131	122	115	112	103	102
Airspeed (km/h)	683	666	661	378	605	620	576	603

a Determine the value of Pearson's correlation coefficient.

b Classify the relationship between the number of passenger seats and airspeed according to the table given.

6 The price and age of several second-hand caravans are listed in the table.

Age (years)	Price ($)
7	4 800
7	3 900
8	4 275
9	3 900
4	6 900
8	6 500
1	11 400

Age (years)	Price ($)
10	8 700
9	1 950
9	3 300
11	1 650
3	9 600
4	8 400
7	6 600

a Determine the value of Pearson's correlation coefficient.

b Classify the relationship between price and age according to the table given.

7 Given are the scores for a group of 12 students who each had two attempts at a test (out of 70).

Attempt 1	53	56	57	49	44	69	66	40	53	43	68	64
Attempt 2	63	66	67	58	54	70	70	55	63	53	70	70

a Construct a scatterplot of these data, and describe the relationship between scores on Attempt 1 and Attempt 2.

b Is it appropriate to calculate the value of Pearson's correlation coefficient for these data? Give reasons for your answer.

8 This table represents the results of two different tests for a group of students.

Student	Test 1	Test 2
1	214	216
2	281	270
3	212	281
4	324	326
5	240	243
6	208	213
7	303	311
8	278	290
9	311	320

a Construct a scatterplot of these data, and describe the relationship between scores on Test 1 and Test 2.

b Is it appropriate to calculate the value of Pearson's correlation coefficient for these data? Give reasons for your answer.

c Determine the values of the q-correlation coefficient and Pearson's correlation coefficient, r.

d Classify the relationship between Test 1 and Test 2, using both the q-correlation coefficient and Pearson's correlation coefficient, r, and compare.

e It turns out that when the data were entered into the student records, the result for Test 2, Student 9 was entered as 32 instead of 320.

- **i** Re-calculate the values of the q-correlation coefficient and Pearson's correlation coefficient, r, with this new data value.
- **ii** Compare these values with the ones calculated in part **c**.

5.13 Extension: Lines on scatterplots

If a linear relationship exists between two variables it is possible to predict the value of the dependent variable from the value of the independent variable. The stronger the relationship between the two variables the better the prediction that is made. To make the prediction it is necessary to determine an equation that relates the variables and this is achieved by fitting a line to the data. Fitting a line to data is often referred to as **regression**, which comes from a Latin word *regressum* that means 'moved back'.

The simplest equation relating two variables x and y is a linear equation of the form

$$y = a + bx$$

where a and b are constants. This is similar to the general equation of a straight line, where a represents the coordinate of the point where the line crosses the y-axis (the y-axis intercept), and b represents the slope of the line. In order to summarise any particular (x, y) data set, numerical values for a and b are needed that will make the line pass close to the data. There are several ways in which the values of a and b can be found, of which the simplest is to find the straight line by placing a ruler on the scatter diagram and drawing a line **by eye** that appears to follow the general trend of the data.

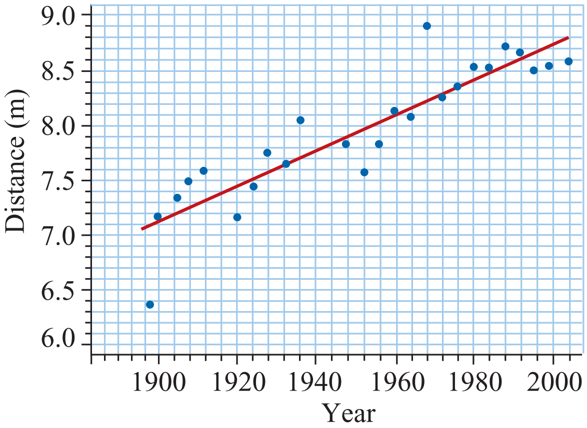
Example 24

The table below gives the gold medal winning distance, in metres, for the men's long jump for the Olympic games for the years 1896 to 1996. (Some years were missing owing to the two world wars.)

Find a straight line that fits the general trend of the data, and use it to predict the winning distance in the year 2008.

Year	1896	1900	1904	1908	1912	1920	1924	1928	1932	1936	1948	1952	1956
Distance (m)	6.35	7.19	7.34	7.49	7.59	7.16	7.44	7.75	7.65	8.05	7.82	7.57	7.82
Year	1960	1964	1968	1972	1976	1980	1984	1988	1992	1996	2000	2004	
Distance (m)	8.13	8.08	8.92	8.26	8.36	8.53	8.53	8.72	8.67	8.50	8.55	8.59	

Solution



Note that this scatterplot does not start at the origin. Since the values of the coordinates that are of interest on both axes are a long way from zero, it is sensible to plot the graph for that range of values only. In fact, any values before 1896 on the horizontal axis are meaningless in this context.

The line shown on the scatterplot is only one of many that could be drawn. To enable the line to be used for prediction it is necessary to find its equation. To do this, first determine the coordinates of any two points through which it passes on the scatterplot. Appropriate points are (1932, 7.65) and (1976, 8.36). The equation of the straight line is then found by substituting the formula that gives the equation for a straight line between two points.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8.36 - 7.65}{1976 - 1932} \\ &= \frac{0.71}{44} \approx 0.016 \\ y - y_1 &= m(x - x_1) \\ y - 7.65 &= 0.016(x - 1932) \\ y &= 0.016x - 23.26 \\ \text{or distance} &= -23.26 + 0.016 \times \text{year} \end{aligned}$$

The intercept for this equation is −23.26 m. In theory, this is the winning distance for the year 0! In practice, there is no meaningful interpretation for the y-axis intercept in this situation. But the same cannot be said about the slope. A slope of 0.016 means that, on average, the gold medal winning distance increases by about 1.6 cm at each successive games.

Using this equation the gold medal winning distance for the long jump in 2008 would be predicted as

$$y = -23.26 + 0.016 \times 2008 \approx 8.87 \text{ m}$$

Obviously, attempting to project too far into the future may give us answers that are not sensible. When using an equation for prediction, derived from data, it is sensible to use values of the explanatory variable that are within a reasonable range of the data. The relationship between the variables may not be linear if we move too far from the known values.

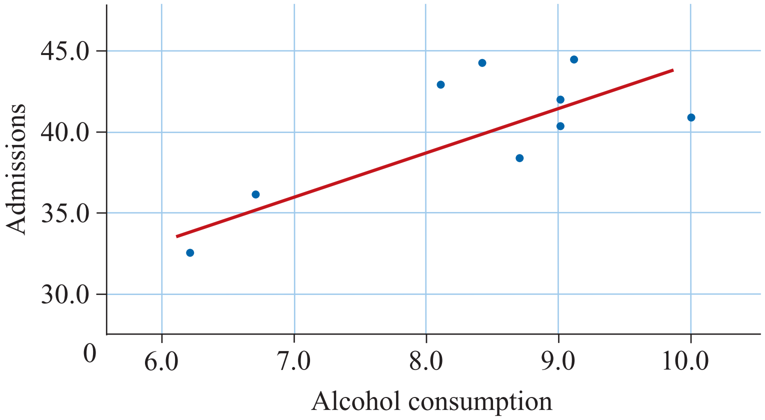
Example 25

This table gives the alcohol consumption per head (in litres) and the hospital admission rate to each of the regions of Victoria in 1994–95.

Region	Per capita consumption (litres of alcohol)	Hospital admissions per 1000 residents
Loddon–Mallee	9.0	42.0
Grampians	8.4	44.7
Barwon	8.7	38.6
Gippsland	9.1	44.7
Hume	10.0	41.0
Western Metropolitan	9.0	40.4
Northern Metropolitan	6.7	36.2
Eastern Metropolitan	6.2	32.3
Southern Metropolitan	8.1	43.0

Find a straight line that fits the general trend of the data, and interpret the intercept and slope.

Solution



One possible line passes through the points (7, 36) and (9, 42).

$$
\begin{aligned}
\text{Thus} \qquad m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{42 - 36}{9 - 7} \\
&= \frac{6}{2} \\
&= 3 \\
y - 36 &= 3(x - 7) \\
y &= 3x + 15
\end{aligned}
$$

or admission rate $= 15 + 3 \times$ alcohol consumption

The intercept for this equation is 15, implying that we predict a hospital admission rate would be 15 per 1000 residents for a region with 0 alcohol consumption. Although this is interpretable, it would be a brave prediction, as it is well out of the range of the data. A slope of 3 means that, on average, the admission rate rises by 3 per 1000 residents for each additional litre of alcohol consumed per capita.

Exercise 5M

Example 24 **1** Plot the following set of data points on graph paper:

x	0	1	2	3	4	5	6	7	8
y	1	3	6	7	7	11	13	18	17

Draw a straight line that fits the data by eye, and find an equation for this line.

2 Plot the following set of points on graph paper:

x	−3	−2	−1	0	1	2	3	4
y	5	2	0	−6	−7	−11	−13	−20

Draw a straight line that fits the data by eye, and find an equation for this line.

Example 25

3 The numbers of burglaries during two successive years for various districts in one State are given in the table.

a Make a scatterplot of the data.

b Find the equation of a straight line that relates the two variables.

c Describe the trend in burglaries in this State.

District	Year 1 (x)	Year 2 (y)
A	3233	2709
B	2363	2208
C	4591	3685
D	4317	4038
E	2474	2792
F	3679	3292
G	5016	4402
H	6234	5147
I	6350	5555
J	4072	4004
K	2137	1980

4 The data below give a girl's height (in cm) between the ages of 36 months and 60 months.

Age (x)	36	40	44	52	56	60
Height (y)	84	87	90	92	94	96

a Make a scatterplot of the data.

b Find the equation of a straight line that relates the two variables.

c Interpret the intercept and slope, if appropriate.

d Use your equation to estimate the girl's height at age:

i 42 months **ii** 18 years

e How reliable are your answers to part **d**?

5 The following table gives the adult heights (in cm) of ten pairs of mothers and daughters:

Mother (x)	170	163	157	165	175	160	164	168	152	173
Daughter (y)	178	175	165	173	168	152	163	168	160	178

a Make a scatterplot of the data.

b Find the equation of a straight line that relates the two variables.

c Estimate the adult height of a girl whose mother is 170 cm tall.

6 The manager of a company that manufactures MP3 players keeps a weekly record of the cost of running the business and the number of units produced. The figures for a period of eight weeks are shown in the table.

Number of MP3 players produced (x)	100	160	80	100	220	150	170	200
Cost in \$000s (y)	2.5	3.3	2.4	2.6	4.1	3.1	3.5	3.8

a Make a scatterplot of the data.

b Find the equation of a straight line that relates the two variables.

c What is the manufacturer's fixed cost for operating the business each week?

d What is the cost of production of each unit, over and above this fixed operating cost?

7 The amount of a particular pain relief drug given to each patient and the time taken for the patient to experience pain relief are shown.

Patient	1	2	3	4	5	6	7	8	9	10
Drug dose (mg)	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	4.0
Response time (min)	65	35	15	10	22	16	10	18	70	20

a Find the equation of a straight line that relates the two variables.

b Interpret the intercept and slope if appropriate.

c Use your equation to predict the time taken for the patient to experience pain relief if 6 mg of the drug is given. Is this answer realistic?

8 The proprietor of a hairdressing salon recorded the amount spent on advertising in the local paper and the business income for each month for a year, with the results shown.

a Find the equation of a straight line that relates the two variables.

b Interpret the intercept and slope, if appropriate.

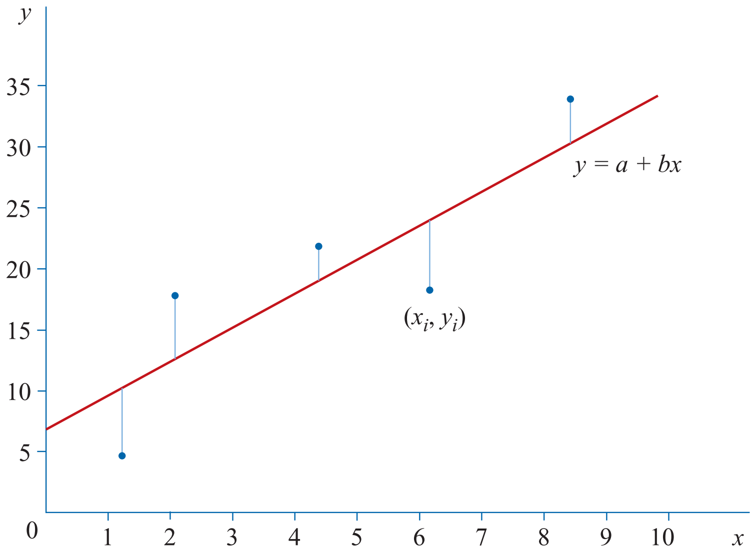
c Use your equation to predict the business income that would be attracted if the proprietor of the salon spent the following amounts on advertising:

i \$1000 **ii** \$0

Month	Advertising (\$)	Business (\$)
1	350	9 450
2	450	10 070
3	400	9 380
4	500	9 110
5	250	5 220
6	150	3 100
7	350	8 060
8	300	7 030
9	550	11 500
10	600	12 870
11	550	10 560
12	450	9 850

5.14 Extension: The least squares regression line

Fitting a line to a scatterplot by eye is not generally the best way of modelling a relationship. What is required is a method that uses a more objective criterion. A simple method is to divide the data set into two halves on the basis of the median *x* value, and to fit a line that passes through the mean *x* and *y* values of the lower half, and the mean *x* and *y* values of the upper half. This is called the two-mean line but, although easy to determine, it is not very widely used. The most common procedure is the method of least squares. The **least squares regression line** is the line for which the sum of squares of the vertical deviations from the data to the line (as indicated in the diagram) is a minimum. These deviations are called the **residuals**.



Consider the line $y = a + bx$.

We would like to find a and b such that the sum of the residuals is zero. That is,

$$\sum_{i=1}^{n}(y_i - a - bx_i) = 0 \qquad \textbf{(1)}$$

and the sum of the residuals squared is as small as possible. That is,

$$\sum_{i=1}^{n}(y_i - a - bx_i)^2 \text{ is a minimum} \qquad \textbf{(2)}$$

We will use the symbol S to denote $\sum_{i=1}^{n}(y_i - a - bx_i)^2$.

From equation **1**,

$$\begin{aligned} &\sum_{i=1}^{n}(y_i - a - bx_i) = 0 && \text{(expand)} \\ \therefore\ &\sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i = 0 && (\div\, n) \\ \therefore\ &\bar{y} - a - b\bar{x} = 0 \\ \therefore\ &a = \bar{y} - b\bar{x} && \textbf{(3)} \end{aligned}$$

Substituting this relationship into equation **2**:

$$\begin{aligned} S &= \sum_{i=1}^{n}[y_i - (\bar{y} - b\bar{x}) - bx_i]^2 \\ &= \sum_{i=1}^{n}[(y_i - \bar{y}) - b(x_i - \bar{x})]^2 \\ &= \sum_{i=1}^{n}[(y_i - \bar{y})^2 - 2b(x_i - \bar{x})(y_i - \bar{y}) + b^2(x_i - \bar{x})^2] \end{aligned}$$

This can be thought of as a quadratic expression in b.

In order to find the value of b that minimises S, we will differentiate with respect to b and set the derivative equal to zero. (See Chapter 9 for differentiation.)

$$\begin{aligned}\frac{dS}{db} &= -2\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) + 2b\sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= 0\end{aligned}$$

Simplifying gives

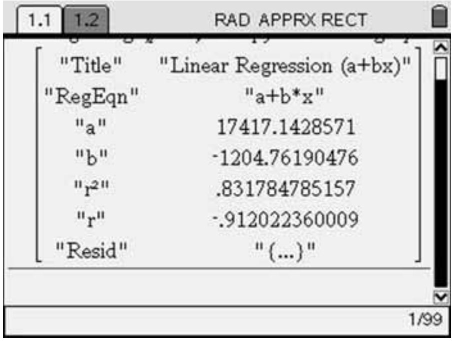
$$b = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \textbf{(4)}$$

Equations **3** and **4** can then be used to calculate the least squares estimates of the y-axis intercept and the slope.

Using technology

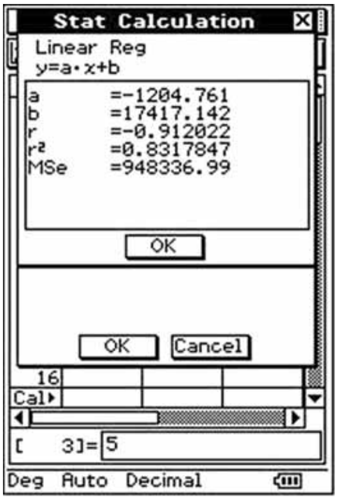
The values of the intercept and slope and the value of the Pearson product-moment correlation coefficient, r, can be determined at the same time.

Using the TI-Nspire:
Once the data has been entered into two **named** columns, select *Linear Regression* ($a + bx$) from the Statistics menu in the calculator application.



```
"Title"    "Linear Regression (a+bx)"
"RegEqn"   "a+b*x"
"a"        17417.1428571
"b"        -1204.76190476
"r²"       .831784785157
"r"        -.912022360009
"Resid"    "{...}"
```

Using the ClassPad:
Once the data has been entered into list1 and list2, select *Linear Reg* from the Calc menu and ensure XList is set to list1 and YList is set to list2.



```
Linear Reg
y=a·x+b
a     =-1204.761
b     =17417.142
r     =-0.912022
r²    =0.8317847
MSe   =948336.99
[   3]=5
```

After the equation of the least squares line has been determined, we can interpret the intercept and slope in terms of the problem at hand, and use the equation to make predictions. The method of least squares is also sensitive to any outliers in the data.

Example 26

Consider again the gold medal winning distance, in metres, for the men's long jump for the Olympic games for the years 1896 to 2004.

Find the equation of the least squares regression line for these data, and use it to predict the winning distance for the year 2008.

Year	1896	1900	1904	1908	1912	1920	1924	1928	1932	1936	1948	1952	1956
Distance (m)	6.35	7.19	7.34	7.49	7.59	7.16	7.44	7.75	7.65	8.05	7.82	7.57	7.82
Year	1960	1964	1968	1972	1976	1980	1984	1988	1992	1996	2000	2004	
Distance (m)	8.13	8.08	8.92	8.26	8.36	8.53	8.53	8.72	8.67	8.50	8.55	8.59	

Solution

Using a calculator or computer, the equation is found to be

$$\begin{aligned}\text{Distance} &= -23.87 + 0.0163 \times \text{year} \\ \text{Distance} &= -23.87 + 0.0163 \times 2008 \approx 8.86\text{ m}\end{aligned}$$

The predicted distance for the year 2008 is approximately 8.86 m.

Example 27

Consider again the data from Example 25, which related alcohol consumption per head (in litres) and the hospital admission rate to each of the regions of Victoria in 1994–95.

Find the equation of the least squares regression line that fits these data.

Region	Per capita consumption (litres of alcohol)	Hospital admissions per 1000 residents
Loddon–Mallee	9.0	42.0
Grampians	8.4	44.7
Barwon	8.7	38.6
Gippsland	9.1	44.7
Hume	10.0	41.0
Western Metropolitan	9.0	40.4
Northern Metropolitan	6.7	36.2
Eastern Metropolitan	6.2	32.3
Southern Metropolitan	8.1	43.0

Solution

Using a calculator or computer, the equation is found to be

$$\text{Admissions} = 19.9 + 2.45 \times \text{alcohol}$$

Correlation and causation

The existence of even a strong linear relationship between two variables is not, in itself, sufficient to imply that altering one variable *causes* a change in the other. It implies only that this *might* be the explanation. It may be that both the measured variables are affected by a third and different variable. For example, if data about crime rates and unemployment in a range of cities were gathered, a high correlation would be found. But could it be inferred that high unemployment causes a high crime rate? The explanation could be that both of these variables are dependent on other factors, such as home circumstances, peer group pressure, level of education or economic conditions, all of which may be related to both unemployment and crime rates. These two variables may vary together, without one being the direct cause of the other.

Exercise 5N

Example 26 **1** The following data give a girl's height (in cm) between the ages of 36 months and 60 months:

Age (x)	36	40	44	52	56	60
Height (y)	84	87	90	92	94	96

a Using the method of least squares, find the equation of a straight line that relates the two variables.

b Interpret the intercept and slope, if appropriate.

c Use your equation to estimate the girl's height at age:

i 42 months **ii** 18 years

d How reliable are your answers to part **c**?

2 The number of burglaries during two successive years for various districts in one State are given in the table.

Using the method of least squares, find the equation of a straight line that relates the two variables.

District	Year 1 (x)	Year 2 (y)
A	3 233	2 709
B	2 363	2 208
C	4 591	3 685
D	4 317	4 038
E	2 474	2 792
F	3 679	3 292
G	5 016	4 402
H	6 234	5 147
I	6 350	5 555
J	4 072	4 004
K	2 137	1 980

Example 27 **3** The following table gives the adult heights (in cm) of ten pairs of mothers and daughters:

Mother (x)	170	163	157	165	175	160	164	168	152	173
Daughter (y)	178	175	165	173	168	152	163	168	160	178

Extension

a Using the method of least squares, find the equation of a straight line that relates the two variables.

b Interpret the slope in this context.

c Estimate the adult height of a girl whose mother is 170 cm tall.

4 The manager of a company that manufactures MP3 players keeps a weekly record of the cost of running the business and the number of units produced. The figures for a period of eight weeks are:

Number of MP3 players produced (x)	100	160	80	100	220	150	170	200
Cost in \$000s (y)	2.5	3.3	2.4	2.6	4.1	3.1	3.5	3.8

a Using the method of least squares, find the equation of a straight line that relates the two variables.

b What is the manufacture's fixed cost for operating the business each week?

c What is the cost of production of each unit, over and above this fixed operating cost?

5 The amount of a particular pain relief drug given to each patient and the time taken for the patient to experience pain relief are shown.

Patient	1	2	3	4	5	6	7	8	9	10
Drug dose (mg)	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	4.0
Response time (min)	65	35	15	10	22	16	10	18	70	20

a Using the method of least squares, find the equation of a straight line that relates the two variables.

b Interpret the intercept and slope if appropriate.

c Use your equation to predict the time taken for the patient to experience pain relief if 6 mg of the drug is given. Is this answer realistic?

6 The proprietor of a hairdressing salon recorded the amount spent on advertising in the local paper and the business income for each month for a year, with the results shown.

Month	Advertising (\$)	Business (\$)
1	350	9 450
2	450	10 070
3	400	9 380
4	500	9 110
5	250	5 220
6	150	3 100
7	350	8 060
8	300	7 030
9	550	11 500
10	600	12 870
11	550	10 560
12	450	9 850

a Using the method of least squares, find the equation of a straight line that relates the two variables.

b Interpret the intercept and slope, if appropriate.

c Use your equation to predict the volume that would be attracted if the proprietor of the salon spent the following amounts on advertising:

i \$1000 **ii** \$0

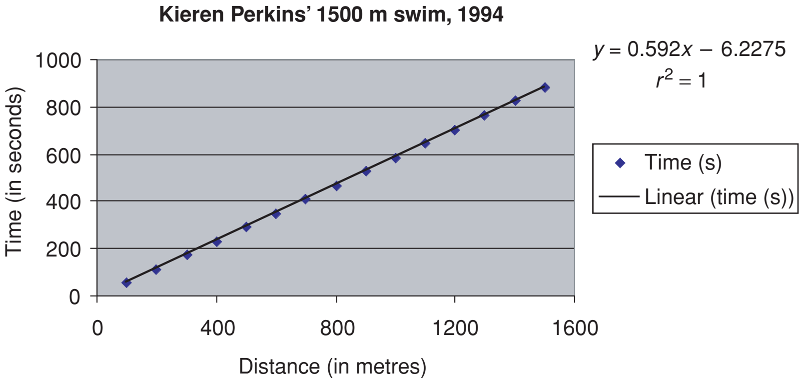
Example 28

In Victoria, Canada, in 1994, Kieren Perkins swam a world record time for the 1500 m Freestyle. The 100 m split times are shown below.

End of lap	Split (min sec)
100	52.9725
200	112.1725
300	171.3725
400	230.5725
500	289.7725
600	348.9725
700	408.1725
800	467.3725
900	526.5725
1000	585.7725
1100	644.9725
1200	704.1725
1300	763.3725
1400	822.5725
1500	881.7725

Use Excel to create a mathematical model of Kieren Perkins' 1500 m swim. Use this model to predict his times for the half nautical mile and the nautical mile.

Solution



The equation given by Excel is $y = 0.592x - 6.2275$. This means that $t = 0.592d - 6.2275$.

There are 1852 m in a nautical mile. Therefore, there are 926 m in half a nautical mile.

$$\begin{aligned} d = 926, \qquad t &= 0.592\,(926) - 6.2275 \\ &\approx 541.96 \text{ seconds} \\ &= 9 \text{ minutes } 1.96 \text{ seconds} \end{aligned}$$

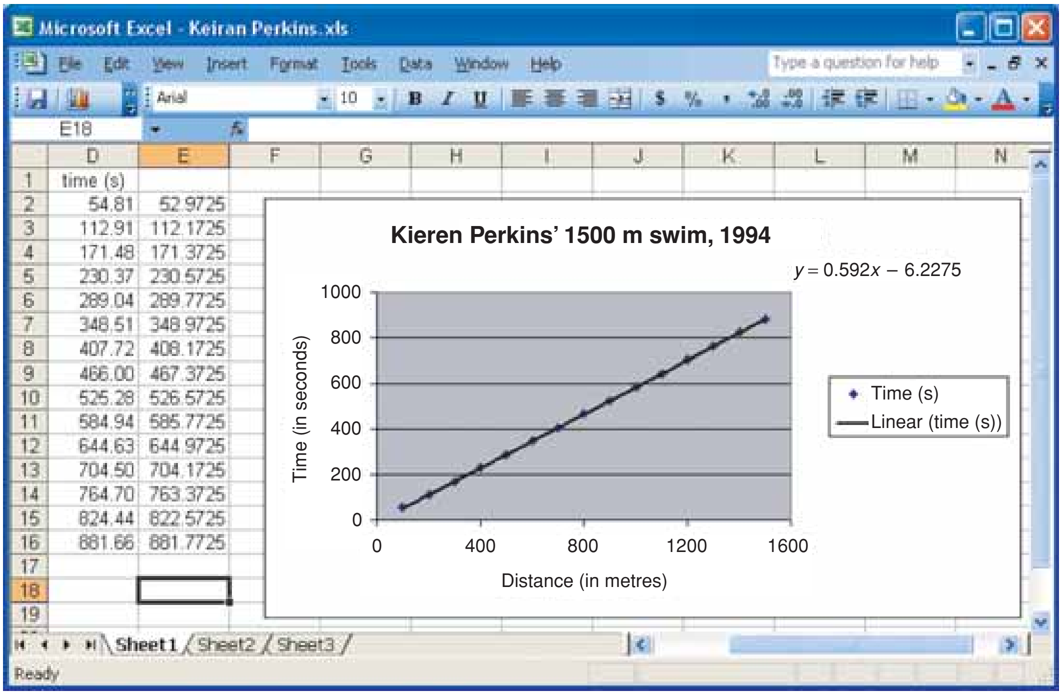
$\therefore$ He would have taken 9 minutes 1.96 seconds to swim half a nautical mile.

$$\begin{aligned} d = 1852, \qquad t &= 0.592\,(1852) - 6.2275 \\ &\approx 1090.16 \text{ seconds} \\ &= 18 \text{ minutes } 10.16 \text{ seconds} \end{aligned}$$

$\therefore$ He would have taken 18 minutes 10.16 seconds to swim a nautical mile.

The prediction for the half nautical mile time is reasonable because it is an **interpolation** of the data and the r^2 value is approximately 1. The prediction for the nautical mile time is not as reliable. It is an **extrapolation** of the data and is based on the **assumption** that Perkins could have continued to swim at the same rate for a further 352 m, although he may not have been able to.

On the use of Excel

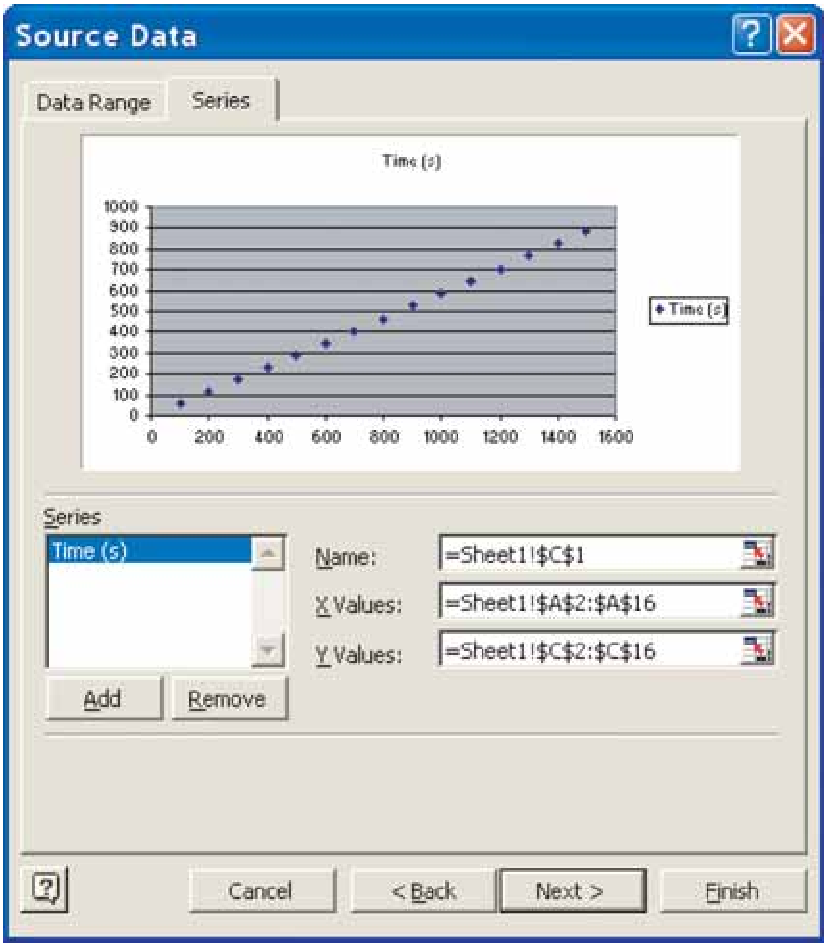


	D	E
1	time (s)	
2	54.81	52.9725
3	112.91	112.1725
4	171.48	171.3725
5	230.37	230.5725
6	289.04	289.7725
7	348.51	348.9725
8	407.72	408.1725
9	466.00	467.3725
10	525.28	526.5725
11	584.94	585.7725
12	644.63	644.9725
13	704.50	704.1725
14	764.70	763.3725
15	824.44	822.5725
16	881.66	881.7725
17		
18		
19		

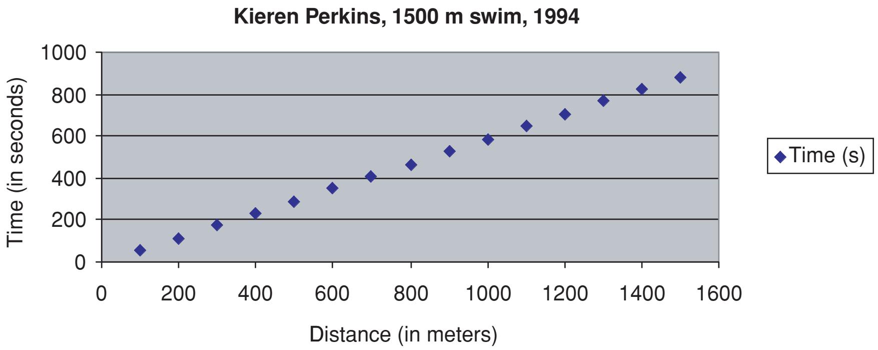
To produce a scatter plot using Excel, select **XY (Scatter)** from the **Chart Type** window.

In the **Source Data** window select the Y values only as the **Data range** and then go to the **Series** page.

In the **Series** page complete the **Name** and **X Values** boxes. (If you want to include more than one graph then **Add** the other **Series** at this point.)



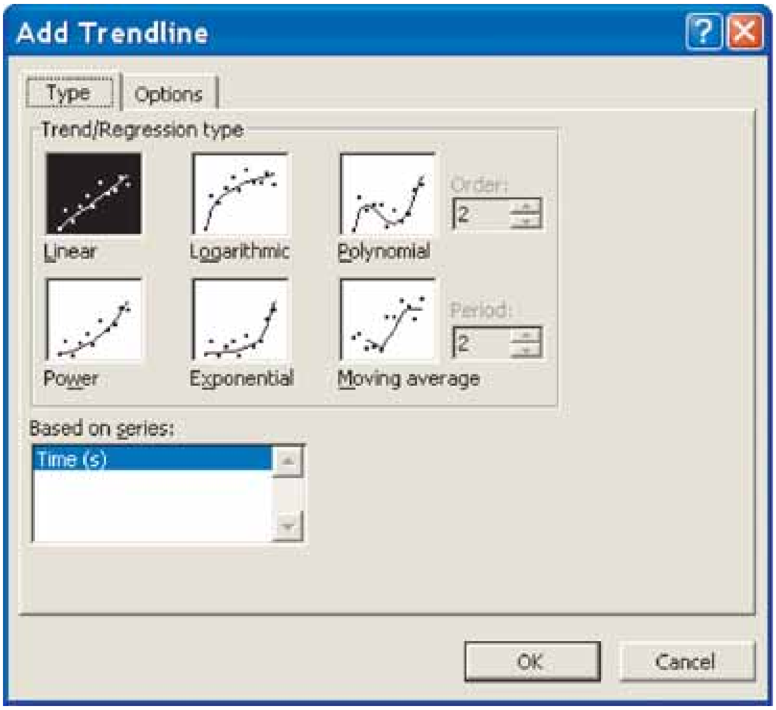
Name: =Sheet1!$C$1
X Values: =Sheet1!$A$2:$A$16
Y Values: =Sheet1!$C$2:$C$16



To draw a line (or curve) of best fit on an Excel graph, right-click on a data point on the graph and select **Add Trendline**.

Then select the **Trend/Regression type** required. In this case, **Linear**.

Now go to **Options** and tick **Display equation on chart** and **Display R-squared value on chart**.

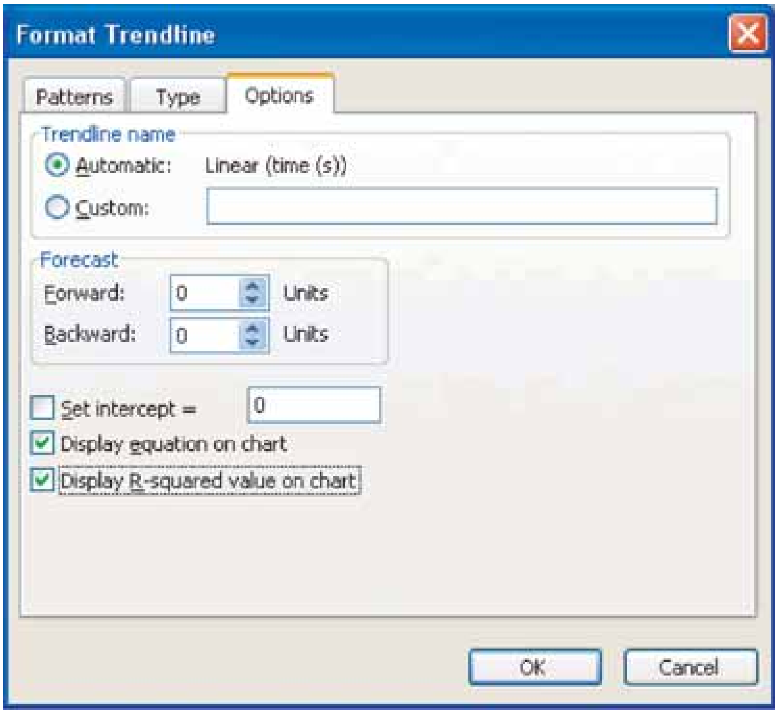


Trend/Regression type: Linear

Order: 2

Period: 2

Based on series: Time (s)



Trendline name

● Automatic: Linear (time (s))

○ Custom:

Forecast

Forward: 0 Units

Backward: 0 Units

☐ Set intercept = 0

☑ Display equation on chart

☑ Display R-squared value on chart

When asked to make a prediction:

- Create a mathematical model for the data and use it to make the prediction.
- Calculate a correlation coefficient for the data.
- Conclude with a paragraph discussing the reliability of your prediction. Ensure you include a discussion about your correlation coefficient in this paragraph.

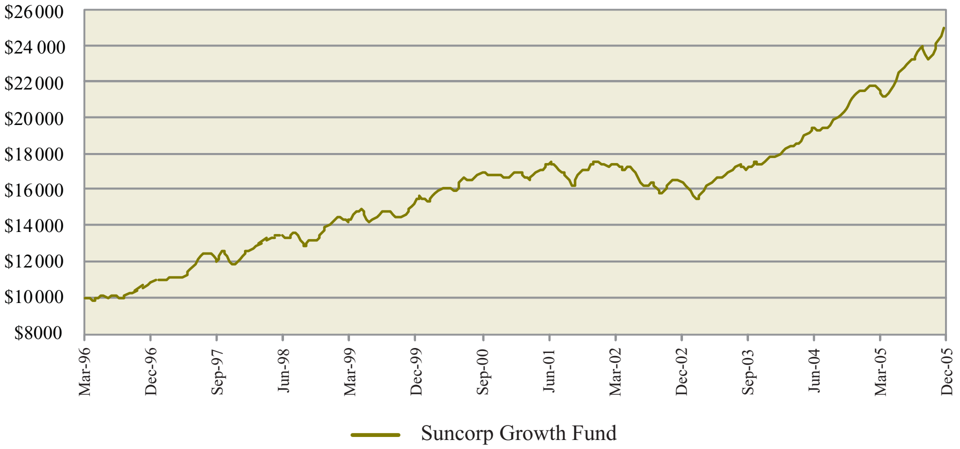
5.15 Modelling and problem solving

Exercise 50

1 On a recent test, the mean score of eight girls in class Year 11A was 74.5% and the mean score of the ten boys was 71.2%. Find the mean score of the whole class on the test.

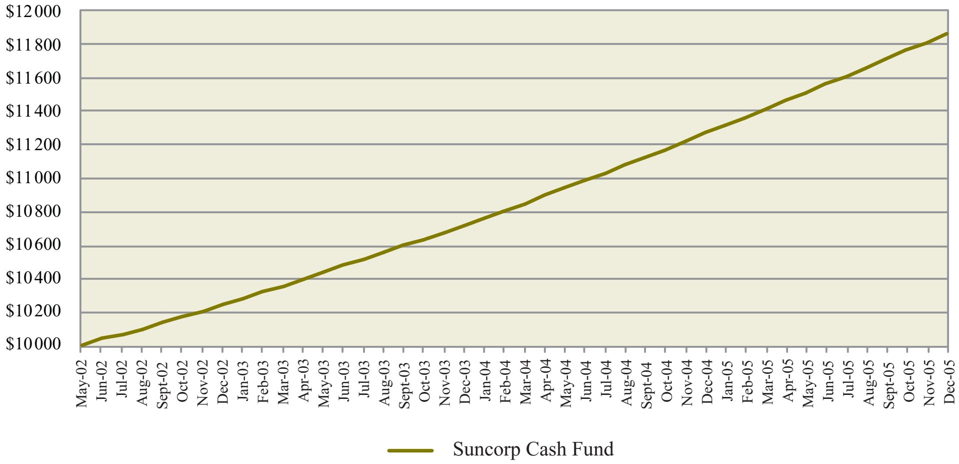
*Use the graphs shown to answer questions **2–6**.*

Value of a $10 000 investment in Suncorp Growth Fund as at 31 December 2005



Source: www.suncorp.com.au

Value of a $10 000 investment in Suncorp Cash Fund as at 31 December 2005



Source: www.suncorp.com.au

2 If you invest money in the Suncorp Growth Fund in February 1996, how long does it take to double your money?

3 A sum of \$20 000 is invested in the Suncorp Growth Fund in February 1996. What is the value of this investment in October 2000?

4 A sum of \$10 000 is invested in the Suncorp Growth Fund in October 2000. What is the value of this investment in November 2004?

5 An investor puts \$5000 in the Suncorp Cash Fund and \$12 000 in the Suncorp Growth Fund in February 2003. What is the total value of their investment in November 2004?

6 What is the average annual growth rate over the life of the Suncorp Growth Fund?

7 In a Year 11 Maths B class at an all-boys school in Brisbane, the students measure the heights of all of the students in their class. What method should they use to answer each of these questions and how precisely can they answer them?

- **a** What is the standard deviation of heights of students in that class?
- **b** What is the average height of Year 11 students in that school?
- **c** What is the standard deviation of heights of Year 11 students in that school?
- **d** What is the average height of Year 11 students in Queensland?

The formula $\sigma_n = \sqrt{\dfrac{n\sum x^2 - \left(\sum x\right)^2}{n^2}}$ *is used by Excel to calculate the standard deviation of a set of data. Use this formula to answer questions* ***8–10***.

8 Show that the formula above gives the correct value of σ_n for the data set $\{2, 2, 4, 5, 6\}$.

9 The mean of a set of four scores is 4.5 and the standard deviation is $\frac{\sqrt{21}}{2}$. Evaluate Σx^2 for this data set.

10 Show that the formula above and the one stated earlier in this chapter both lead to the same value of σ_n for the data set $\{a, b, c\}$.

11 The heights, in centimetres, of Year 11A and B students are shown below.

Year 11A	176	180	182	175	187	181	170	175	176	181	172	183
	180	181	173	180	163	165	168	179	171	182	190	
Year 11B	173	172	172	185	171	179	176	190	181	192	175	160
	182	180	179	183	174	171	178					

Compare the heights of the two classes.

12 In a small company, upper management wants to know if there is a difference in the three methods used to train its machine operators and, if so, which is more effective in training its staff. One method uses a hands-on approach. A second method uses a combination of classroom instruction and on-the-job training. The third method is based completely on classroom training. Fifteen trainees are assigned to each training technique.

The following data are the results of a practical test undertaken by the machine operators after completion of one of the different training methods.

Method 1	Method 2	Method 3
98	79	70
100	62	74
89	61	60
90	89	72
81	69	65
85	99	49
97	87	71
95	62	75
87	65	55
70	88	65
69	98	70
75	79	59
91	73	77
92	96	67
93	83	80

Report to upper management on the differences in performance arising from the three methods and advise them on which method is more effective in training its staff.

13 It has been argued that there is a relationship between a child's level of independence and the order in which they were born in the family. Suppose that the children in thirteen three-children families are rated on a 50-point scale of independence. This is done when all children are adults, thus eliminating age effects. The results are presented.

Family	1	2	3	4	5	6	7	8	9	10	11	12	13
First-born	38	45	30	29	34	19	35	40	25	50	44	36	26
Second-born	9	40	24	16	16	21	34	29	22	29	20	19	18
Third-born	12	12	12	25	9	11	20	12	10	20	16	13	10

Compare the independence scores of first-, second- and third-born children.

Example 28 **14** To investigate the relationship between marks on an assignment and the final examination mark a sample of 10 students was taken. The table indicates the marks for the assignment and the final exam mark for each individual student.

a The school principal said, 'Good final exam marks are the result of good assignment marks.' Do the results given bear this out?

b Predict the final exam mark for a student who scored 50 on the assignment.

Assignment mark (max = 80)	Final exam mark (max = 90)
80	83
77	83
71	79
78	75
65	68
80	84
68	71
64	69
50	66
66	58

15 A marketing firm wanted to investigate the relationship between airplay and CD sales (in the following week) of newly released songs. Data were collected on a random sample of ten songs.

Predict the weekly sales for a song that was played 60 times.

No. of times the song was played	Weekly sales of the CD
47	3950
34	2500
40	3700
34	2800
33	2900
50	3750
28	2300
53	4400
25	2200
46	3400

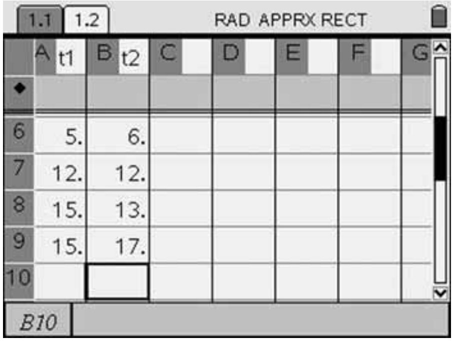
Using technology

How to construct a scatterplot

Using the TI-Nspire:

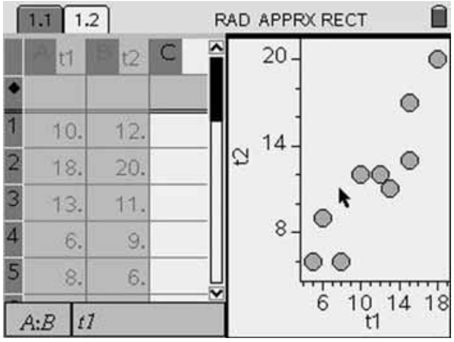
1 Enter the Test 1 scores into a column called **t1**.

2 Enter the Test 2 scores into a column called **t2**.



	A t1	B t2
◆		
6	5.	6.
7	12.	12.
8	15.	13.
9	15.	17.
10		

3 Highlight both columns, press menu, then select *Quick Graph* from the Data submenu.



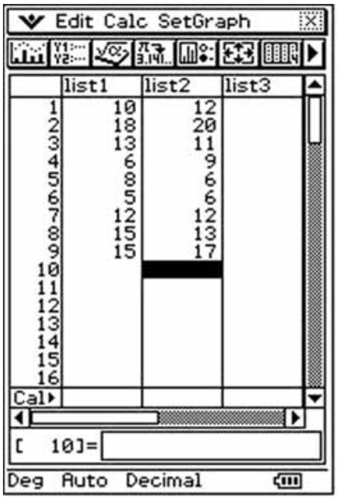
	A t1	B t2
◆		
1	10.	12.
2	18.	20.
3	13.	11.
4	6.	9.
5	8.	6.

(The default graph for two data sets is a scatterplot, thus there is no need to change the plot type.)

Using the ClassPad:

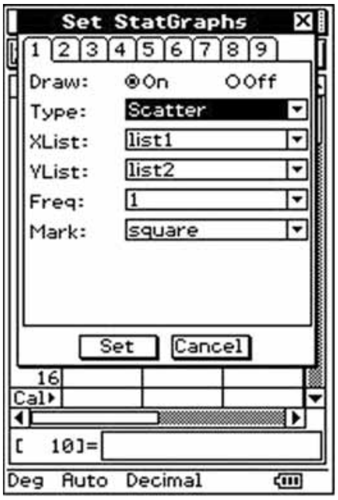
1 Enter the Test 1 scores into list1.

2 Enter the Test 2 scores into list2.



	list1	list2	list3
1	10	12	
2	18	20	
3	13	11	
4	6	9	
5	8	6	
6	5	6	
7	12	12	
8	15	13	
9	15	17	
10			
11			
12			
13			
14			
15			
16			

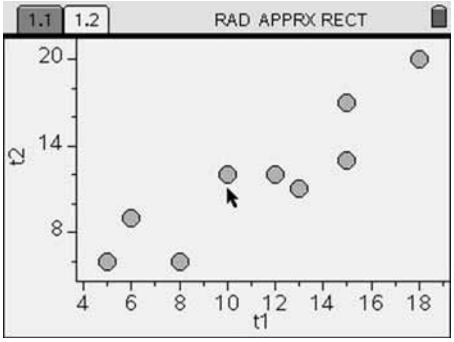
3 To ensure a scatterplot is drawn, tap SetGraph then tap on Setting. . . and set the following:



Draw: ◉On ○Off
Type: Scatter
XList: list1
YList: list2
Freq: 1
Mark: square

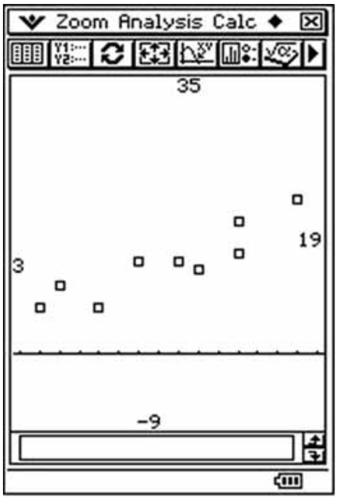
Tap SET to save the changes.

4 For a full-screen view of the scatterplot, press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, home. Scroll to *Page Layout* ⟶ *Select Layout* ⟶ *1: Layout 1*.



4 To see the scatterplot, tap ▦.

5 Tap Resize for a full-screen view of the scatterplot.

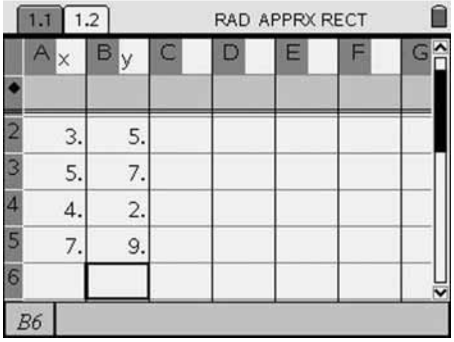


How to calculate the correlation coefficient, *r*

Using the TI-Nspire:

1 Enter the *x* values into a column called **x**.

2 Enter the *y* values into a column called **y**.



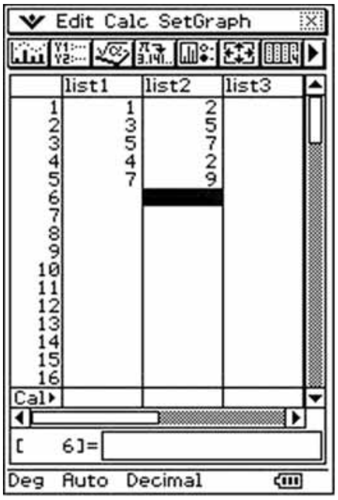
	A x	B y	C	D	E	F	G
◆							
2	3.	5.					
3	5.	7.					
4	4.	2.					
5	7.	9.					
6							

3 Highlight both columns, press menu, and navigate to *Linear Regression* (*a* + *bx*) from the Statistics menu.

Using the ClassPad:

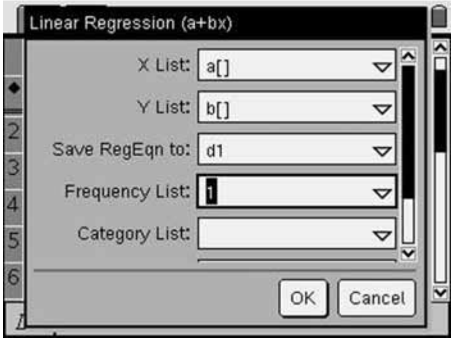
1 Enter the *x* values into list1.

2 Enter the *y* values into list2.

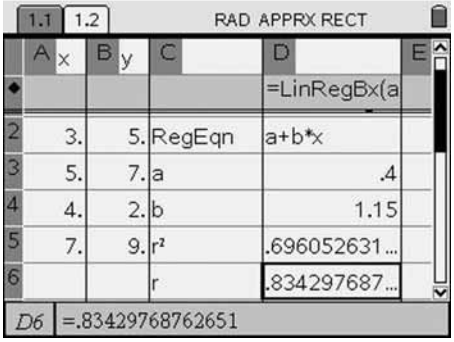


	list1	list2	list3
1	1	2	
2	3	5	
3	5	7	
4	4	2	
5	7	9	
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			

4 Set the following and then press OK.



X List: a[]
Y List: b[]
Save RegEqn to: d1
Frequency List: 1
Category List:

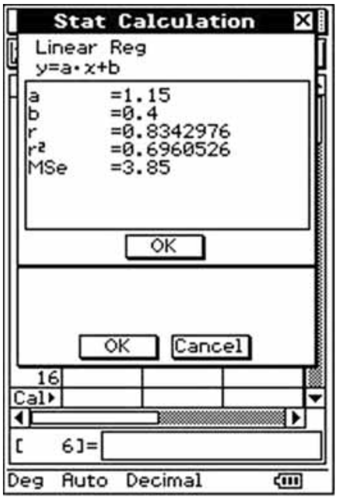


	A x	B y	C	D
◆				=LinRegBx(a
2	3.	5.	RegEqn	a+b*x
3	5.	7.	a	.4
4	4.	2.	b	1.15
5	7.	9.	r²	.696052631…
6			r	.834297687…

D6 =.83429768762651

3 Select *Linear Reg* from the Calc menu, set the following, then tap OK.

Linear Reg
XList: list1
YList: list2
Freq: 1
Copy Formula: Off
Copy Residual: Off



Linear Reg
y=a·x+b
a =1.15
b =0.4
r =0.8342976
r² =0.6960526
MSe =3.85

How to determine the equation of a least squares regression

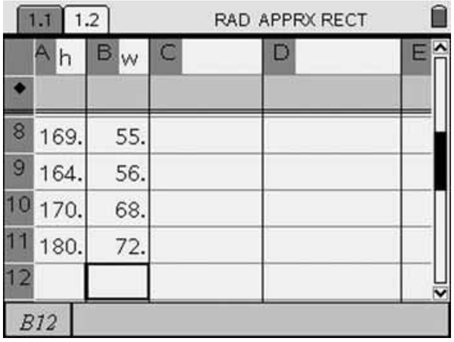
Using the TI-Nspire:

1 Enter the height values into a column called **h**.

Using the ClassPad:

1 Enter the height values into list1.

2 Enter the weight values into a column called **w**.

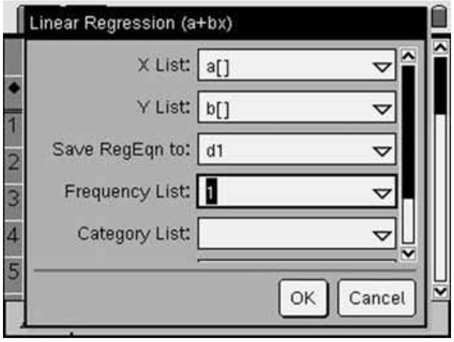


	A h	B w	C	D	E
8	169.	55.			
9	164.	56.			
10	170.	68.			
11	180.	72.			
12					

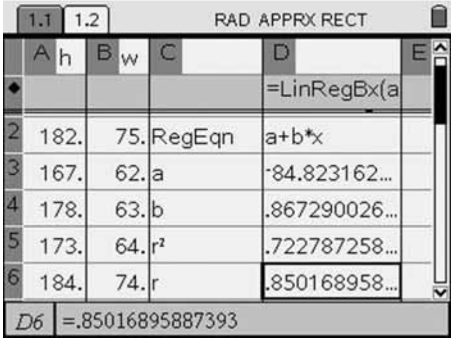
B12

3 Highlight both columns, press menu, and navigate to *Linear Regression* ($a + bx$) from the Statistics menu.

4 Set the following and then press OK.



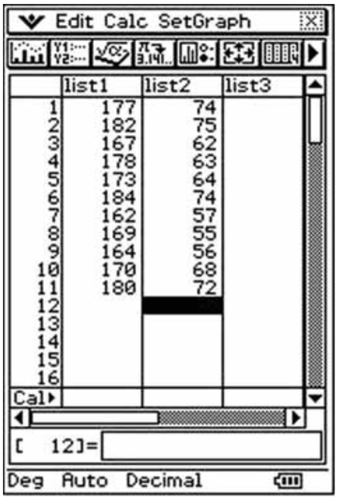
X List: a[]
Y List: b[]
Save RegEqn to: d1
Frequency List: 1
Category List:



	A h	B w	C	D	E
◆				=LinRegBx(a	
2	182.	75.	RegEqn	a+b*x	
3	167.	62.	a	-84.823162...	
4	178.	63.	b	.867290026...	
5	173.	64.	r²	.722787258...	
6	184.	74.	r	.850168958...	

D6 =.85016895887393

2 Enter the weight values into list2.

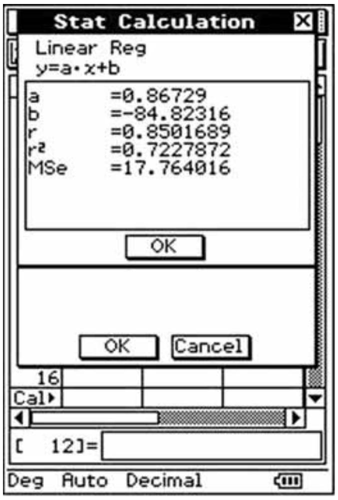


	list1	list2	list3
1	177	74	
2	182	75	
3	167	62	
4	178	63	
5	173	64	
6	184	74	
7	162	57	
8	169	55	
9	164	56	
10	170	68	
11	180	72	
12			
13			
14			
15			
16			

[12]=

3 Select *Linear Reg* from the Calc menu, set the following, then tap OK.

Linear Reg
XList: list1
YList: list2
Freq: 1
Copy Formula: Off
Copy Residual: Off



Linear Reg
y=a·x+b
a =0.86729
b =-84.82316
r =0.8501689
r² =0.7227872
MSe =17.764016

Chapter summary

- Variables may be classified as **categorical** or **numerical**. Numerical data may be **discrete** or **continuous**. Categorical data may be **nominal** or **ordinal**.
- Examination of a data set should always begin with a visual display.
- A **sector graph** is an appropriate visual display for nominal data.
- A **bar chart** is the appropriate visual display for ordinal data.
- When a data set is small, a **stem-and-leaf plot** is the most appropriate visual display for numerical data.
- When a data set is larger, a **histogram**, **frequency polygon** or **boxplot** is a more appropriate visual display for numerical data.
- Cumulative frequency distributions and cumulative relative frequency distributions are useful for answering questions about the number or proportion of data values greater than or less than a particular value. These are graphically represented in **cumulative frequency polygons** or **cumulative relative frequency polygons**.
- From a stem-and-leaf plot, histogram or boxplot, insight can be gained into the **shape**, **centre** and **spread** of the distribution, and whether or not there are any **outliers**.
- An **outlier** is a value that sits away from the main body of the data in a plot. It is formally defined as a value more than 1.5 IQR below Q_1, or more than 1.5 IQR above Q_3.
- For numerical data it is also very useful to calculate some summary statistics.
- The **mean** is defined as $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and is an unbiased estimate of the mean of the underlying population.
- To compute the **median**:
  - Arrange all the observations in ascending order, according to size.
  - Calculate $\frac{n+1}{2}$.
  - Count along the ordered row to the $\frac{n+1}{2}$th place. If $\frac{n+1}{2}$ is a whole number then you have the median. If $\frac{n+1}{2}$ is not a whole number, then the counting will finish between *two* numbers. *Average* them to find the median.
- The **mode** is the most common observation in a group of data.
- To find the **interquartile range** of a distribution:
  - Arrange all observations in order, according to size.
  - Divide the observations into two equal-sized groups. If n, the number of observations, is odd, then the median is omitted from both groups.
  - Locate Q_1, the first quartile, which is the median of the lower half of the observations, and Q_3, the third quartile, which is the median of the upper half of the observations.
  - The interquartile range (IQR) is defined as the difference between the quartiles. That is, IQR $= Q_3 - Q_1$

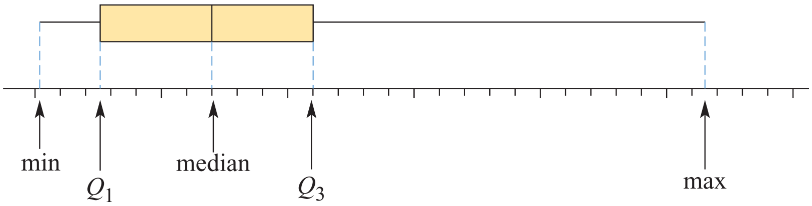
- Use σ_n, the **population standard deviation**, when data of the entire population is known.

$$\sigma_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

- Use σ_{n-1}, the **sample standard deviation**, when the data is a sample taken from a larger population and an unbiased estimate of the standard deviation of the underlying population is being made.

$$\sigma_{n-1} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

- The **five-figure summary** of a set of data consists of the: minimum, Q_1, median, Q_3 and the maximum. A **boxplot** is a diagrammatic representation of this; for example:



- When the data set is symmetrical any of the summary statistics are appropriate.
- When the data set is not symmetrical or when there are outliers the median and the interquartile range are the preferred summary statistics.
- In general, 95% of the values of the data set will fall within two standard deviations of the mean.
- When comparing the distribution of two or more data sets the comparison should be made in terms of the shape, centre, spread and outliers for each distribution.
- **Bivariate data** arises when measurements on two variables are collected for each subject.
- A **scatterplot** is an appropriate visual display of bivariate data if both of the variables are numerical.
- A scatterplot of the data should always be constructed to assist in the identification of outliers and illustrate the association (positive, negative or none).
- Two variables are **positively associated** when larger values of y are associated with larger values of x. Two variables are **negatively associated** when larger values of y are associated with smaller values of x. There is **no association** between two variables when the values of y are not related to the values of x.
- When constructing the scatterplot, the independent or explanatory variable is plotted on the horizontal (x) axis, and the dependent or response variable is plotted on the vertical (y) axis.
- If a linear relationship is indicated by the scatterplot a measure of its strength can be found by calculating the ***q*-correlation coefficient**, or **Pearson's product-moment correlation coefficient**, r.

- If the values on a scatterplot are divided by lines representing the median of x and the median of y into four quadrants A, B, C and D, with a, b, c, d representing the number of points in each quadrant, respectively, then the ***q*-correlation coefficient** is given by

$$q = \frac{(a+c)-(b+d)}{a+b+c+d}$$

- **Pearson's product-moment correlation**, r, is a measure of strength of linear relationship between two variables, x and y. If we have n observations then for this set of observations

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{s_x}\right)\left(\frac{y_i-\bar{y}}{s_y}\right)$$

  where $\bar{x}$ and s_x are the mean and standard deviation of the x scores and $\bar{y}$ and s_y are the mean and standard deviation of the y scores.
- For these correlation coefficients

$$-1 \le q \le 1$$
$$-1 \le r \le 1$$

  with values close to ±1 indicating strong correlation, and those close to 0 indicating little correlation.
- If a linear relationship is indicated from the scatterplot a straight line may be fitted to the data, either 'by eye' or using the **least squares regression** method.
- The **least squares regression line** is the line for which the sum of squares of the vertical deviations from the data to the line is a minimum.
- The value of the slope (b) gives the extent of the change in the dependent variable associated with a unit change in the independent variable.
- Once found, the equation to the straight line may be used to predict values of the response variable (y) from the explanatory variable (x). The accuracy of the prediction depends on how closely the straight line fits the data, and an indication of this can be obtained from the correlation coefficient.
- When asked to make a **prediction**:
  - Create a mathematical model for the data and use it to make the prediction.
  - Calculate a correlation coefficient for the data.
  - Conclude with a paragraph discussing the reliability of your prediction. Ensure you include a discussion about your correlation coefficient in this paragraph.

Multiple-choice questions

1 In a survey a number of subjects were asked to indicate how much they exercise by selecting one of the following options.

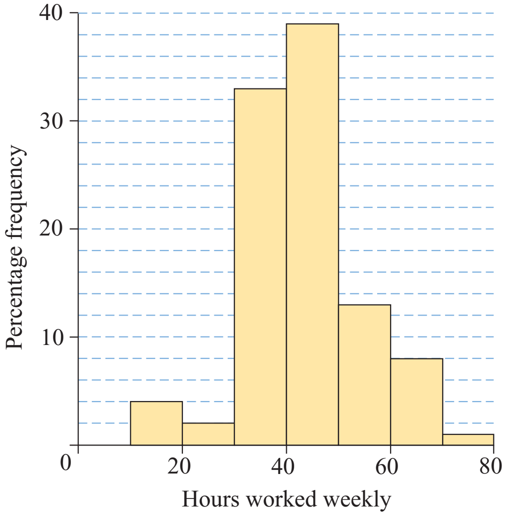
1 Never **2** Seldom **3** Occasionally **4** Regularly

The resulting variable was named *Level of Exercise*, and the level of measurement of this variable is

A variable **B** numerical **C** constant **D** categorical **E** metric

*Questions **2** and **3** relate to the following information.*

The numbers of hours worked per week by employees in a large company are shown in this percentage frequency histogram.



2 The percentage of employees who work from 20 to fewer than 30 hours per week is closest to

A 1% **B** 2% **C** 6% **D** 10% **E** 33%

3 The median number of hours worked is in the interval

A 10 to fewer than 20 **B** 20 to fewer than 30

C 30 to fewer than 40 **D** 40 to fewer than 50

E 50 to fewer than 60

*Questions **4** and **5** relate to the following information.*

A group of 19 employees of a company was asked to record the number of meetings that they attended in the last month. Their responses are summarised in the following stem-and-leaf plot:

0	1	1	2	3	4	5	5	6	6	7	9
1	0	2	4	4	6						
2	2	3									
3											
4	4										

4 The median number of meetings is

A 6 **B** 6.5 **C** 7 **D** 7.5 **E** 9

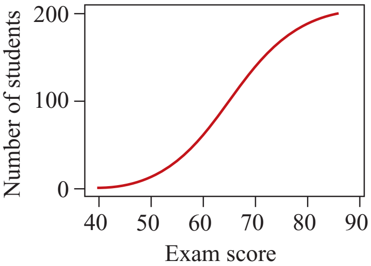
5 The interquartile range (IQR) of number of meetings is

A 0 **B** 4 **C** 9.5 **D** 10 **E** 14

6 The cumulative frequency polygon shown gives the examination scores in Mathematics for a group of 200 students.

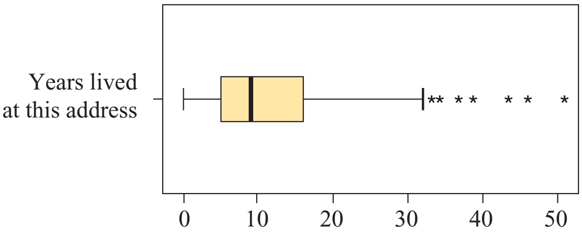
The number of students who scored less than 70 on the examination is closest to

A 30 **B** 100 **C** 150 **D** 175 **E** 200



Questions 7 *and* **8** *relate to the following information.*

The number of years that a sample of people has lived at their current address is summarised in this boxplot.



7 The shape of the distribution of years lived at this address is

A positively skewed **B** negatively skewed **C** bimodal

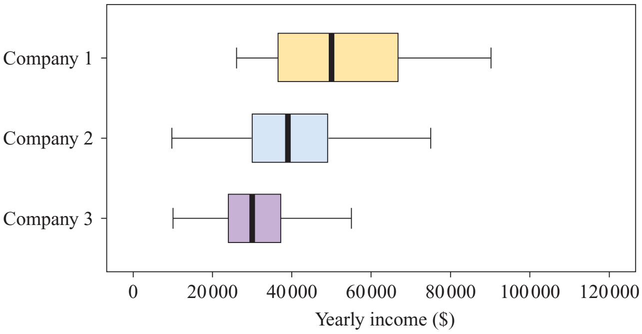
D symmetrical **E** symmetrical with outliers

8 The interquartile range years lived at this address is approximately equal to

A 5 **B** 8 **C** 17 **D** 12 **E** 50

Questions **9** *and* **10** *relate to the following data.*

The amount paid per week to the employees of each of three large companies are shown in the given boxplots.



9 The company with the lowest typical wage is

A company 1 **B** company 2 **C** company 3

D company 1 and company 2 **E** company 2 and company 3

10 The company with the largest variation in wage is

A company 1 **B** company 2 **C** company 3

D company 1 and company 2 **E** company 2 and company 3

11 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

A eye colour (blue, green, brown, other) and hair colour (black, brown, blonde, red, other)

B score out of 100 on a test for a group of Year 9 students and a group of Year 11 students

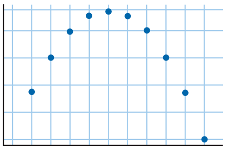
C political party preference (Labor, Liberal, Other) and age in years

D age in years and blood pressure in mm Hg

E height in cm and gender (male, female)

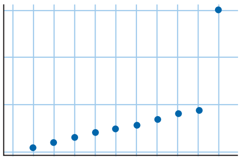
12 For which one of the following plots would it be appropriate to calculate the value of the *q*-correlation coefficient?

A

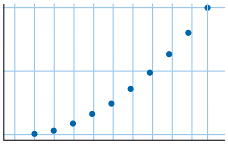


B

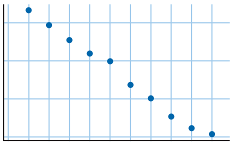
C



D



E



13 A *q*-correlation coefficient of 0.32 would describe a relationship classified as

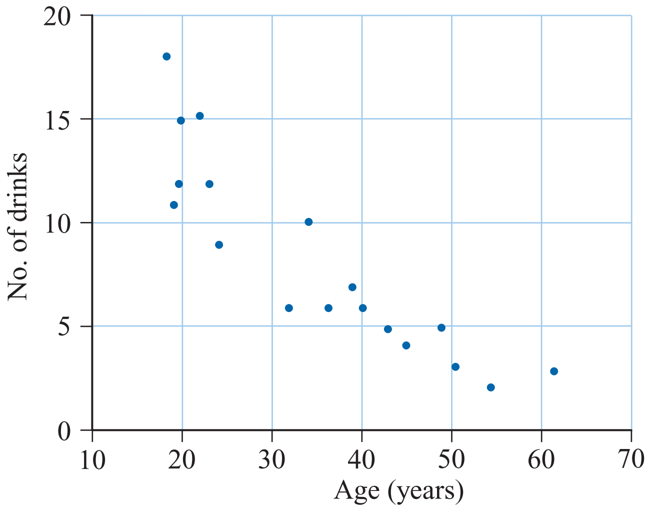
A weak positive **B** moderate positive **C** strong positive

D close to zero **E** moderately strong

14 The scatterplot shows the relationship between age and the number of alcoholic drinks consumed on the weekend by a group of people.

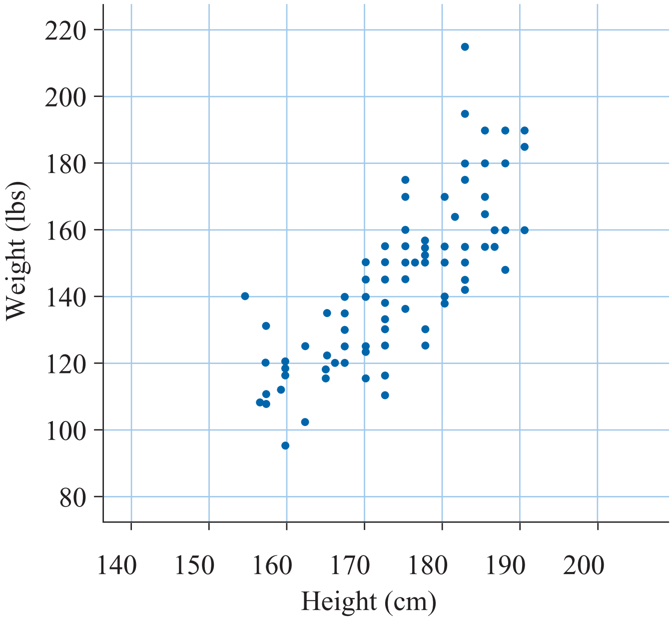
The value of the *q*-correlation coefficient is closest to

A -1 **B** $-\frac{7}{9}$ **C** $-\frac{5}{6}$ **D** $\frac{7}{9}$ **E** 1



15 The scatterplot shows the relationship between height and weight for a group of people. The value of the Pearson's product-moment correlation coefficient, r, is closest to

A 1 **B** 0.8 **C** 0.5 **D** 0.3 **E** 0



*Questions **16** and **17** relate to the following information.*

The weekly income and weekly expenditure on food for a group of ten university students is given in the table below.

Weekly income ($)	150	250	300	600	300	380	950	450	850	1000
Weekly food expenditure ($)	40	60	70	120	130	150	200	260	460	600

16 The value of the Pearson product-moment correlation coefficient, r, for these data is closest to

A 0.2 **B** 0.4 **C** 0.6 **D** 0.7 **E** 0.8

17 The least squares regression line which would enable expenditure on food to be predicted from weekly income is closest to

A $0.482 + 42.864 \times$ weekly income
B $0.482 - 42.864 \times$ weekly income
C $-42.864 + 0.482 \times$ weekly income
D $239.868 + 1.355 \times$ weekly income
E $1.355 + 239.868 \times$ weekly income

*Questions **18** and **19** relate to the following information.*

Suppose that the least squares regression line which would enable expenditure on entertainment (in dollars) to be predicted from weekly income is given by

$$\text{Weekly expenditure on entertainment} = 40 + 0.10 \times \text{weekly income}$$

18 Using this rule, the expenditure on entertainment by an individual with an income of $600 per week is predicted to be

A $40 **B** $24 060 **C** $100 **D** $46 **E** $240

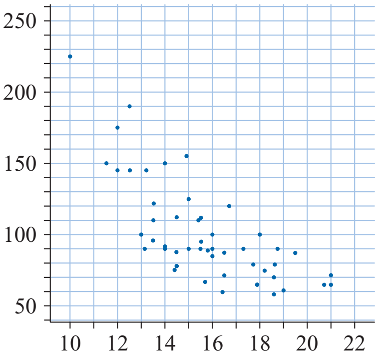
19 From this rule which of the following statements is correct?

- **A** On average, for each extra dollar of income an extra 10 cents is spent on entertainment.
- **B** On average, for each extra 10 cents in income an extra $1 is spent on entertainment.
- **C** On average, for each extra dollar of income an extra 40 cents is spent on entertainment.
- **D** On average, people spend $40 per week on entertainment.
- **E** On average, people spend $50 per week on entertainment.

20 For the scatterplot shown, the line of best fit would have a slope closest to

A 0.1 **B** −0.1 **C** 10

D −10 **E** 200



Short-response questions

1 Classify the data that arise from the following situations as either nominal, ordinal, discrete or continuous:

- **a** The time taken by each competitor to complete the Swimming 50 m butterfly.
- **b** The placing your team members achieved in each of the events at the swimming carnival.
- **c** The number of swimmers who recorded under 60 seconds for the 100 m freestyle at each of the Friday night swimming carnivals.
- **d** The colour of swimmers worn by each of the competitors is recorded.

2 Classify the data that arise from the following situations as categorical or numerical.

- **a** The number of phone calls a hotel receptionist receives each day.
- **b** Interest in politics on a scale from 1 to 5, where 1 = very interested, 2 = quite interested, 3 = somewhat interested, 4 = not very interested and 5 = uninterested.

3 Draw appropriate graphs for each of the following sets of data:

- **a** The class is made up of 15 girls and 10 boys.
- **b** To a survey, 16 people responded ‘agree’, 10 responded ‘disagree’ and 8 were ‘neutral’.

4 A researcher asked a group of people to record how many cigarettes they had smoked on a particular day. Below are her results.

0	0	9	10	23	25	0	0	34	32	0	0	30	0	4
5	0	17	14	3	6	0	33	23	0	32	13	21	22	6

Using an appropriate class interval, construct a histogram of these data.

5 A teacher recorded the time taken (in minutes) by each of a class of students to complete a test.

56	57	47	68	52	51	43	22	59	51	39
54	52	69	72	65	45	44	55	56	49	50

a Make a stem-and-leaf plot of these times, using one row per stem.
b Use this stem-and-leaf-plot to find the median and quartiles for the time taken.

6 The weekly rentals, in dollars, for apartments in a particular suburb are given in the following table:

285	185	210	215	320	680	280
265	300	210	270	190	245	315

Find the mean and the median of the weekly rental.

7 Geoff decided to record the time it takes him to complete his mail delivery round each working day for four weeks. His data are recorded in the following table:

170	189	201	183	168	182	161	166	167	173
164	176	161	187	180	201	147	188	186	176
182	167	188	211	174	193	185	183		

The mean of the time taken, $\bar{x}$, is 179 and the standard deviation, s, is 14.

a Determine the percentage of observations falling within two standard deviations of the mean.
b Is this what you would expect to find?

8 A group of students were asked to record the number of SMS messages that they sent in one 24-hour period, and the following five-figure summary was obtained from the data set. Use it to construct a boxplot of these data.

$$\text{Min} = 0, \quad Q_1 = 3, \quad \text{Median} = 5, \quad Q_3 = 12, \quad \text{Max} = 24$$

9 The following table gives the number of students absent each day from a large secondary college on each of 36 randomly chosen school days.

7	22	12	15	21	16	23	23	17	23	8	16
7	3	21	30	13	2	7	12	18	14	14	0
15	16	13	21	10	16	11	4	3	0	31	44

Construct a boxplot of these data, with outliers.

10 The divorce rates (in percentages) of 19 countries are

27	18	14	25	28	6	32	44	53	0
26	8	14	5	15	32	6	19	9	

a What is the level of measurement of the variable 'divorce rate'?

b Construct an ordered stem-and-leaf plot of divorce rates, with one row per stem.

c What shape is the divorce rates?

d What percentage of countries have divorce rates greater than 30?

e Calculate the mean and median of the divorce rates for the 19 countries.

f Construct a histogram of the data with class intervals of width 10.

  i What is the shape of the histogram?

  ii How many countries have divorce rates from 10% to less than 20%?

g Construct a cumulative percentage frequency polygon of divorce rates.

  i What percentage of countries has divorce rates less than 20%?

  ii Use the cumulative frequency distribution to estimate the median percentage divorce rate.

11 Hillside Trains have decided to improve their service on the Lilydale line. Trains were timed on the run from Lilydale to Flinders Street, and their times recorded over a period of six weeks at the same time each day. The time taken for each journey is shown below.

60	61	70	72	68	80	76	65	69	79	82
90	59	86	70	77	64	57	65	60	68	60
63	67	74	78	65	68	82	89	75	62	64
58	64	69	59	62	63	89	74	60		

a Construct a histogram of the times taken for the journey from Lilydale to Flinders Street, using class intervals 55–59, 60–64, 65–69 etc.

  i On how many days did the trip take from 65–69 minutes?

  ii What shape is the histogram?

  iii What percentage of trains take less than 65 minutes to reach Flinders Street?

b Calculate the following summary statistics for the time taken (correct to 2 decimal places):

$\bar{x}$ σ_n Min Q_1 Median Q_3 Max

c Use the summary statistics to complete the following report:

  i The mean time taken from Lilydale to Flinders Street (in minutes) is . . .

  ii 50% of the trains take more than . . . minutes to travel from Lilydale to Flinders Street.

  iii The range of travelling times is . . . minutes and the interquartile range is . . . minutes.

(cont'd)

iv 25% of trains take more than . . . minutes to travel to Flinders Street.

v The standard deviation of travelling times of train times surveyed is . . .

vi Approximately 95% of trains take between . . . and . . . minutes to travel to Flinders Street.

c Summary statistics for the year before Hillside Trains took over the Lilydale line from the Met are indicated below:

$$\text{Min} = 55 \quad Q_1 = 65 \quad \text{Median} = 70 \quad Q_3 = 89 \quad \text{Max} = 99$$

Draw simple boxplots for the last year the Met ran the line and the data from Hillside trains on the same axis.

d Use the information from the boxplots to compare travelling times for the two transport corporations in terms of shape, centre and spread.

Technology is required to answer some of the following questions.

12 The table gives the number of times the ball was inside the 50 metre line in an AFL football game, and the team's score in that game

a Plot the score against the number of Inside 50s.

b From the scatterplot, describe any association between the two variables.

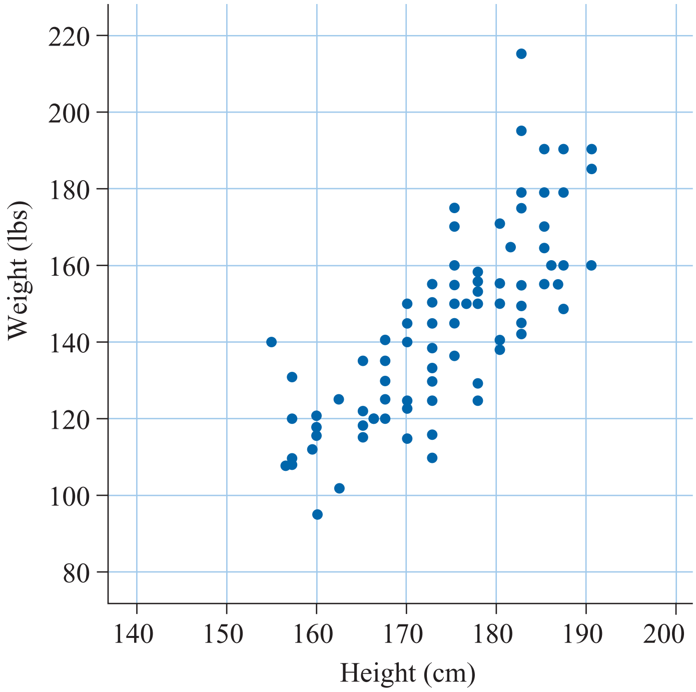
c Use the scatterplot constructed in part **a** to determine the q-correlation between the score and the number of Inside 50s.

Inside 50	Score (points)
64	90
57	134
34	76
61	92
51	93
52	45
53	120
51	66
64	105
55	108
58	88
71	133

13 The distance travelled to work and the time taken for a group of company employees are given in the following table. Determine the value of the Pearson product-moment correlation, r, for these data.

Distance (km)	12	50	40	25	45	20	10	3	10	30
Time (min)	15	75	50	50	80	50	10	5	10	35

14 The following scatterplot shows the relationship between height and weight for a group of people. Draw a straight line which fits the data by eye, and find an equation for this line.



15 The time taken to complete a task, and the number of errors on the task, were recorded for a sample of ten primary school children. Determine the equation of the least squares regression line which fits these data.

Time (seconds)	Errors
22.6	2
21.7	3
21.7	3
21.3	4
19.3	5
17.6	5
17.0	7
14.6	7
14.0	9
8.8	9

16 For the data in Question **15**:

a Interpret the intercept and slope of the least squares regression line.

b Use the least squares regression line to predict the number of errors which would be observed for a child who took 10 seconds to complete the task.

17 A marketing company wishes to predict the likely number of new clients each of its graduates will attract to the business in their first year of employment, by using their scores on a marketing exam in the final year of their course.

a Which is the independent variable and which is the dependent variable?

b Construct a scatterplot of these data.

c Describe the association between ‘Number of new clients’ and ‘Exam score’.

d Determine the value of the q-correlation coefficient for these data, and classify the strength of the relationship.

e Determine the value of the Pearson product-moment correlation coefficient for these data and classify the strength of the relationship.

Exam score	Number of new clients
65	7
72	9
68	8
85	10
74	10
61	8
60	6
78	10
70	5
82	11

f Determine the equation for the least squares regression line and write it down in terms of the variables ‘Number of new clients’ and ‘Exam score’.

g Interpret the intercept and slope of the least squares regression line in terms of the variables in the study.

h Use your regression equation to predict, to the nearest whole number, the number of new clients for a person who scored 100 on the exam.

i How reliable is the prediction made in part **h**?

CHAPTER 6

More functions

Objectives

- To be able to work problems involving **direct and inverse variation**.
- To sketch **hyperbolae**.
- To **divide** polynomials.
- To **factorise polynomials**.
- To sketch **cubic and quartic polynomial** functions.
- To use **transformations** as an aid to graphing.
- To sketch graphs of **absolute value functions**.

6.1 Direct and inverse variation (Proportion)

Direct proportion

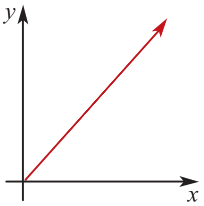
The statement ‘The circumference of a circle is **proportional** to its radius’ describes a function that is an example of direct variation. An alternative way of saying this is: ‘The circumference of a circle **varies directly** with its radius’.

The statement ‘y is proportional to x’ is written as $y \propto x$. In all cases of direct variation the functions can be written as linear functions of the form $y = kx$.

The pronumeral k in the equation $y = kx$ is called the **constant of proportionality** or the **constant of variation**. In the case of the circle, the constant of proportionality is 2π, so $y = 2\pi x$.

It is worth noting that:

- All functions involving direct variation appear as straight lines passing through the origin when graphed on the Cartesian plane.
- The constant of proportionality is the gradient of the straight line.



Inverse proportion

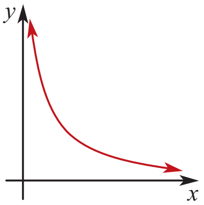
The statement ‘The length of time it takes to get to the beach is **inversely proportional** to the speed of travel’ describes a function that is an example of inverse variation. An alternative way of saying this is: ‘The length of time it takes to get to the beach **varies inversely** with the speed of travel’.

The statement ‘y is inversely proportional to x’ is written as $y \propto \frac{1}{x}$. In all cases of inverse variation the functions can be written as functions of the form $y = \frac{k}{x}$.

The pronumeral k in the equation $y = \frac{k}{x}$ is called the **constant of proportionality** or the **constant of variation**.

It is worth noting that:

- As one variable gets bigger the other gets smaller. In fact double one and you halve the other.
- All functions involving inverse variation appear as hyperbolae on the Cartesian plane. They never cross either axis.



Example 1

a Show that w is proportional to t in the function shown in the table below.

t	2.3	4.1	6.7
w	5.75	10.25	16.75

b Show that r is inversly proportional to i in the function shown in the table below.

r	1.6	4	6.4
i	5	2	1.25

Solution

a If $y \propto t$, then $w = kt$.

When $t = 2.3$, $w = 5.75$

i.e. $5.75 = k \times 2.3 \quad (\div 2.3)$

$k = 2.5$

i.e. $w = 2.5t$

When $t = 4.1$

$$\begin{aligned} w &= 2.5 \times 4.1 \\ &= 10.25 \end{aligned}$$

When $t = 6.7$

$$\begin{aligned} w &= 2.5 \times 6.7 \\ &= 16.75 \end{aligned}$$

All points in the table satisfy $w = 2.5t$.

$\therefore w \propto t$

b If $i \propto \frac{1}{r}$ then $i = \frac{k}{r}$.

When $r = 1.6$, $i = 5$

i.e. $5 = \frac{k}{1.6} \quad (\times 1.6)$

$k = 8$

i.e. $i = \frac{8}{r}$

When $r = 4$

$$\begin{aligned} i &= \frac{8}{4} \\ &= 2 \end{aligned}$$

When $r = 6.4$

$$\begin{aligned} i &= \frac{8}{6.4} \\ &= 1.25 \end{aligned}$$

All points in the table satisfy $i = \frac{8}{r}$.

$\therefore \quad i \propto \frac{1}{r}$

Example 2

John was paid \$25.80 for 3 hours work last week. No deductions come out of John's pay and so his pay is directly proportional to the number of hours he works. Find what he would be paid for $7\frac{1}{2}$ hours work.

Solution

$$\begin{aligned} Pay &\propto Hours \\ \therefore Pay &= k \times Hours \\ 25.80 &= k \times 3 \qquad (\div 3) \\ k &= 8.6 \\ \therefore Pay &= 8.6 \times Hours \\ Hours &= 7.5 \\ \therefore Pay &= 8.6 \times 7.5 \\ &= 64.5 \end{aligned}$$

i.e. John will be paid \$64.50 for $7\frac{1}{2}$ hours work.

Note: The constant k in this question is the gradient and shows that John's pay rate is \$8.60 per hour.

Exercise 6A

Example 1

1 Show that:

a v is proportional to t in the function shown in the table below.

t	1.6	3.4	7.6
v	4.16	8.84	19.76

b x is inversely proportional to t in the function shown in the table below.

t	2.5	4.5	12
x	3.6	2	0.75

2 Show that :

a y is proportional to x in the function shown in the table below.

x	3.4	4.1	6.2
y	4.42	5.33	8.06

b y is inversely proportional to t in the function shown in the table below.

t	2.5	6	12
y	1.92	0.8	0.4

Example 2

3 Peita was paid \$19.80 for $2\frac{1}{2}$ hours work last week. No deductions come out of Peita's pay and so her pay is directly proportional to the number of hours she works. Find what she would be paid for 7 hours work.

4 Rice retails at \$1.60 per 250 g bag. What is the constant of variation?

5 In an electrical circuit connected to a battery, the resistance is inversely proportional to the current. What is the constant of variation if the resistance is 7.2 ohms when the current is 5.4 amps?

6 In the burning of propane the amount of oxygen needed is proportional to the amount of propane burned. As by-products, the amount of carbon dioxide produced and the amount of water produced are proportional to the amount of propane burned. For complete combustion of propane, 1 mole of propane C_3H_8 reacts with 5 moles of oxygen O_2 to produce 3 moles of CO_2 and 4 moles of H_2O.

- **a** Write the amount of water produced as a function of the amount of oxygen used.
- **b** How much oxygen is needed for complete combustion of 12 moles of propane?
- **c** How much carbon dioxide is produced if 18 moles of oxygen are used in burning propane?

7 A model put forward by a researcher is: ‘The area of an oil spill on land varies directly as the volume of oil spilled’. The experimental data showed that 12 barrels of crude oil would cover an area of 320 m^2. How much crude oil was spilt if the area covered was 100 m^2?

8 The perimeter of a semicircle is directly proportional to the diameter of the semicircle. What is the constant of variation?

9 The area of a rectangle is 25 cm^2. Show that its length is inversely proportional to its width.

10 The area of a triangle is 7 cm^2. The length of the base of a triangle is inversely proportional to its perpendicular height. Find the constant of proportionality.

11 A cylindrical pipe has an internal diameter of 95 mm. Explain why the volume of water held by the pipe is directly proportional to the length of the pipe.

*The following note relates to questions **12** and **13**.*

Note: When a variable x is proportional to two other variables, such as y and z, we write $x \propto yz$. The area of a triangle is proportional to the length of the base and the perpendicular height.

$$A \propto bh, \text{ leading to } A = \frac{1}{2}bh.$$

In this case, the constant of proportionality is $\frac{1}{2}$.

12 Express the variables in these sentences in algebraic terms and find the constant of proportionality.

- **a** The area (A) of a rectangle is proportional to its length (l) and its breadth (b).
- **b** The volume (V) of a cone is proportional to its height (h) and the square of the radius of its base (r).
- **c** The area (A) of a semicircle is proportional to the square of its diameter (d).

13 The gravitational force (F) between two objects (with masses m_1 and m_2) is proportional to each of the masses and inversly proportional to the square of the distance (d) between them.

- **a** Write this statement in algebraic terms.
- **b** Explain the effect on the gravitational force of doubling the size of both of the masses and tripling the distance between them.

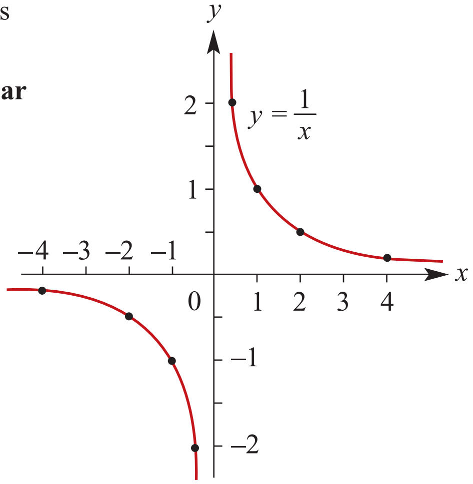
6.2 Rectangular hyperbolae

Although equations of the form $y = \dfrac{a}{x-h} + k$ have appeared in earlier chapters, they are treated again here.

Consider the rule $y = \dfrac{1}{x} = x^{-1}$, for $x \neq 0$.

We can construct a table of values for $y = \dfrac{1}{x}$ for values of x between −4 and 4, as follows.

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



We can plot these points and then connect the dots to produce a continuous curve.

A graph of this type is an example of a **rectangular hyperbola**.

It should be noted that when $x = 0$, y is undefined and there is no x value that will produce the value $y = 0$.

As x approaches infinity in either direction, the value of y approaches zero. The following notation will be used to state this.

As $x \to \infty$, $y \to 0^+$.

As $x \to -\infty$, $y \to 0^-$.

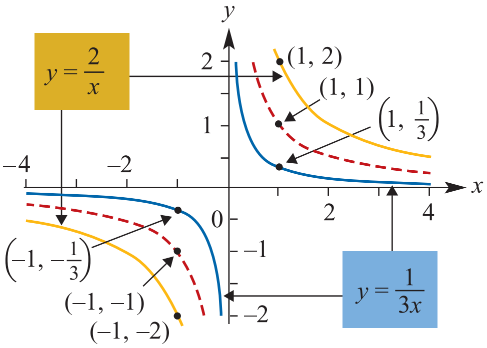
These are read: ‘As x approaches infinity, y approaches 0 from the positive side’ and ‘As x approaches negative infinity, y approaches 0 from the negative side’.

As x approaches zero from either direction, the magnitude of y becomes very large. The following notation will be used to state this.

As $x \to 0^+$, $y \to \infty$ and as $x \to 0^-$, $y \to -\infty$.

These are read: ‘As x approaches zero from the positive side, y approaches infinity’ and ‘As x approaches zero from the negative side, y approaches negative infinity’.

The graph approaches both the x-axis (the line $y = 0$) and the y-axis (the line $x = 0$) but does not cross either of these lines.



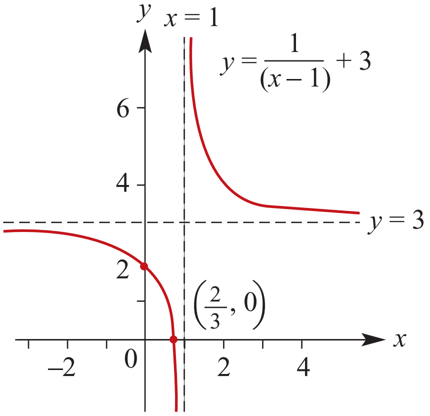
We refer to these lines as **asymptotes**.

Hence, for the graph of $y = \frac{1}{x}$, the equations of the asymptotes are $y = 0$ and $x = 0$.

In the diagram on the right, the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ are shown.

The asymptotes are the x-axis and the y-axis, and they have equations $y = 0$ and $x = 0$, respectively.

As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same ‘shape’ and asymptotes as $y = \frac{1}{x}$. A closer look at this will be taken later in this chapter, when discussing transformations.



Now, let us consider the graph of $y = \frac{1}{(x-1)} + 3$.

The basic graph of $y = \frac{1}{x}$, has been translated 1 unit to the right and 3 units up. The equation of the vertical asymptote is now $x = 1$ and the equation of the horizontal asymptote is now $y = 3$.

The graph now has x-axis and y-axis intercepts. These can be calculated in the usual way to give further detail to the graph.

***x*-axis intercept:** let $y = 0$

$$\begin{aligned} 0 &= \frac{1}{x-1} + 3 \\ -3 &= \frac{1}{x-1} \\ -3(x-1) &= 1 \\ x &= \frac{2}{3} \end{aligned}$$

$\therefore$ Intercept is $\left(\frac{2}{3}, 0\right)$.

***y*-axis intercept:** let $x = 0$

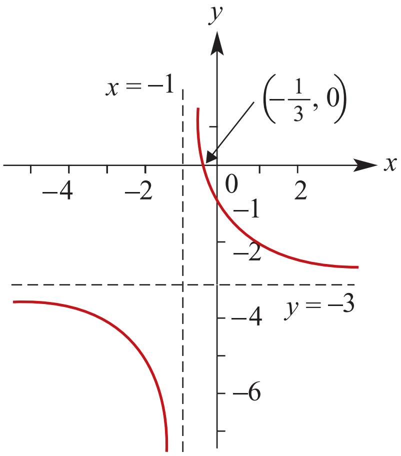
$$\begin{aligned} y &= \frac{1}{(0-1)} + 3 \\ y &= 2 \end{aligned}$$

$\therefore$ Intercept is (0, 2).

Using the technique above, we are therefore able to sketch graphs of all rectangular hyperbolae of the form $y = \frac{a}{x-h} + k$.

Example 3

Sketch the graph of $y = \dfrac{2}{x+1} - 3$.



Solution

The graph of $\dfrac{2}{x}$ has been translated 1 unit to the left and 3 units down.

Asymptotes have equations $x = -1$ and $y = -3$.

y-axis intercept: $(0, -1)$

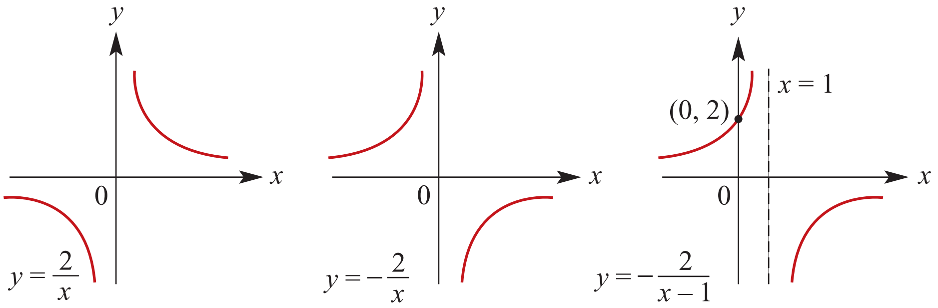
x-axis intercept: $\left(-\dfrac{1}{3}, 0\right)$

Example 4

Sketch the graph of $y = \dfrac{-2}{x-1}$.

Solution

The graph of $y = -\dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{2}{x}$ by reflection in the x-axis. This graph is then translated 1 unit to the right to obtain the graph of $y = \dfrac{-2}{x-1}$.



Exercise 6B

Examples 3, 4

1 Sketch the graphs below of the following, showing all important features of the graphs:

a $y = \dfrac{1}{x}$ **b** $y = \dfrac{2}{x}$ **c** $y = \dfrac{1}{2x}$ **d** $y = \dfrac{-3}{x}$

e $y = \dfrac{1}{x} + 2$ **f** $y = \dfrac{1}{x} - 3$ **g** $y = \dfrac{2}{x} - 4$ **h** $y = \dfrac{-1}{2x} + 5$

i $y = \dfrac{1}{x-1}$ **j** $y = \dfrac{-1}{x+2}$ **k** $y = \dfrac{1}{x+1} + 3$ **l** $y = \dfrac{-2}{x-3} - 4$

2 Write down the equations of the asymptotes for each of the graphs in Question **1**.

6.3 Introduction to polynomial functions

Linear and quadratic functions were considered in Chapter 1. These functions are examples of what are generally called **polynomials**. $f(x) = 6$, $f(x) = 3x - 7$ and $f(x) = 4x^2 + 2x + 5$ are all examples of polynomial functions. A polynomial of the form $f(x) = ax^2 + bx + c$ is said to have **degree** 2 because the highest power of x in the function is 2. $f(x) = ax + b$ has degree 1 and $f(x) = a$ has degree 0.

A third degree polynomial is called a cubic and is a function, f, with rule

$$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$$

A fourth degree polynomial is called a quartic and is a function, f, with rule

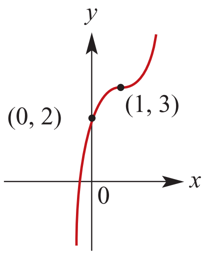
$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$$

In Chapter 1 it was shown that all quadratic functions could be written in ‘perfect square’ form and that the graph of a quadratic has one basic form, the parabola.

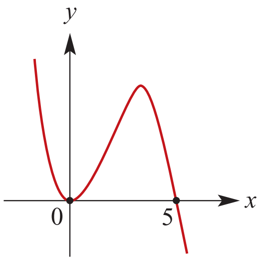
This is not true of cubic or quartic functions.

Two examples of graphs of cubic functions and two examples of quartic functions are shown.

Cubic functions

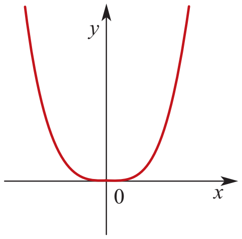


$$f(x) = (x - 1)^3 + 3$$
$$= x^3 - 3x^2 + 3x + 2$$

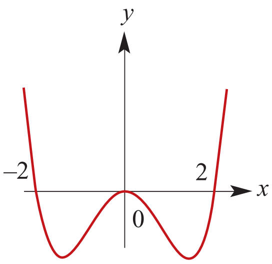


$$f(x) = x^2(5 - x)$$
$$= -x^3 + 5x^2$$

Quartic functions



$y = 2x^4$



$y = x^4 - 4x^2 = x^2(x^2 - 4)$

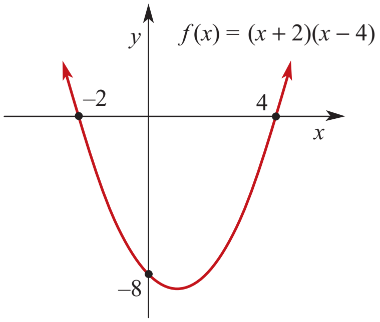
Sketching polynomials

The method used to sketch polynomials in this text is the method of using the x- and y-intercepts, discussed in Chapter 1. In the language of Chapter 4, this method went as follows:

1. Find the y-intercept by evaluating $f(0)$.
2. Find the x-intercept by solving $f(x) = 0$.
3. Find another point if $f(0) = 0$.
4. Sketch the curve.

Example 5

Sketch $f(x) = (x + 2)(x - 4)$.



Solution

$$f(0) = -8$$

i.e. y-intercept is $(0, -8)$.

$$f(x) = 0 \qquad \therefore\ 0 = (x+2)(x-4)$$

$$(x+2)(x-4) = 0$$

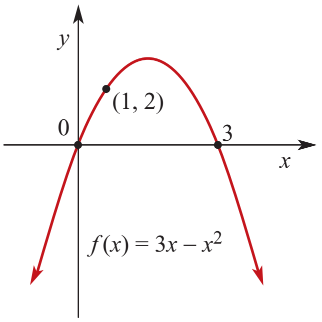
Null factor

$$x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$

$$\therefore\ x = -2 \text{ or } x = 4$$

Example 6

Sketch $f(x) = 3x - x^2$.



Solution

$$f(0) = 0$$

i.e. y-intercept is $(0, 0)$.

$$f(x) = 0 \qquad \therefore\ 0 = 3x - x^2$$

$$3x - x^2 = 0$$

$$x(3 - x) = 0$$

Null factor

$$x = 0 \quad \text{or} \quad 3 - x = 0$$

$$\therefore\ x = 0 \text{ or } \ x = 3$$

i.e. x-intercepts are $(0, 0)$ and $(3, 0)$.

$$f(1) = 3 \times 1 - 1^2$$
$$= 2$$

i.e. $(1, 2)$ is on the graph.

Note: The point $(1, 2)$ was found because the graph passed through the origin.

Note: Factorising the polynomial is a key step in the process of sketching polynomials of order 2 or higher.

Exercise 6C

1 Sketch each of the following:

a $f(x) = 2x - 3$ **b** $f(x) = 5 - x$ **c** $f(x) = 3$

d $f(x) = \frac{2}{3}x$ **e** $f(x) = -2$ **f** $f(x) = 4 - \frac{3}{4}x$

Example 5 **2** Sketch each of the following:

a $f(x) = (x-2)(x+1)$ **b** $f(x) = (x-3)(x+1)$ **c** $f(x) = (x+2)(x+3)$

d $f(x) = (2-x)(x-3)$ **e** $f(x) = (x-2)(2x+3)$ **f** $f(x) = x(5-2x)$

Example 6 **3** Sketch:

a $f(x) = x^2 - 4x$ **b** $f(x) = x^2 - 4$ **c** $f(x) = x^2 - 4x + 3$

d $f(x) = x^2 - 2x - 8$ **e** $f(x) = x^2 - 6x + 8$ **f** $f(x) = 2x^2 + 6x + 4$

g $f(x) = 9 - 4x^2$ **h** $f(x) = 6x^2 - 7x - 10$

6.4 Division of polynomials

A key stage in sketching polynomials is ‘Find the x-intercepts by solving $f(x) = 0$’. Although there is no simple method for solving cubics and quartics (and no general method for solving higher polynomials), a number of useful techniques are treated in this text.

Reviewing the process of long division; for example, 206 ÷ 13, gives

$$\begin{array}{r}
15 \\
13\,\overline{)206} \\
\underline{13} \\
76 \\
\underline{65} \\
11
\end{array}$$

$$\therefore\ \frac{206}{13} = 15 \text{ and } 11 \text{ remainder, i.e } 15\frac{11}{13}.$$

We note that as there is a remainder, it can be seen that 13 is not a factor of 206.

The process of dividing a polynomial by a linear factor follows very similar steps.

For example, $(x^2 + 7x + 11) \div (x - 2)$ gives

$$\begin{array}{rl}
x + 9 & \\
x - 2\,\overline{)\,x^2 + 7x + 11} & \text{Divide } x^2 \text{ by } x. \\
\underline{x^2 - 2x\phantom{{}+11}} & \text{Multiply } (x-2) \text{ by } x \text{ and subtract from } x^2 + 7x + 11. \\
9x + 11 & \text{This leaves } 9x + 11;\, x \text{ into } 9x \text{ goes 9 times.} \\
\underline{9x - 18} & \text{Multiply } (x-2) \text{ by 9 and subtract from } 9x + 11. \\
29 & \text{This leaves 29 remainder.}
\end{array}$$

Thus, $(x^2 + 7x + 11) \div (x - 2) = x + 9$ with remainder 29.

$$\therefore\ \frac{x^2 + 7x + 11}{x - 2} = x + 9 + \frac{29}{x - 2}$$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$.

Example 7

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r l}
 & \phantom{x^3 +{}} x^2 - x - 12 \\
x + 2 \big) & \overline{x^3 + x^2 - 14x - 24} \\
 & \underline{x^3 + 2x^2} \\
 & -x^2 - 14x \\
 & \underline{-x^2 - 2x} \\
 & -12x - 24 \\
 & \underline{-12x - 24} \\
 & 0
\end{array}$$

Thus, $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \frac{x^3 + x^2 - 14x - 12}{x + 2} = x^2 - x - 12$$

Note: In this example, we see that $x + 2$ *is* a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero.

Example 8

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

$$\begin{array}{r l}
 & \phantom{3x^3 +{}} 3x^2 + 6x + 13 \\
x - 2 \big) & \overline{3x^3 + 0x^2 + x - 3} \\
 & \underline{3x^3 - 6x^2} \\
 & \phantom{3x^3 +{}} 6x^2 + x \\
 & \phantom{3x^3 +{}} \underline{6x^2 - 12x} \\
 & \phantom{3x^3 + 6x^2 +{}} 13x - 3 \\
 & \phantom{3x^3 + 6x^2 +{}} \underline{13x - 26} \\
 & \phantom{3x^3 + 6x^2 + 13x -{}} 23
\end{array}$$

$$\begin{aligned}
\text{Thus, } \frac{3x^3 + x - 3}{x - 2} &= 3x^2 + 6x + 13 \text{ with a remainder of } 23. \\
&= 3x^2 + 6x + 13 + \frac{23}{x - 2}
\end{aligned}$$

Note: Here there is no term in x^2, however, we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

Example 9

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{rl}
 & \;\;\frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
2x+1 & \overline{)\,3x^3 + 2x^2 - x - 2} \\
 & \underline{\;\;3x^3 + \frac{3}{2}x^2} \\
 & \qquad\;\; \frac{1}{2}x^2 - x \\
 & \qquad\;\; \underline{\frac{1}{2}x^2 + \frac{1}{4}x} \\
 & \qquad\qquad\quad -\frac{5}{4}x - 2 \\
 & \qquad\qquad\quad \underline{-\frac{5}{4}x - \frac{5}{8}} \\
 & \qquad\qquad\qquad\quad -1\frac{3}{8}
\end{array}
$$

$$\therefore \frac{3x^3 + 2x^2 - x - 2}{2x + 1} = \frac{3x^2}{2} + \frac{x}{4} - \frac{5}{8} - \frac{11}{8(2x + 1)}$$

Exercise 6D

Examples 7, 8 **1** For each of the following, divide the polynomial by the accompanying linear expression:

- **a** $x^3 + x^2 - 2x + 3,\ x - 1$
- **b** $2x^3 + x^2 - 4x + 3,\ x + 1$
- **c** $3x^3 - 4x^2 + 2x + 1,\ x + 2$
- **d** $x^3 + 3x - 4,\ x + 1$
- **e** $2x^3 - 3x^2 + x - 2,\ x - 3$
- **f** $2x^3 + 3x^2 + 17x + 15,\ x + 4$
- **g** $x^3 + 4x^2 + 3x + 2,\ x + 3$

Example 9 **2** For each of the following, divide the polynomial by the accompanying linear expression:

- **a** $x^3 + 6x^2 + 8x + 11,\ 2x + 5$
- **b** $2x^3 + 5x^2 - 4x - 5,\ 2x + 1$
- **c** $x^3 - 3x^2 + 1,\ 3x - 1$
- **d** $x^3 - 3x^2 + 6x + 5,\ x - 2$
- **e** $2x^3 + 3x^2 - 32x + 15,\ 2x - 1$
- **f** $x^3 + 2x^2 - 1,\ 2x + 1$

6.5 Factorising cubics

Factor theorem

In order for $(x - a)$ to be a factor of $P(x)$, $P(a)$ must equal zero. This result is known as the factor theorem.

> If, for a polynomial, $P(x)$, $P(a) = 0$ then $x - a$ is a factor.
> Conversely, if $x - a$ is a factor of $P(x)$ then $P(a) = 0$.

More generally:

If $ax + b$ is a factor of $P(x)$ then $P\left(-\frac{b}{a}\right) = 0$.

Conversely, if $P\left(-\frac{b}{a}\right) = 0$ then $ax + b$ is a factor of $P(x)$.

Example 10

Show that $(x + 1)$ is a factor of $x^3 - 4x^2 + x + 6$ and, hence, find the other linear factors.

Solution

$$\begin{aligned} \text{Let } P(x) &= x^3 - 4x^2 + x + 6 \\ P(-1) &= (-1)^3 - 4(-1)^2 + (-1) + 6 \\ &= 0 \end{aligned}$$

$\therefore$ From the factor theorem $(x - (-1)) = (x + 1)$ is a factor.

$$\begin{array}{r@{}l} & x^2 - 5x + 6 \\ x + 1 \,\big) & \overline{x^3 - 4x^2 + x + 6} \\ & \underline{x^3 + x^2} \\ & -5x^2 + x \\ & \underline{-5x^2 - 5x} \\ & 6x + 6 \\ & \underline{6x + 6} \\ & 0 \end{array}$$

$$\begin{aligned} \therefore \quad x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) \\ &= (x + 1)(x - 3)(x - 2) \end{aligned}$$

$\therefore$ The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Note: When factorising $P(x)$, find the factors of the constant term in $P(x)$ and substitute them into $P(x)$. This works because:

$$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc$$

Example 11

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

The factors of 6 are ±1, ±2, ±3, ±6.

$$\begin{aligned} P(-1) &= (-1)^3 - 2(-1)^2 - 5(-1) + 6 \neq 0 \\ P(1) &= (1)^3 - 2(1)^2 - 5(1) + 6 = 0 \\ P(1) &= 1 - 2(1) - 5 + 6 = 0 \end{aligned}$$

$\therefore$ $(x - 1)$ is a factor.

$$\begin{array}{r|l}
 & x^2 - x - 6 \\ \hline
x - 1 & x^3 - 2x^2 - 5x + 6 \\
 & x^3 - x^2 \\ \hline
 & \quad - x^2 - 5x \\
 & \quad - x^2 + x \\ \hline
 & \qquad\quad - 6x + 6 \\
 & \qquad\quad - 6x + 6 \\ \hline
 & \qquad\qquad\quad 0
\end{array}$$

$$\begin{aligned}
\therefore \quad x^3 - 2x^2 - 5x + 6 &= (x - 1)(x^2 - x - 6) \\
&= (x - 1)(x - 3)(x + 2)
\end{aligned}$$

$\therefore$ The factors of $x^3 - 2x^2 - 5x + 6$ are $(x - 1)$, $(x - 3)$ and $(x + 2)$.

Special cases: Sums and differences of cubes

Example 12

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$

$P(3) = 27 - 27 = 0$

$\therefore (x - 3)$ is a factor.

$$\begin{array}{r|l}
 & x^2 + 3x + 9 \\ \hline
x - 3 & x^3 + 0x^2 + 0x - 27 \\
 & x^3 - 3x^2 \\ \hline
 & \qquad 3x^2 \\
 & \qquad 3x^2 - 9x \\ \hline
 & \qquad\qquad 9x - 27 \\
 & \qquad\qquad 9x - 27 \\ \hline
 & \qquad\qquad\qquad 0
\end{array}$$

$$\therefore \quad x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$

In general, if $P(x) = x^3 - a^3$ then $(x - a)$ is a factor and by division

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$ then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 13

Factorise $8x^3 + 64$.

Solution

$$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \\ &= 8(x + 1)(x^2 - 2x + 4) \end{aligned}$$

Exercise 6E

Examples 10, 11

1 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$ **b** $x^3 + 3x^2 + 3x + 1$ **c** $6x^3 - 13x^2 + 13x - 6$

d $x^3 - 21x + 20$ **e** $2x^3 + 3x^2 - 1$ **f** $x^3 - x^2 - x + 1$

g $4x^3 + 3x - 38$ **h** $4x^3 + 4x^2 - 11x - 6$

Examples 12, 13

2 Factorise each of the following:

a $x^3 - 1$ **b** $x^3 + 64$ **c** $27x^3 - 1$ **d** $64x^3 - 125$

e $1 - 125x^3$ **f** $8 + 27x^3$ **g** $64m^3 - 27n^3$ **h** $27b^3 + 8a^3$

3 Factorise each of the following:

a $x^3 + x^2 - x + 2$ **b** $3x^3 - 7x^2 + 4$

c $x^3 - 4x^2 + x + 6$ **d** $6x^3 + 17x^2 - 4x - 3$

MAPS

4 Find the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

6.6 Graphs of cubic functions

Example 14

Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$.

Solution

$$P(0) = -6$$

$\therefore$ y-intercept is -6.

Let $P(x) = x^3 + 2x^2 - 5x - 6$

$$P(1) = 1 + 2 - 5 - 6 \neq 0$$

$$P(-1) = -1 + 2 + 5 - 6 = 0$$

$\therefore$ $(x + 1)$ is a factor.

By division

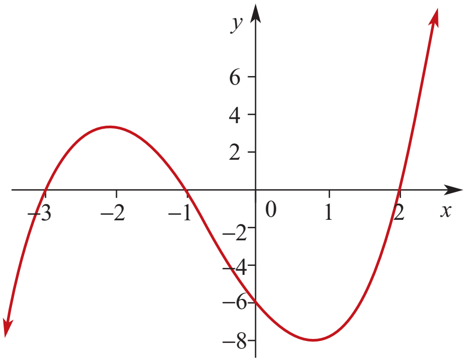
$y = (x + 1)(x - 2)(x + 3)$

For $(x + 1)(x - 2)(x + 3) = 0$

$x + 1 = 0$ or $x - 2 = 0$ or $x + 3 = 0$

$\therefore x = -1, 2$ or -3

$\therefore x$-intercepts are -1, 2 and -3.

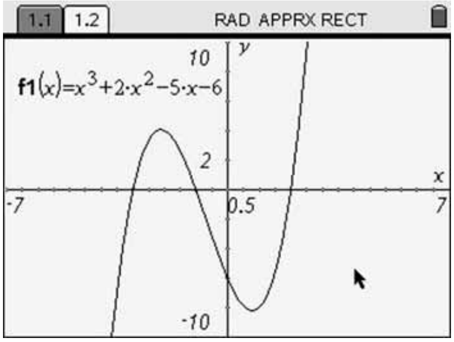


At this stage the location of the turning points is unspecified. It is important, however, to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. It will be shown later in the course how to determine the turning points.

Using technology

Using the TI-Nspire:

1 Enter **x^3 + 2x^2 – 5x – 6** into $f1(x)$ and choose a suitable window.

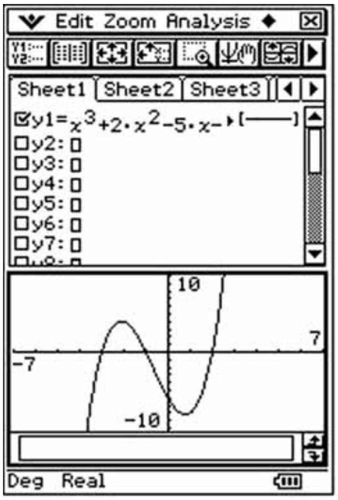


2 Press menu and select *Graph Trace* from the Trace submenu.

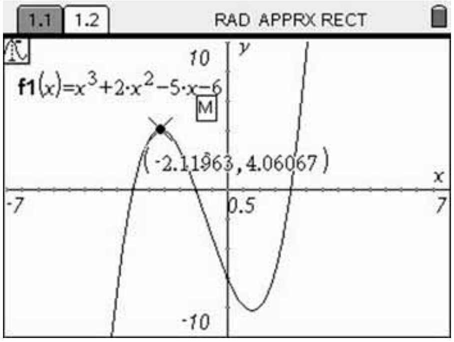
Using the ClassPad:

1 Enter **x^3+ 2x^2 – 5x – 6** into $y1$ and press EXE.

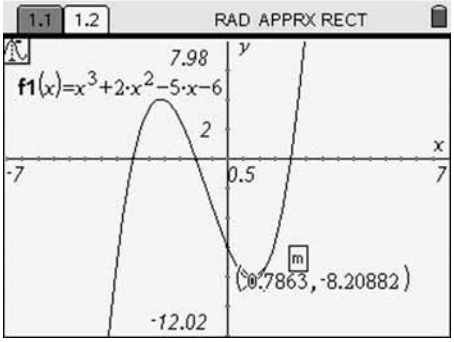
2 Tap and choose a suitable window.



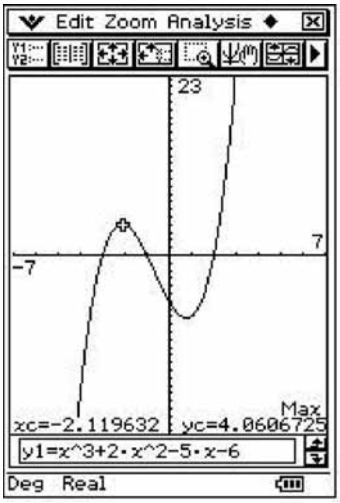
3 Tap the left and right buttons and move the cursor to the maximum until $\boxed{\text{M}}$ is displayed on the screen.



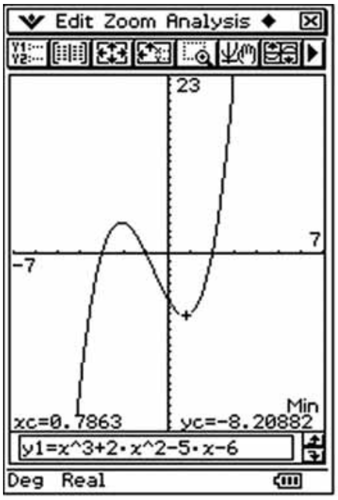
4 Move the cursor to the minimum until $\boxed{\text{m}}$ is displayed on the screen.



3 To display the coordinates of the maximum, tap Analysis and select *Max* from the G-Solve submenu.



4 To display the coordinates of the minimum, tap Analysis and select *Min* from the G-Solve submenu.



Example 15

Sketch the graph of $y = (x - 1)(x + 2)(x + 1)$. Do not give coordinates of turning points.

Solution

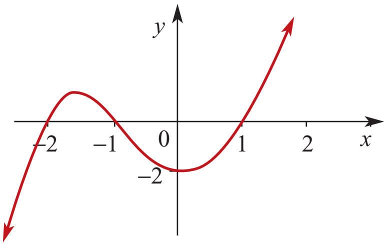
Let $y = 0$

$$0 = (x - 1)(x + 2)(x + 1)$$

Using the null factor law, the x-axis intercepts are 1, -1 and -2.

To find the y-axis intercept, let $x = 0$.

$$\begin{aligned} y &= (0 - 1)(0 + 2)(0 + 1) \\ &= -2 \end{aligned}$$



If the factorised cubic has a repeated factor there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 16

Sketch the graph of $y = x^2(x - 1)$.

Solution

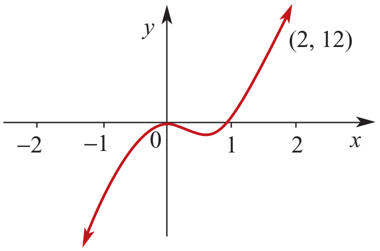
To find the x-axis intercepts, let $y = 0$.

Then $\quad x^2(x - 1) = 0$

The x-axis intercepts are at $x = 0$ and 1 and, because the repeated factor is x^2, there is also a turning point at $x = 0$.

The y-axis intercept (letting $x = 0$) is at $y = 0$.

When $x = 2$, $y = 12$.



Some cubics will have only one x-axis intercept. This is because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 17

Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor of $x^2 + 4x + 5$ cannot be factorised further.

$$\begin{aligned}\Delta &= b^2 - 4ac\\ \Delta &= 4^2 - 4(1)(5)\\ &= -4\end{aligned}$$

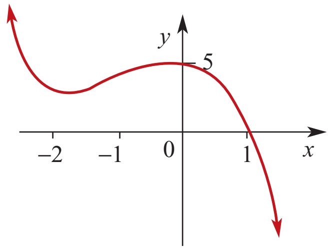
$\therefore$ There are no linear factors.

Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

$\therefore$ x-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$.

$$-((0) - 1)((0)^2 + 4(0) + 5) = 5$$



Note: Using a calculator, it is found that the turning points are at (−0.18, 5.09) and (−1.82, 2.91), where the values for the coordinates of the second point are given to 2 decimal places.

Exercise 6F

Examples 14–17

1 Sketch the graphs for each of the following. Label your sketch, showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

a $y = x(x - 1)(x - 3)$

b $y = (x - 1)(x + 1)(x + 2)$

c $y = (x - 1)(x - 2)(x - 3)$

d $y = (2x - 1)(x - 2)(x + 3)$

e $y = x^3 - 9x$

f $y = x^3 + x^2$

g $y = x^3 - 5x^2 + 7x - 3$

h $y = x^3 - 4x^2 - 3x + 18$

i $y = -x^3 + x^2 + 3x - 3$

j $y = 3x^3 - 4x^2 - 13x - 6$

k $y = 6x^3 - 5x^2 - 2x + 1$

2 Sketch the graphs of each of the following, using a graphics calculator to find the coordinates of axes intercepts and local maximum and local minimum values:

a $y = -4x^3 - 12x^2 + 37x - 15$

b $y = -4x^3 + 19x - 15$

c $y = -4x^3 + 0.8x^2 + 19.8x - 18$

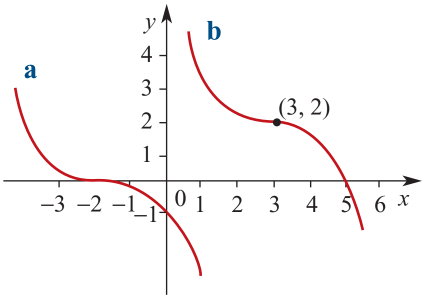
d $y = 2x^3 + 11x^2 + 15x$

e $y = 2x^3 + 6x^2$

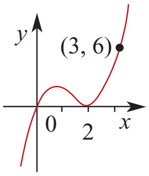
f $y = 2x^3 + 6x^2 + 6$

3 Show that the graph of f, where $f(x) = x^3 - x^2 - 5x - 3$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

4 The graphs shown are similar to the basic curve $y = -x^3$. Find possible cubic functions that define each of the curves.



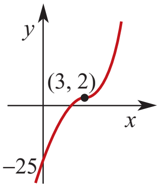
5 Find the equation of the cubic function for which the graph is shown.



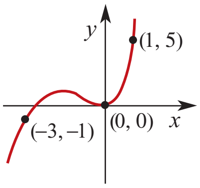
6 Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

7 The graphs below have equations of the form shown. In each case, determine the equation.

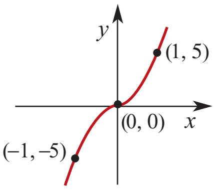
a $y = a(x - h)^3 + k$



b $y = ax^3 + bx^2$



c $y = ax^3$



6.7 Graphs of quartic functions

The techniques for graphing quartic functions are very similar to those employed for cubic functions. This section will first consider solving simple quartic equations. A graphics calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

Example 18

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a
$$\begin{aligned} x^4 - 8x &= 0 \\ x(x^3 - 8) &= 0 \\ \therefore\quad x &= 0 \text{ or } x^3 - 8 = 0 \end{aligned}$$
Thus, $x = 2$ or $x = 0$.

b
$$\begin{aligned} 2x^4 - 8x^2 &= 0 \\ 2x^2(x^2 - 4) &= 0 \\ \therefore\quad 2x^2 &= 0 \text{ or } x^2 - 4 = 0 \end{aligned}$$
Thus, $x = 2$ or $x = -2$ or $x = 0$.

c
$$\begin{aligned} x^4 - 2x^3 - 24x^2 &= 0 \\ x^2(x^2 - 2x - 24) &= 0 \\ \therefore\quad x^2 &= 0 \text{ or } x^2 - 2x - 24 = 0 \\ \text{i.e. } x &= 0 \text{ or } (x - 6)(x + 4) = 0 \end{aligned}$$
Thus, $x = 0$ or $x = 6$ or $x = -4$.

In general, quartic equations can be solved by techniques similar to those used for solving cubic functions, but in this course only those equations that can be solved by 'simple' techniques are dealt with.

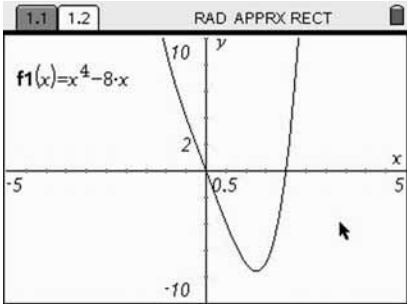
Using technology

Example 19

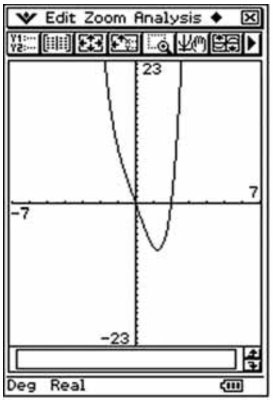
Use a graphics calculator to graph each of the following quartic equations:

a $f(x) = x^4 - 8x$ **b** $f(x) = 2x^4 - 8x^2$ **c** $f(x) = \frac{1}{2}(x^4 - 2x^3 - 24x^2)$

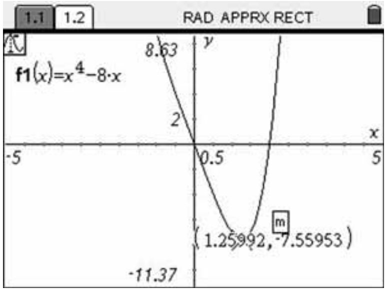
a Using the TI-Nspire:



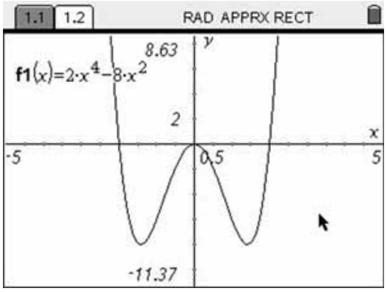
a Using the ClassPad:



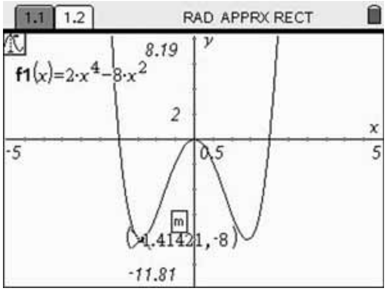
This is done through *Graph Trace* from the Trace menu.



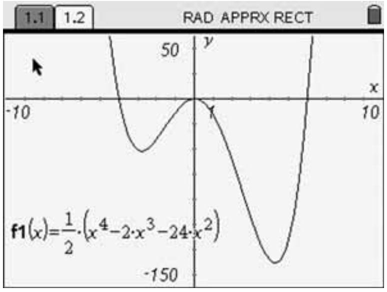
b Using the TI-Nspire:



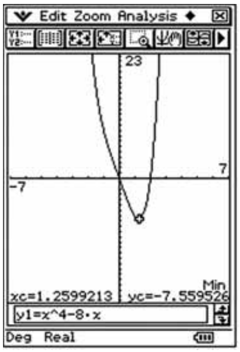
This is done through *Graph Trace* from the Trace menu.



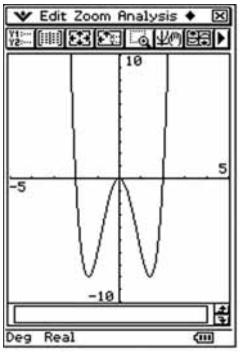
c Using the TI-Nspire:



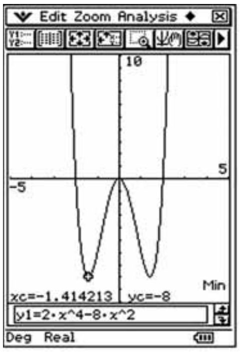
This is done using *Min* from the G-Solve submenu.



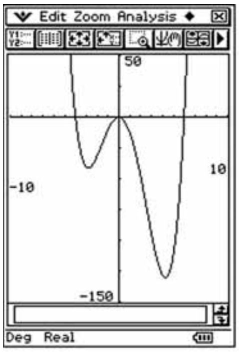
b Using the ClassPad:



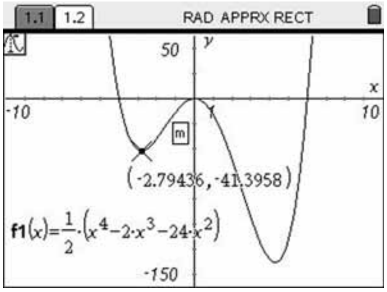
This is done using *Min* from the G-Solve submenu.



c Using the ClassPad:

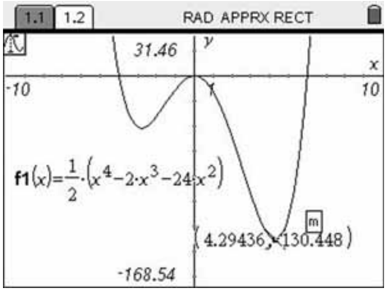


This is done through *Graph Trace* from the Trace menu.



(-2.79436, -41.3958)

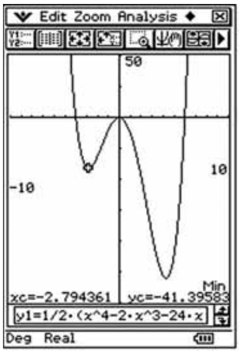
$f1(x)=\frac{1}{2}\cdot(x^4-2\cdot x^3-24\cdot x^2)$



$f1(x)=\frac{1}{2}\cdot(x^4-2\cdot x^3-24\cdot x^2)$

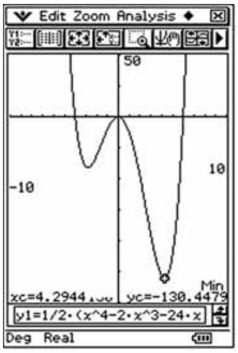
(4.29436, -130.448)

This is done using *Min* from the G-Solve submenu.



xc=-2.794361 yc=-41.39583

y1=1/2·(x^4-2·x^3-24·x



xc=4.2944… yc=-130.4479

y1=1/2·(x^4-2·x^3-24·x

Exercise 6G

Example 18

1 Solve each of the following equations for x:

a $x^4 - 27x = 0$

b $(x^2 - x - 2)(x^2 - 2x - 15) = 0$

c $x^4 + 8x = 0$

d $x^4 - 6x^3 = 0$

e $x^4 - 9x^2 = 0$

f $81 - x^4 = 0$

g $x^4 - 16x^2 = 0$

h $x^4 - 7x^3 + 12x^2 = 0$

i $x^4 - 9x^3 + 20x^2 = 0$

j $(x^2 - 4)(x^2 - 9) = 0$

k $(x - 4)(x^2 + 2x + 8) = 0$

l $(x + 4)(x^2 + 2x - 8) = 0$

Example 19

2 Use a graphics calculator to help draw the graphs of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points are to be given correct to 2 decimal places.)

a $y = x^4 - 125x$
b $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$
d $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$
f $y = 16 - x^4$
g $y = x^4 - 81x^2$
h $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$
j $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$
l $y = (x + 4)(x^2 + 2x - 35)$

6.8 Applying translations to curve sketching

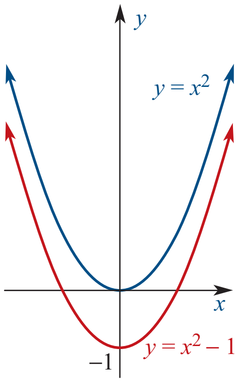
Curve sketching can be made simpler by recognising that, in many cases, the graph required is a **translation**, horizontally, vertically or both, of a simpler known graph.

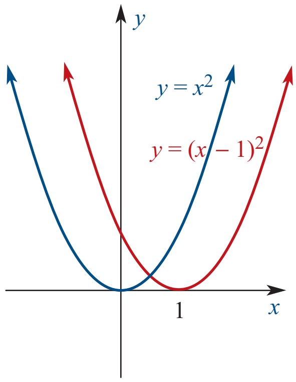
The graphs of $y = x^2 - 1$, $y = (x - 1)^2$ and $y = (x + 1)^2 + 2$ are simply translations of $y = x^2$.

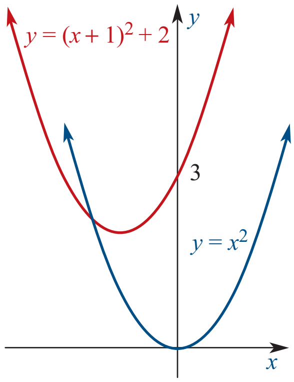
$y = x^2 - 1$ is $y = x^2$ translated 1 unit downwards.

$y = (x - 1)^2$ is $y = x^2$ translated 1 unit to the right.

$y = (x + 1)^2 + 2$ is $y = x^2$ translated 1 unit to the left and 2 units upwards.







In general, $y = f(x - h) + k$ is a translation, h to the right and k upwards, of the graph of $y = f(x)$.

Example 20

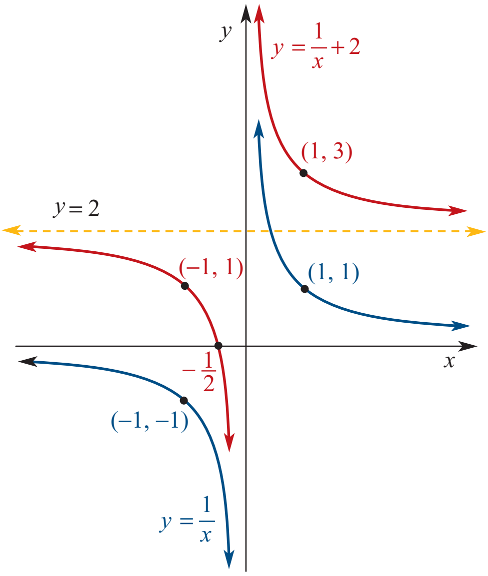
Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = \frac{1}{x}$ and $y = \frac{1}{x} + 2$

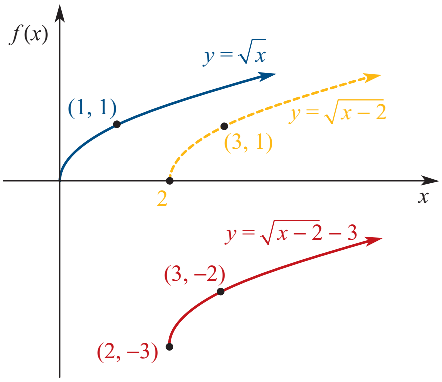
b $y = \sqrt{x}$ and $y = \sqrt{x - 2} - 3$

Solution

a +2 is a translation upwards of 2 units.



b −2 is a translation to the right of 2 units.
−3 is a translation downwards of 3 units.

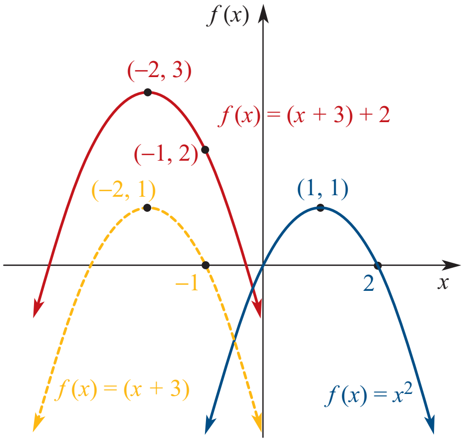


Example 21

$f(x) = 2x - x^2$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = f(x + 3) + 2$.

Solution

+3 is a translation to the left of 3 units.
+2 is a translation upwards of 2 units.



Exercise 6H

Example 20 **1** Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = x^2$ and $y = (x + 2)^2$

b $y = \frac{4}{x}$ and $y = \frac{4}{x} - 2$

c $y = x^2 - 2x$ and $y = (x + 3)^2 - 2(x + 3)$

d $y = \sqrt{x}$ and $y = \sqrt{x} - 4$

e $y = \log_{10} x$ and $y = \log_{10} x - 2$

f $y = 2^x$ and $y = 2^{x+1}$

2 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = x^2$ and $y = (x + 1)^2 - 3$

b $y = \sqrt{x}$ and $y = \sqrt{x - 2} + 3$

c $y = \frac{2}{x}$ and $y = \frac{2}{x + 2} - 1$

d $y = -x^2$ and $y = 4 - (x - 1)^2$

e $y = \log_{10} x$ and $y = \log_{10}(x - 1) - 2$

f $y = 3^x$ and $y = 3^{x-2} + 1$

Example 21 **3** $f(x) = x^2$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = f(x - 3) - 4$.

4 $g(x) = x^2 + 2x$. Show, with a sketch on the Cartesian plane, how $y = g(x)$ is transformed into $y = g(x + 3) + 1$.

5 $h(x) = \begin{cases} 2x, & x < 0 \\ x^2 - 2x, & x \geq 0 \end{cases}$. Show, with a sketch on the Cartesian plane, how $y = h(x)$ is transformed into $y = h(x + 2) - 3$.

6 $p(x) = \begin{cases} 2x - 1, & x < 1 \\ \frac{1}{x}, & x \geq 1 \end{cases}$. Show, with a sketch on the Cartesian plane, how $y = p(x)$ is transformed into $y = p(x - 3) + 2$.

6.9 Applying dilations and reflections to curve sketching

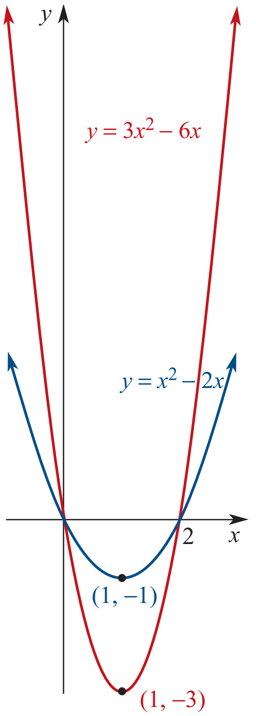
Curve sketching can be made simpler by also recognising that, in many cases, the graph required is a **dilation from the *x*-axis**, a **dilation from the *y*-axis** or **reflection** of a simpler known graph.

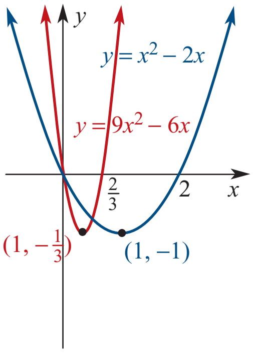
The graphs of $y = 3x^2 - 6x$, $y = 9x^2 - 6x$ and $y = -x^2 + 2x$ are a dilation from the x-axis, a dilation from the y-axis and a reflection of $y = x^2 - 2x$.

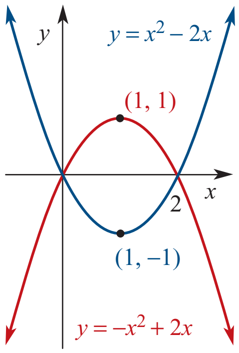
$y = 3x^2 - 6x$ is a dilation from the x-axis by a factor of 3 of the curve $y = x^2 - 2x$.

$y = 9x^2 - 6x$ is a dilation from the y-axis by a factor of $\frac{1}{3}$ of the curve $y = x^2 - 2x$.

$y = -x^2 + 2x$ is a reflection about the x-axis of the curve $y = x^2 - 2x$.







In general, $y = af(x)$ is a dilation from the x-axis of the graph of $y = f(x)$ by a factor of a. This means that $y = f(x)$ is stretched away from the x-axis by a factor of a. In the illustration above, note that $3x^2 - 6x = 3\left(x^2 - 2x\right)$ and that all points on $y = 3x^2 - 6x$ are 3 times as far from the x-axis as their corresponding point on $y = x^2 - 2x$.

In general, $y = f(ax)$ is a dilation from the y-axis of the graph of $y = f(x)$ by a factor of $\frac{1}{a}$. This means that $y = f(x)$ is compressed towards the y-axis by a factor of a (i.e. the stretch factor is $\frac{1}{a}$). In the illustration above, note that $9x^2 - 6x = (3x)^2 - 2(3x)$ and all points on $y = 9x^2 - 6x$ are $\frac{1}{3}$ as far from the y-axis as their corresponding point on $y = x^2 - 2x$.

Also, in general $y = -f(x)$ is a reflection of the graph of $y = f(x)$ about the x-axis. The impact of this is that all points above the x-axis in $y = f(x)$ end up below it in $y = -f(x)$. And all points below the x-axis in $y = f(x)$ end up above it in $y = -f(x)$.

Note: Any points on the x-axis are unaffected by dilation from the x-axis or reflection about the x-axis.

Note: Any points on the y-axis are unaffected by dilation from the y-axis.

Example 22

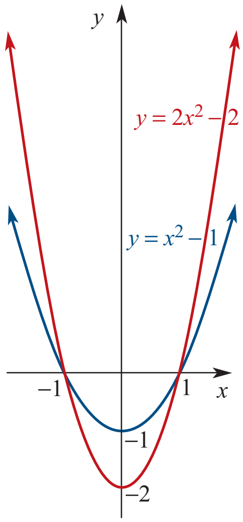
Show, with a sketch on the Cartesian plane, how $y = x^2 - 1$ is transformed into:

a $y = 2x^2 - 2$ **b** $y = 4x^2 - 1$ **c** $y = 1 - x^2$

Solution

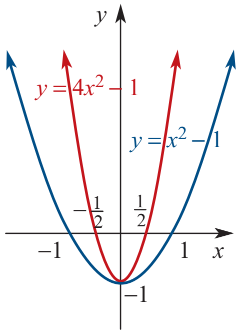
a $2x^2 - 2 = 2\left(x^2 - 1\right)$

$\therefore$ Dilate from the x-axis by a factor of 2.



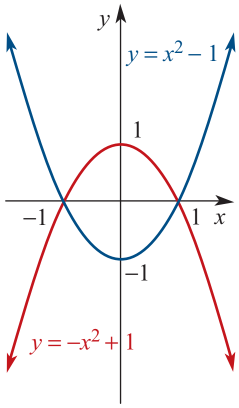
b $4x^2 - 1 = (2x)^2 - 1$

$\therefore$ Dilate from the y-axis by a factor of $\frac{1}{2}$.



c $1 - x^2 = -(x^2 - 1)$

$\therefore$ Reflect about the x-axis.

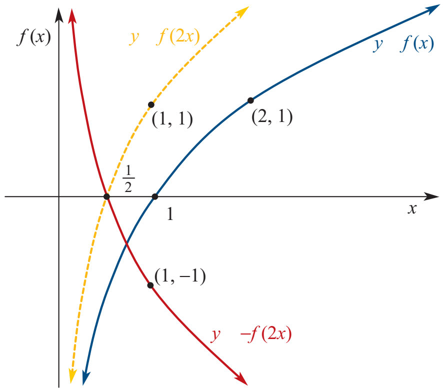


Example 23

$f(x) = \log_2 x$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = -f(2x)$.

Solution

2 is a dilation by a factor of $^1\!/_2$ from the y-axis. The negative (−) sign is a reflection about the x-axis.



Exercise 6I

Example 22

1 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second function.

a $y = x^2$ and $y = -x^2$

b $y = \dfrac{1}{x}$ and $y = \dfrac{4}{x}$

c $y = x^2 + x$ and $y = 3x^2 + 3x$

d $y = \sqrt{x}$ and $y = \sqrt{3x}$

e $y = \log_2 x$ and $y = -\log_2 x$

f $y = 2^x$ and $y = 3 \times 2^x$

Example 23

2 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second function.

a $y = x^2$ and $y = -3x^2$

b $y = \sqrt{x}$ and $y = -\sqrt{3x}$

c $y = \dfrac{4}{x}$ and $y = \dfrac{-2}{x}$

d $y = -x^2$ and $y = \dfrac{-x^2}{2}$

e $y = \log_3 x$ and $y = 3\log_3(2x)$

f $y = 2^x$ and $y = -2^{3x}$

3 $f(x) = x + 1$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = -3f(x)$.

4 $g(x) = x^2 + 3x$. Show, with a sketch on the Cartesian plane, how $y = g(x)$ is transformed into $y = -\frac{1}{2}g(x)$.

5 $h(x) = \begin{cases} 2x - x^2, & x < 2 \\ -x, & x \geq 2 \end{cases}$ Show, with a sketch on the Cartesian plane, how $y = h(x)$ is transformed into $y = -h(2x)$.

6 $p(x) = \begin{cases} 2x - 1, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$ Show, with a sketch on the Cartesian plane, how $y = p(x)$ is transformed into $y = \frac{3}{2}p(x)$.

6.10 Combinations of transformations

Translations, dilations and reflections are collectively referred to as **transformations**.

When applying translations, dilations and reflections:

1. Do what is in the brackets first (i.e. dilation from the y-axis and horizontal translation).
2. Do any multiplication or division (i.e. dilation from the x-axis and reflection).
3. Do any addition or subtraction (i.e. vertical translation using the constant term).

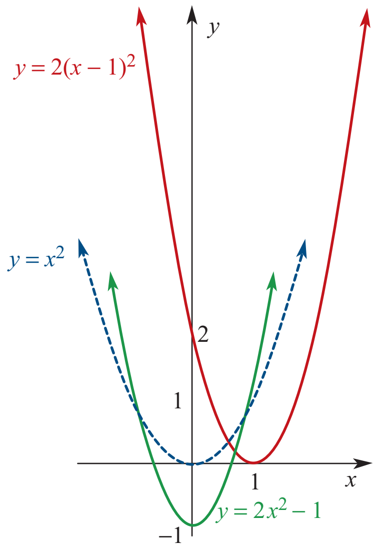
Example 24

Sketch $y = x^2$, $y = 2(x - 1)^2$ and $y = 2x^2 - 1$ on the same set of coordinate axes.

Solution

For $y = 2(x - 1)^2$, translate to the right 1 unit, and then dilate from the x-axis by a factor of 2.

For $y = 2x^2 - 1$, dilate from the x-axis by a factor of 2, and then translate down 1 unit.



Example 25

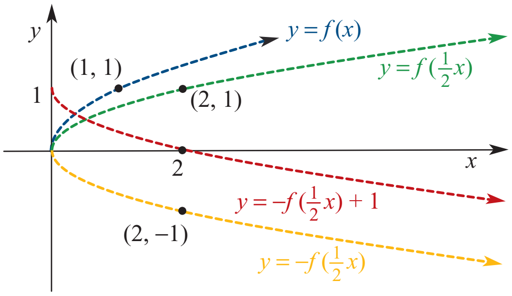
$f(x) = \sqrt{x}$. Sketch $y = f(x)$ and $y = -f\left(\frac{1}{2}x\right) + 1$ on the Cartesian plane. Show the steps in transforming $y = f(x)$ into $y = -f\left(\frac{1}{2}x\right) + 1$.

Solution

Dilate by a factor of 2 from the y-axis.

Reflect about the x-axis.

Translate upwards by 1 unit.



A word of caution

Care must be taken when a dilation from the y-axis and a horizontal translation are being considered; that is, when $y = f(ax \pm b)$.

1 Factorise what is inside the brackets; that is, $y = f\left(a\left(x \pm \frac{b}{a}\right)\right)$.

2 Treat the dilation, then the translation.

Example 26

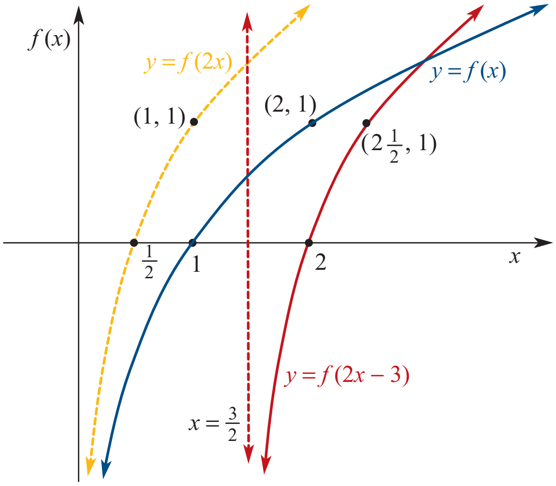
$f(x) = \log_2 x$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = f(2x - 3)$.

Solution

$$f(2x-3) = f\left(2\left(x - \frac{3}{2}\right)\right)$$

2 means dilate from the y-axis by a factor of $\frac{1}{2}$.

$-\frac{3}{2}$ means translate to the right by $\frac{3}{2}$ units.



Example 27

Find a sequence of transformations that will transform the graph of $y = x^2$ to the graph of $y = -3\,(2x + 8)^2 + 1$.

Solution

$(2x + 8) = 2(x + 4)$ $\quad\therefore$ Dilate from the y-axis by a factor of $\frac{1}{2}$.
Translate to the left by 4 units.
-3 $\quad\therefore$ Dilate from the x-axis by a factor of 3.
Reflect about the x-axis.
$+1$ $\quad\therefore$ Translate up 1 unit.

Exercise 6J

Example 24 **1** Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

- **a** $y = x^2$ and $y = 2(x+1)^2 + 3$
- **b** $y = \sqrt{x}$ and $y = -2\sqrt{x} - 4$
- **c** $y = \dfrac{1}{x}$ and $y = \dfrac{-1}{x-2} + 3$
- **d** $y = x^2$ and $y = -3(x-1)^2$
- **e** $y = \log_{10} x$ and $y = \log_{10}(2x) - 3$
- **f** $y = 3^x$ and $y = 3^{2x} - 4$

Example 25 **2** $f(x) = x^2$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = 3f(x+2) - 4$.

3 $g(x) = \log_{10} x$. Show, with a sketch on the Cartesian plane, how $y = g(x)$ is transformed into $y = -2g(x-1) + 3$.

4 $h(x) = \begin{cases} x, & x < 0 \\ 2x - x^2, & x \geq 0 \end{cases}$. Show, with a sketch on the Cartesian plane, how $y = h(x)$ is transformed into $y = 3h(2x) - 4$.

Example 26 **5** Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

- **a** $y = x^2$ and $y = (2x-2)^2 + 3$
- **b** $y = \log_{10} x$ and $y = \log_{10}(2x+6) - 1$
- **c** $y = 3^x$ and $y = 2 \times 3^{2x+1}$

Example 27 **6** **a** Find a sequence of transformations that will transform the graph of $y = x^2$ to the graph of $y = 4(3x-6)^2 + 1$.

b Find a sequence of transformations that will transform the graph of $y = \sqrt{x}$ to the graph of $y = -8\sqrt{2x+10} - 7$.

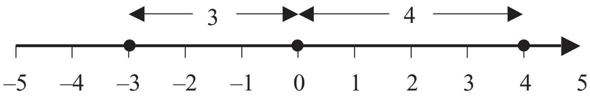
c Find a sequence of transformations that will transform the graph of $y = 4^x$ to the graph of $y = 2 \times 4^{6x-6} - 5$.

6.11 The absolute value function

The absolute value of a number is its distance from 0 on the number line.

-3 is 3 units from 0; that is, the absolute value of -3 is 3.

4 is 4 units from 0; that is, the absolute value of 4 is 4.



In shorthand we write: $|-3| = 3$ and $|4| = 4$.

We say: 'absolute value of -3 equals 3' and 'absolute value of 4 equals 4'. Said more mathematically:

$$\begin{aligned} |x| &= -x, \quad x < 0 \\ |x| &= x, \quad x \geq 0 \end{aligned}$$

Example 28

Evaluate:

a $|-5|$ **b** $|-9+7|$ **c** $|3| - |-7|$

Solution

a 5

b $$\begin{aligned} &|-9+7| \\ &= |-2| \\ &= 2 \end{aligned}$$

c $$\begin{aligned} &|3| - |-7| \\ &= 3 - 7 \\ &= -4 \end{aligned}$$

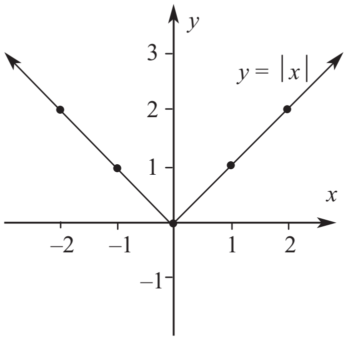
Example 29

Complete the table of values for $y = |x|$ and, hence, sketch $y = |x|$ on the Cartesian plane.

x	-2	-1	0	1	2
y					

Solution

x	-2	-1	0	1	2
y	2	1	0	1	2



Example 30

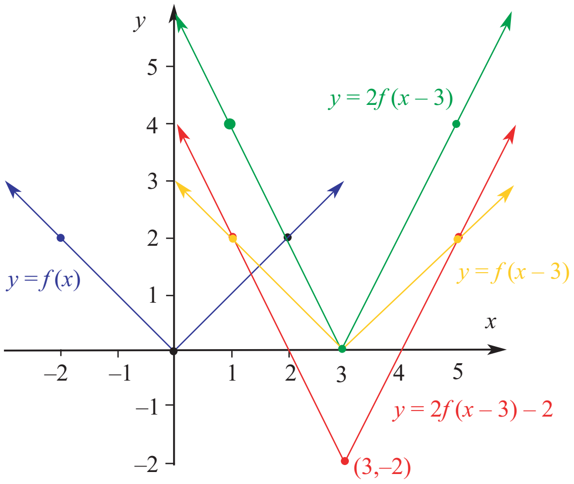
$f(x) = |x|$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = 2f(x-3) - 2$.

Solution

-3 is a translation to the right of 3.

2 is a dilation by a factor of 2 from the x-axis.

-2 is a translation downwards by 2.



Exercise 6K

Example 28 **1** Evaluate:

a $|-7|$ **b** $|6|$ **c** $|-8+3|$

d $|5-11|$ **e** $|-5|+|-6|$ **f** $|-3|-|7|$

Example 29 **2** **a** Complete the table of values for $y = 3|x|$ and, hence, sketch $y = 3|x|$ on the Cartesian plane.

x	-2	-1	0	1	2
y					

b Complete the table of values for $y = 3 - |x|$ and, hence, sketch $y = 3 - |x|$ on the Cartesian plane.

x	-2	-1	0	1	2
y					

Example 30 **3** Show, with a sketch on the Cartesian plane, how $y = |x|$ is transformed into:

a $y = |x| + 3$ **b** $y = 4|x|$ **c** $y = |x + 2|$

d $y = |4x|$ **e** $y = 2|x| - 4$ **f** $y = |x - 2| + 1$

g $y = 2|x - 2|$ **h** $y = |2x - 2|$ **i** $y = 4 - |x|$

j $y = -3|x + 2|$

4 **a** $f(x) = |x|$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = 3f(x + 2)$.

b $g(x) = |x|$. Show, with a sketch on the Cartesian plane, how $y = g(x)$ is transformed into $y = 3 - 2g(x - 4)$.

c $h(x) = |x + 2|$. Show, with a sketch on the Cartesian plane, how $y = h(x)$ is transformed into $y = 3h(x + 2)$.

d $p(x) = |x|$. Show, with a sketch on the Cartesian plane, how $y = p(x)$ is transformed into $y = 3p(2x - 4)$.

5 **a** Sketch on the Cartesian plane $y = |x| - x$.

b Sketch on the Cartesian plane $y = |x^2 - 2x - 3|$.

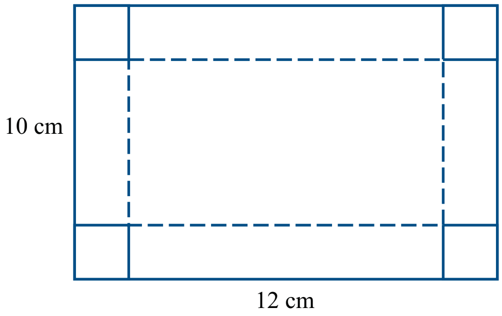
6.12 Modelling and problem solving

MAPS

Exercise 6L

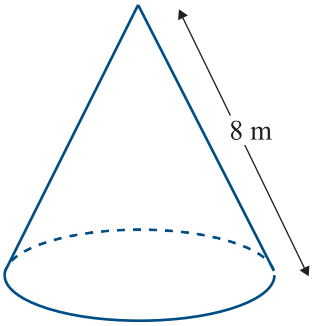
1 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from each corner of the sheet, as shown in the figure. The sides are then folded up to make a tray.

Find all possible sizes of the cut that give the volume of the tray as 72 cm^2.



2 The figure shows a conical heap of gravel. The slant height of the heap is 8 m.

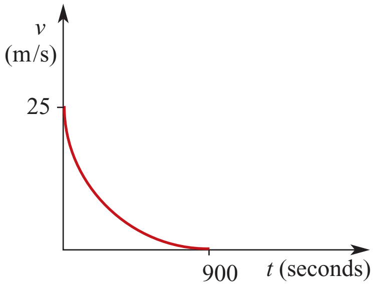
Find the height of the cone for which the volume of the heap is 55π cm^3.



3 There is a proposal to provide a quicker, more efficient and more environmentally 'friendly' system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can be driven only for a limited time before requiring energy replenishment.

The graph shows the speed, v m/s, which the taxi will maintain if it is driven at constant speed in such a way that it uses up all its energy in t seconds. The curve is a section of a parabola that touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.

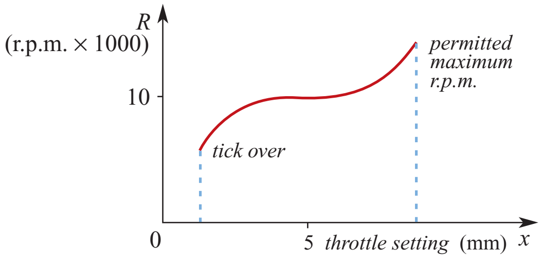
Construct a rule for v in terms of t.



4 The figure shows part of a cubic graph that represents the relationship between the engine speed, R r.p.m., and the throttle setting, x mm from the closed position, for a new engine.

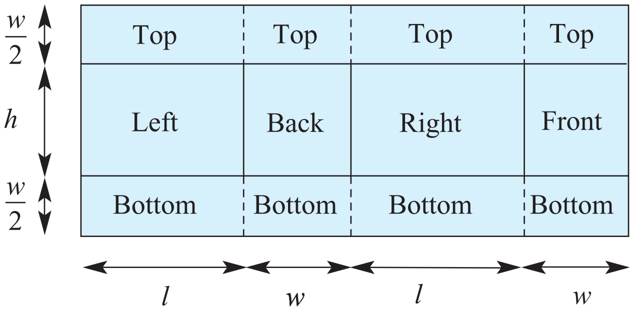
It can be seen from the graph that the engine has a 'flat spot' where an increase in x has very little effect on R.

Assuming that the curve passes through the origin, develop a cubic expression for R in terms of x of the form $R = a(x - h)^3 + k$.



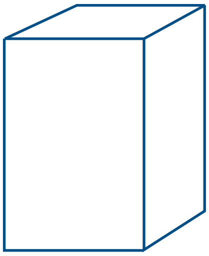
5 A net for making a cardboard box with overlapping flaps and an area of 1458 cm^2 is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded. For stacking purposes, it has been decided to make the height and length of the box equal.

Find the height of the box when its volume is 2160 cm^3.



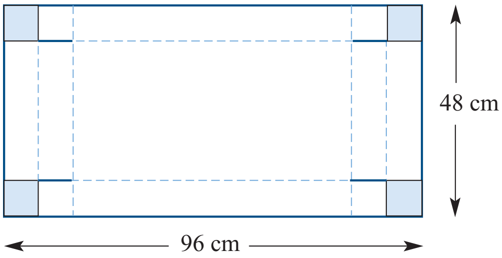
6 The figure shows a rectangular prism with a square base.

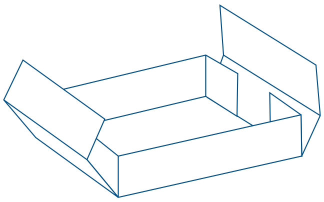
If the sum of the dimensions, length plus width plus height, is 160 cm, find the dimensions for which the volume is 90 000 cm^3.



7 A reinforced box is made by cutting congruent squares from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.

Find the maximum volume possible.





8 Sketch on the Cartesian plane $y = x^2 - | x^2 + 3 |$.

9 To transform $y = f(x)$ into $y = 2f(x) + 1$, dilate by a factor of 2 from the x-axis and then translate upwards 1 unit.

What steps will transform $y = f(x)$ into $y = f(5 - 2x)$?

Chapter summary

- 'y is **proportional** to x' or 'y **varies directly** with x' is written as $y \propto x$.
- 'y is **inversely proportional** to x' or 'y **varies inversely** with x' is written as $y \propto \frac{1}{x}$.
- All functions involving direct variation appear as straight lines passing through the origin when graphed on the Cartesian plane and their equation is $y = kx$.
- All functions appearing as inverse variation appear as hyperbolae when graphed on the Cartesian plane and their equation is $y = \frac{k}{x}$.
- The pronumeral k is called the **constant of proportionality** or the **constant of variation**.
- A function of the form $y = \frac{a}{x-h} + k$ is called an hyperbola.
- Each hyperbola has a vertical and a horizontal asymptote. In the case of $y = \frac{a}{x}$, the asymptotes are the x- and y-axes. In the case of $y = \frac{a}{x-h} + k$, the asymptotes are $x = h$ and $y = k$.
- The **degree** of a **polynomial** is the highest power for x in the polynomial.
- A **cubic** polynomial is of the form $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$.
- A **quartic** polynomial is of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$.
- The factor theorem states, 'If $P(a) = 0$, then $x - a$ is a factor of $P(x)$'.
- $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
- In general, $y = f(x - h) + k$ is a translation, h units to the right and k units up, of the graph of $y = f(x)$.
- In general, $y = af(x)$ is a dilation from the x-axis of the graph of $y = f(x)$ by a factor of a.
- In general, $y = f(ax)$ is a dilation from the y-axis of the graph of $y = f(x)$ by a factor of $\frac{1}{a}$.
- In general, $y = -f(x)$ is a reflection of the graph of $y = f(x)$ about the x-axis.
- Care must be taken when a dilation from the y-axis and a horizontal translation are being considered (i.e. when $y = f(ax \pm b)$. First, factorise what is inside the brackets (i.e. $y = f\left(a\left(x \pm \frac{b}{a}\right)\right)$. Then treat the dilation, followed by the the translation.
- The absolute value of a number is its distance from 0 on the number line.

$$\begin{aligned} |x| &= -x, \quad x < 0 \\ |x| &= x, \quad\;\; x \geq 0 \end{aligned}$$

Multiple-choice questions

1 If $y \propto x$ and $x = 12$ when $y = 4$, then the constant of proportionality, k, is:

A 16 **B** 3 **C** 48 **D** 8 **E** $\frac{1}{3}$

2 If $y \propto \frac{1}{x}$ and $x = 12$ when $y = 4$, then the constant of proportionality, k, is:

A 16 **B** 3 **C** 48 **D** 8 **E** $\frac{1}{3}$

3 The table shows the values of y against x.

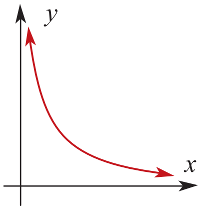
x	3	4	5
y	9	16	23

The relationship between y and x is:

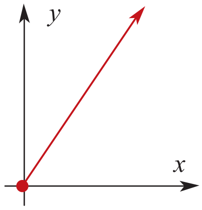
A $y \propto x$ **B** $y \propto \frac{1}{x}$ **C** $y \propto x^2$ **D** $y \propto \frac{1}{x^2}$ **E** none of the above

4 Which of the following graphs cannot show direct or inverse variation?

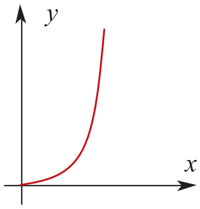
A



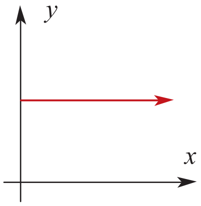
B



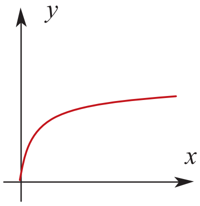
C



D



E



5 Divide $x^3 - 4x^2 - 5x + 3$ by $x - 5$.

A $x^2 - 4x - 5$, remainder 3

B $x^2 + x$, remainder 3

C $x^2 - 4x + 3$, remainder 5

D $-x^2 + 4x + 5$, remainder –3

E $-x^2 - x$, remainder –3

6 Factorise $x^3 - 8$.

A $(x - 2)^3$ **B** $(x - 2)\left(x^2 + 4x + 4\right)$ **C** $(x + 2)\left(x^2 - 4x + 4\right)$

D $(x - 2)\left(x^2 + 2x + 4\right)$ **E** $(x + 2)\left(x^2 - 2x + 4\right)$

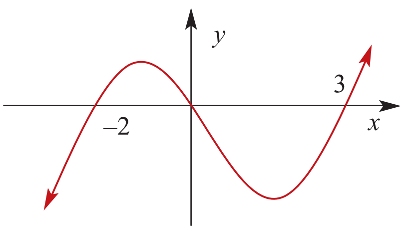
7 Factorise $x^3 + 4x^2 + 3x$.

A $(x + 3)\left(x^2 + 1\right)$ **B** $(x + 1)\left(x^2 + 3\right)$ **C** $x(x + 1)(x + 3)$

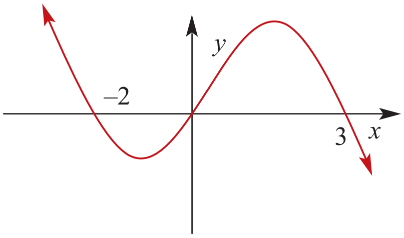
D $x(x - 1)(x - 3)$ **E** $3x(x + 1)(x + 1)$

8 Which of the following is a graph of $y = x^3 - x^2 - 6x$?

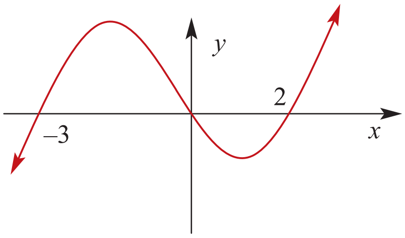
A



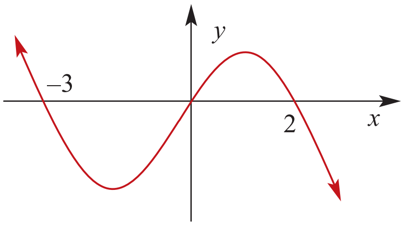
B



C



D



E none of the above

9 To transform $y = x^2$ into $y = 3(x^2 - 2)$:

A shift right 2, then dilate from the y-axis by 3
B shift down 6, then dilate from the x-axis by 3
C dilate from the x-axis by 3, then shift down 6
D dilate from the x-axis by 3, then shift right 2
E shift left 2, then dilate from the x-axis by 3

10 To transform $y = \log_2 x$ into $y = \dfrac{\log_2 (x + 2)}{3}$:

A shift left 2, then dilate from the y-axis by 3
B shift right 2, then dilate from the y-axis by 3
C shift left 2, then dilate from the x-axis by 3
D shift right 2, then dilate from the x-axis by 3
E shift left 2, then dilate from the x-axis by $\frac{1}{3}$

Short-response questions

1 **a** Show that y is proportional to x in the function shown in the table below.

x	2.4	3.5	5.8
y	6.48	9.45	15.66

b Show that y is inversely proportional to x in the function shown in the table below.

x	5	12.5	20
y	10	4	2.5

2 **a** Marie was paid \$25.80 for 3 hours work last week. No deductions come out of Marie's pay and so her pay is directly proportional to the number of hours she works. Find what she would be paid for 6½ hours work.

b The area of a triangle is 11 cm^2. The length of the base of a triangle is inversely proportional to its perpendicular height. Find the constant of proportionality.

3 Sketch:

a $y = \dfrac{2}{x-3}$ **b** $y = \dfrac{1}{x+2} - 3$

4 Sketch each of the following:

a $f(x) = 1$ **b** $f(x) = (3 - x)(x + 2)$ **c** $f(x) = 2x^2 - x - 6$

5 Divide the polynomial by the accompanying linear expression.

a $x^3 + x^2 - 2x + 3, \quad x - 2$

b $2x^3 - 4x + 3, \quad x + 3$

6 Factorise each of the following:

a $x^3 - x^2 - 17x - 15$ **b** $x^3 - 3x^2 - x + 3$ **c** $2x^3 - 16$ **d** $x^3 + 64$

7 Sketch each of the following:

a $y = (x - 2)(x - 3)(x + 1)$

b $y = x(2x - 5)(x + 1)$

c $y = x^3 - 4x$

d $y = x^3 - 4$

e $y = x^3 + 2x^2 - 5x - 6$

8 Sketch each of the following:

a $y = x^4 - 4x^2$

b $y = x^4 - 2x^3$

c $y = \left(4 - x^2\right)\left(x^2 - 1\right)$

9 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = x^2$ and $y = (x - 2)^2 + 1$

b $y = \sqrt{x}$ and $y = \sqrt{x + 1} - 3$

10 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = x^2$ and $y = -2x^2$

b $y = \log_3 x$ and $y = \log_3 (2x)$

11 Show, with a sketch on the Cartesian plane, how the first function is transformed into the second one.

a $y = x^2$ and $y = 2(x + 1)^2$

b $y = x^2$ and $y = (2x + 1)^2$

c $y = 2^x$ and $y = 2^{2x + 5} - 1$

12 Evaluate:

a $|-6|$ **b** $|8-13|$ **c** $|-5|-|2|$

13 Complete the table of values for $y = 2|x|$ and, hence, sketch $y = 2|x|$ on the Cartesian plane.

x	-2	-1	0	1	2
y					

14 **a** Show, with a sketch on the Cartesian plane, how $y = |x|$ is transformed into $y = 3|x+4|$.

b Show, with a sketch on the Cartesian plane, how $y = |x|$ is transformed into $y = 6 - 6|x|$.

c $f(x) = |x|$. Show, with a sketch on the Cartesian plane, how $y = f(x)$ is transformed into $y = 2f(3x+6)$.

CHAPTER

7

Periodic functions and applications

Objectives

- To understand and work with the **graphs of $y = \sin x$, $y = \cos x$** and **$y = \tan x$**.
- To understand the significance of the constants A, B, C and D in the **graphs of $y = A \sin (Bx + C) + D$** and **$y = A \cos (Bx + C) + D$**.
- To understand the significance of the constants A and B in the **graph of $y = A \tan Bx$**.
- To **solve simple trigonometric equations** within a specified domain using the unit circle and graphical methods.
- To understand and **use the complementary angle identity $\sin \left(\frac{\pi}{2} - x\right) = \cos x$**.
- To understand and **use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$**.
- To apply the **periodic functions** to real-life situations and problems.

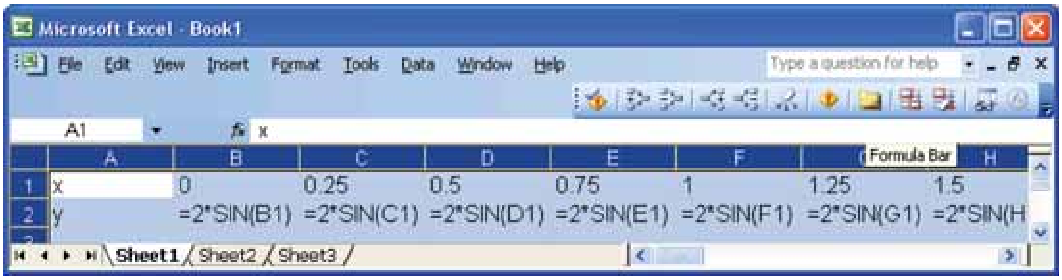
7.1 Introduction to graphing trigonometric functions

The trigonometric functions are an application of trigonometry beyond its use in triangles. It is important to recognise that most of the work relating to trigonometry from this point onwards involves the use of radians and not degrees. Your calculator should be set to **radian mode** most of the time.

As discussed previously in Chapter 4, the fundamental principle behind drawing graphs on the number plane is that of plotting points.

Excel calculates in radians automatically in its trigonometric functions. The graphs and data values in Example 1 have been generated using Excel. Students may consider doing Exercise 7A using Excel.

Reminder: When drawing graphs on the Cartesian plane in Excel use XY Scatter.
The formulae used in Example 1a are shown.



A1 fx x

	A	B	C	D	E	F	G	H
1	x	0	0.25	0.5	0.75	1	1.25	1.5
2	y	=2*SIN(B1)	=2*SIN(C1)	=2*SIN(D1)	=2*SIN(E1)	=2*SIN(F1)	=2*SIN(G1)	=2*SIN(H

Example 1

a On the Cartesian plane accurately graph $y = 2\sin x$, $0 \le x \le 8$. (Use 1 cm = 1 unit.)
b On the Cartesian plane accurately graph $y = \cos 2x$, $0 \le x \le 8$. (Use 1 cm = 1 unit.)

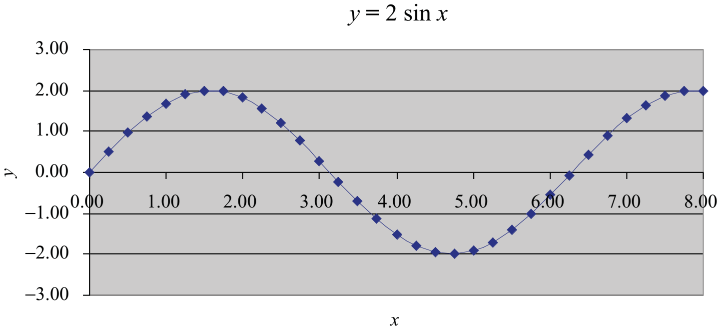
Solution

a

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
y	0.00	0.49	0.96	1.36	1.68	1.90	1.99	1.97	1.82	1.56	1.20

x	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25
y	0.76	0.28	−0.22	−0.70	−1.14	−1.51	−1.79	−1.96	−2.00	−1.92	−1.72

x	5.50	5.75	6.00	6.25	6.50	6.75	7.00	7.25	7.50	7.75	8.00
y	−1.41	−1.02	−0.56	−0.07	0.43	0.90	1.31	1.65	1.88	1.99	1.98

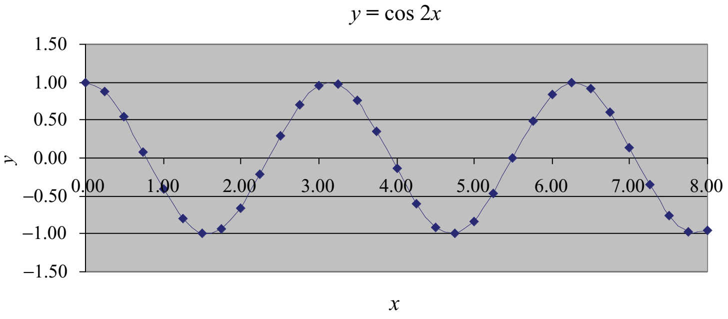


b

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
y	1.00	0.88	0.54	0.07	−0.42	−0.80	−0.99	−0.94	−0.65	−0.21	0.28

x	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25
y	0.71	0.96	0.98	0.75	0.35	−0.15	−0.60	−0.91	−1.00	−0.84	−0.48

x	5.50	5.75	6.00	6.25	6.50	6.75	7.00	7.25	7.50	7.75	8.00
y	0.00	0.48	0.84	1.00	0.91	0.59	0.14	−0.35	−0.76	−0.98	−0.96



Exercise 7A

Example 1 **1** On the same Cartesian plane, graph each of these accurately. (Use 1 cm = 1 unit.)

a $y = \cos x, \quad 0 \le x \le 8$ **b** $y = \sin x, \quad 0 \le x \le 8$

2 On the same Cartesian plane, graph each of these accurately. (Use 1 cm = 1 unit.)

a $y = 3\cos x, \quad 0 \le x \le 8$ **b** $y = 2\cos x + 3, \quad 0 \le x \le 8$

3 On the same Cartesian plane, graph each of these accurately. (Use 1 cm = 1 unit.)

a $y = \sin 3x, \quad 0 \le x \le 8$ **b** $y = \sin 2x - 3, \quad 0 \le x \le 8$

4 On the same Cartesian plane, graph each of these accurately. (Use 1 cm = 1 unit.)

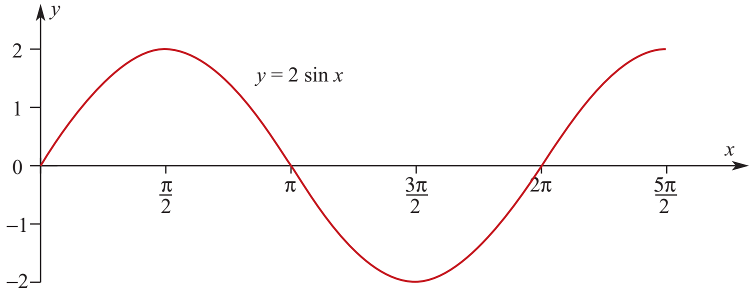
a $y = 2\cos(x - 0.25), \quad 0 \le x \le 8$ **b** $y = 3\cos(x + 0.75), \quad 0 \le x \le 8$

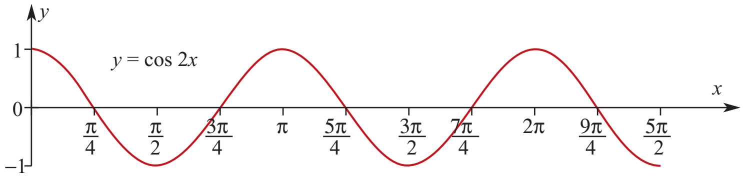
5 On the same Cartesian plane, graph each of these accurately. (Use 1 cm = 1 unit.)

a $y = \cos(3x - 0.75), \quad 0 \le x \le 8$ **b** $y = \sin(2x + 0.5), \quad 0 \le x \le 8$

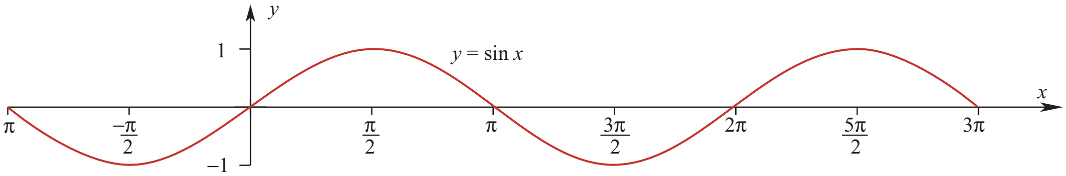
7.2 Graphs of $y = \sin x$ and $y = \cos x$

In Example 1, the graph of $y = 2 \sin x$ crosses the x-axis at 3.14159 . . . and 6.28318 . . . (i.e. at π and 2π) and the graph of $y = \cos 2x$ crosses the x-axis at 0.78539 . . . , 2.35619 . . . , 3.9269 . . . , 5.49778 . . . and 7.0685 . . . (i.e. at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and $\frac{9\pi}{4}$). Many of the trigonometric graphs in this chapter will cross the x-axis at points that are multiples or fractions of π. For convenience of showing the x-intercepts and turning points of trigonometric graphs the scale on the x-axis will appear as fractions of π. The answers to Example 1 would then appear as:





Graphs of sine functions



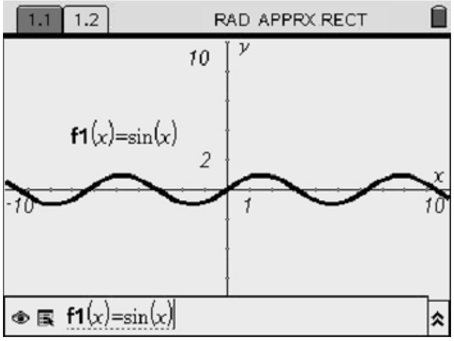
- The graph repeats itself after an interval of 2π units. A function that repeats itself regularly is called a **periodic function** and the interval between the repetitions is called the **period** of the function (also called wavelength). Thus, $y = \sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ (i.e. the y values) are 1 and -1, respectively, so the **range** of the function is $-1 \leq y \leq 1$. Thus, the range is restricted whereas the **domain** is unrestricted (except when specified in the question; in this case, $-\pi \leq x \leq 3\pi$). In general, when the domain is not specified it may be taken as $-\infty \leq x \leq \infty$.
- The horizontal line about which the curve oscillates is called the **equilibrium line**. In this case, it is the x-axis.
- The vertical distance between the equilibrium line and the highest (or lowest) position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

Using technology

Using the TI-Nspire:

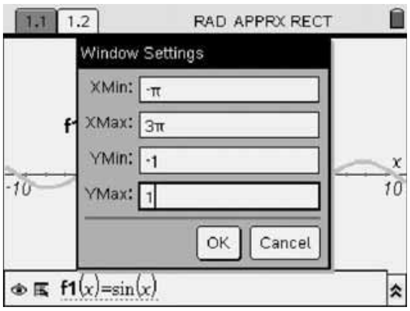
1 Set to Radian mode.

2 Enter **sin(*x*)** into $f1(x)$ then press (enter).



f1(x)=sin(x)

3 Press (menu) and set the Window as shown below.

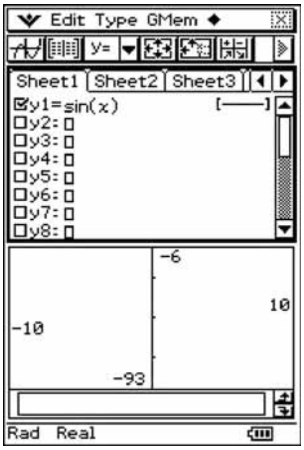


XMin: -π
XMax: 3π
YMin: -1
YMax: 1

Using the ClassPad:

1 Set to Radian mode.

2 Enter **sin(*x*)** into $y1$ then press (EXE).

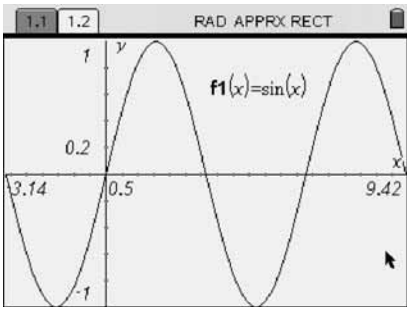


☑y1=sin(x)
☐y2:□
☐y3:□
☐y4:□
☐y5:□
☐y6:□
☐y7:□
☐y8:□

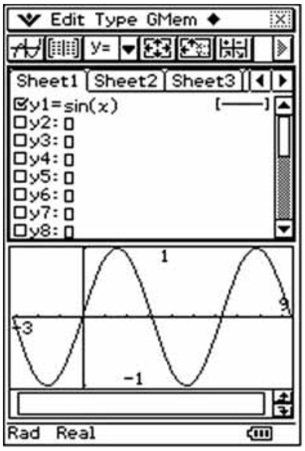
3 Tap and set the Window as shown below.

Memory ●2D ○3D
☐x-log ☐y-log
xmin : −π
max : 3π
scale : 1
dot : 0.12987012987
ymin : −1
max : 1
scale : 1

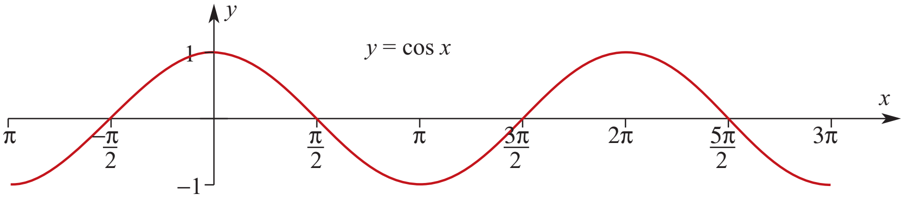
4 Press OK.



4 Tap OK to display the graph.



Graphs of cosine functions



- The graph of $y = \cos x$ is equivalent to the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units to the left.
- Thus period $= 2\pi$
  range is $-1 \leq y \leq 1$
  amplitude $= 1$

Using technology

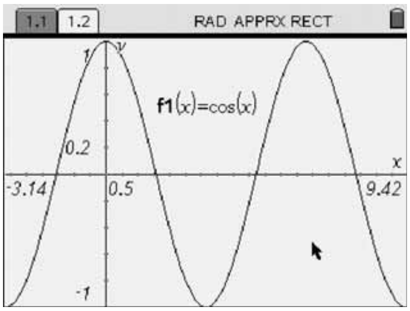
A graph of $y = \cos x$ for $-\pi \leq x \leq 3\pi$ can be produced with a calculator.

Using the TI-Nspire:

1 Enter **cos(*x*)** into $f1(x)$.

2 Set the Window to:

$X_{\text{min}} = -\pi$, $X_{\text{max}} = 3\pi$,
$Y_{\text{min}} = -1$, $Y_{\text{max}} = 1$

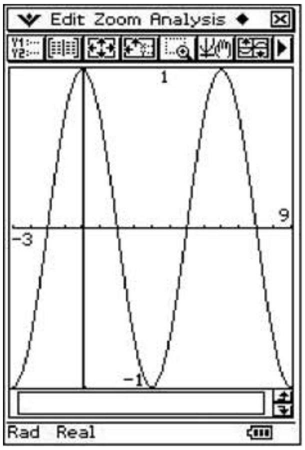


Using the ClassPad:

1 Enter **cos(*x*)** into $y1$.

2 Set the Window to:

$X_{\text{min}} = -\pi$, $X_{\text{max}} = 3\pi$,
$Y_{\text{min}} = -1$, $Y_{\text{max}} = 1$



Graphs of $y = A \sin Bx$, $y = A \cos Bx$

Example 2

On separate axes draw the graphs of the functions:

a $y = 3\sin 2x,\ 0 \le x \le \pi$

b $y = 2\cos 3x,\ 0 \le x \le \frac{2\pi}{3}$

Solution

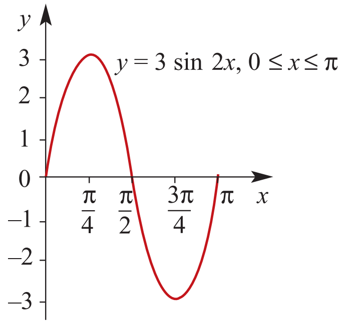
Set up tables of values as shown.

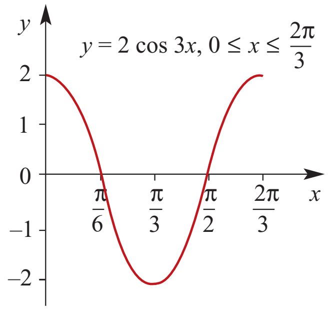
a

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin 2x$	0	3	0	-3	0

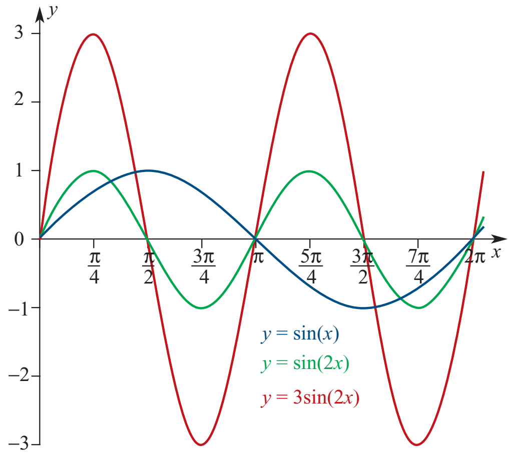
b

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos 3x$	2	0	-2	0	2

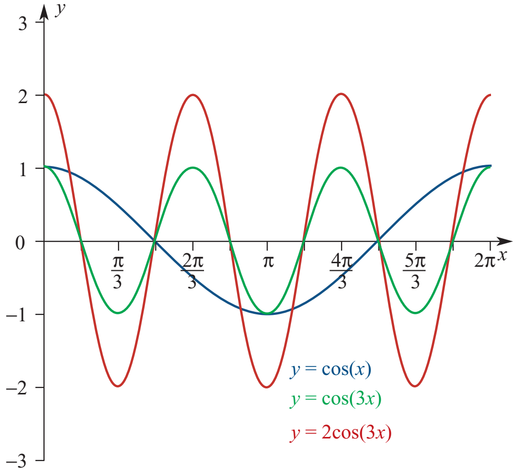




Note: $y = 3\sin\ 2x$ is $y = \sin x$ dilated by $\frac{1}{2}$ from the y-axis and dilated by 3 from the x-axis.



Note: $y = 2\cos 3x$ is $y = \cos x$ dilated by $\frac{1}{3}$ from the y-axis and dilated by 2 from the x-axis.



Although transformations can be used to draw trigonometric graphs, a number of techniques specific to trigonometric functions are discussed in this chapter.

Observations

Function	Amplitude	Period
$y = 3\sin 2x$	3	π
$y = 2\cos 3x$	2	$\frac{2\pi}{3}$

Comparing these results with those for $y = \sin x$ and $y = \cos x$, the following general rules can be stated for A and B positive numbers:

Function	Amplitude	Period
$y = A\sin Bx$	A	$\frac{2\pi}{B}$
$y = A\cos Bx$	A	$\frac{2\pi}{B}$

In general, for A and B positive numbers, the following are important properties of the functions $y = A\sin Bx$ and $y = A\cos Bx$:

- The domain of each of the functions is infinite; that is, all real numbers.
- The amplitude of each of the functions is A, thus the range is $-A \le y \le A$.
- The period of each of the functions is $\frac{2\pi}{B}$.

Using technology

Using the TI-Nspire:

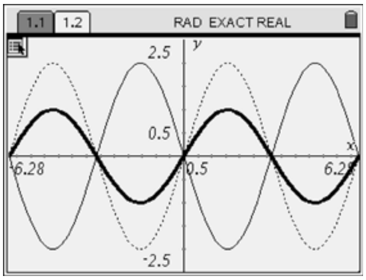
Amplitude

1 Set the calculator to Radian mode.

2 Type **–2sin(x)** into $f1(x)$ and press enter.

3 Type **sin(x)** into $f2(x)$ and press enter. (Make this a bold line to distinguish it from the others.)

4 Type **2sin(x)** into $f3(x)$ and press enter. (Make this a dotted line so comparisons can be made between all three graphs.)



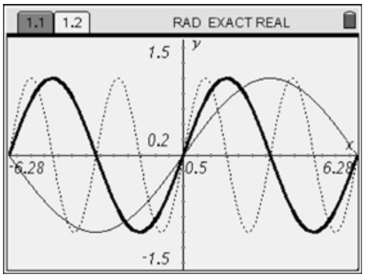
Period

1 Set the calculator to Radian mode.

2 Type **sin(0.5x)** into $f1(x)$ and press enter.

3 Type **sin(x)** into $f2(x)$ and press enter. (Make this a bold line to distinguish it from the others.)

4 Type **sin(2x)** into $f3(x)$ and press enter. (Make this a dotted line so comparisons can be made between all three graphs.)



Using the ClassPad:

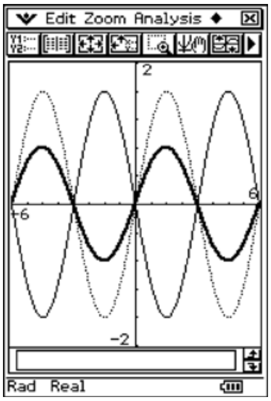
Amplitude

1 Set the calculator to Radian mode.

2 Type **–2sin(x)** into $y1$ and press EXE.

3 Type **sin(x)** into $y2$ and press EXE. (Make this a bold line to distinguish it from the others.)

4 Type **2sin(x)** into $y3$ and press EXE. (Make this a dotted line so comparisons can be made between all three graphs.)



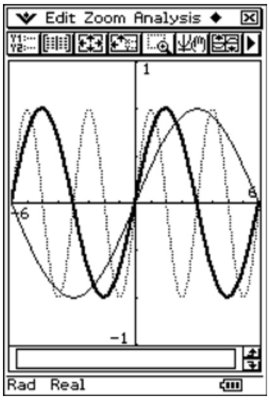
Period

1 Set the calculator to Radian mode.

2 Type **sin(0.5x)** into $y1$ and press EXE.

3 Type **sin(x)** into $y2$ and press EXE. (Make this a bold line to distinguish it from the others.)

4 Type **sin(2x)** into $y3$ and press EXE. (Make this a dotted line so comparisons can be made between all three graphs.)



Example 3

State A, B, amplitude and period for each of the following functions:

a $y = 2\sin(3x)$ **b** $y = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **c** $f(x) = 4\cos(3\pi x)$

Solution

a $A = 2,\ B = 3$

Amplitude is 2

Period $= \frac{2\pi}{3}$

b $A = -\frac{1}{2},\ B = \frac{1}{2}$

Amplitude is $\frac{1}{2}$

$$\text{Period} = 2\pi \div \frac{1}{2}$$
$$= 2\pi \times 2 = 4\pi$$

c $A = 4,\ B = 3\pi$

Amplitude is 4

Period $= 2\pi \div 3\pi = \frac{2}{3}$

Using technology

Example 4

Use a calculator to plot $y = \frac{1}{2}\sin 2x$ over the domain $[0 \leq x \leq 2\pi]$.

Solution

Radian measure is assumed because domain is given in terms of π.

Using the TI-Nspire:

1 Set to Radian mode.

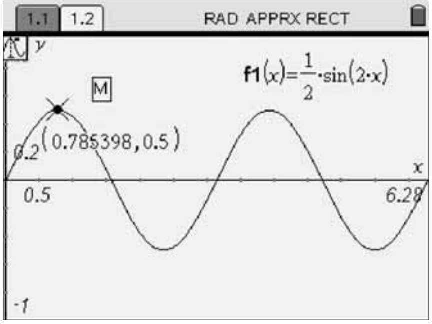
2 Enter $\mathbf{\frac{1}{2}\sin(2x)}$ into $f1(x)$ then press (enter).

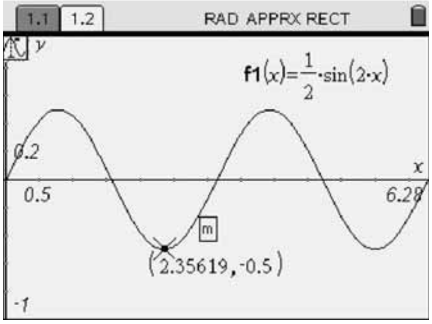
3 Set the Window to:

$X_{\min} = 0,\ X_{\max} = 2\pi,$

$Y_{\min} = -1,\ Y_{\max} = 1$

4 Use *Graph Trace* to identify key x values.





Using the ClassPad:

1 Set to Radian mode.

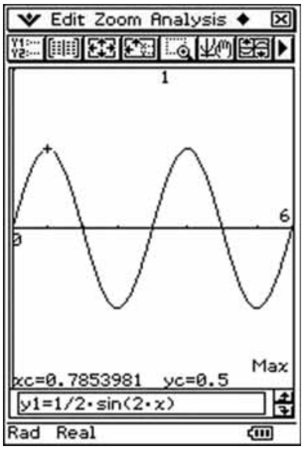
2 Enter $\mathbf{\frac{1}{2}\sin(2x)}$ into $y1$ then press (EXE).

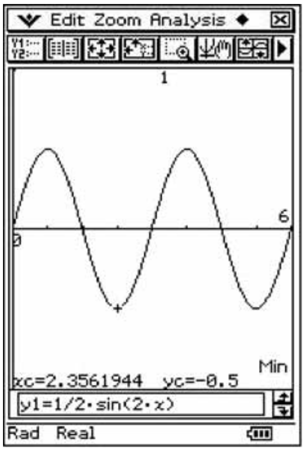
3 Set the Window to:

$X_{\min} = 0,\ X_{\max} = 2\pi,$

$Y_{\min} = -1,\ Y_{\max} = 1$

4 Use *Min* and *Max* from the G-Solve submenu to identify key x values.





xc=2.3561944 yc=-0.5

y1=1/2·sin(2·x)

For example, $\frac{\pi}{4} \approx 0.78539816$ $\qquad$ $\frac{3\pi}{4} \approx 2.356194$

Example 5

Sketch the graphs of each of the following, showing one complete cycle.

a $y = 2\cos 2\theta$ $\qquad$ **b** $y = \frac{1}{2}\sin\frac{x}{2}$

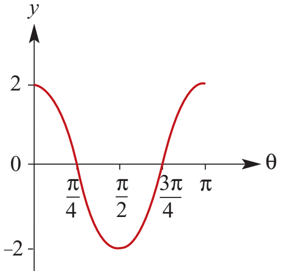
Solution

a $A = 2$ $\qquad \therefore$ Amplitude $= 2$

$$\text{Period} = \frac{2\pi}{B} \qquad B = 2$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

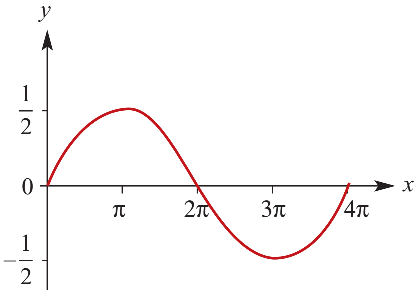


b $A = \frac{1}{2}$ $\qquad \therefore$ Amplitude $= \frac{1}{2}$

$$\text{Period} = \frac{2\pi}{B} \qquad B = \frac{1}{2}$$

$$= \frac{2\pi}{\frac{1}{2}}$$

$$= 4\pi$$



Negative coefficients: $y = -A\sin Bx$, $y = -A\cos Bx$

Example 6

For the domain $[0 \le x \le 4\pi]$, sketch the graphs of the following:

a $f(x) = -2\sin\frac{x}{2}$ $\qquad$ **b** $y = -\cos 2t$

Solution

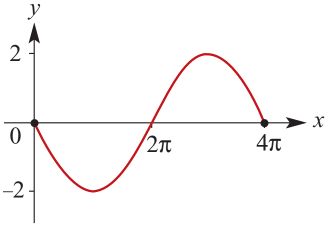
a $A = 2$ $\therefore$ Amplitude $= 2$

$$\text{Period} = \frac{2\pi}{B} \qquad B = \frac{1}{2}$$

$$= \frac{2\pi}{\frac{1}{2}}$$

$$= 4\pi$$

Reflect about the x-axis because of the negative sign.



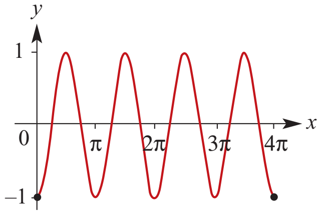
b $A = 1$ Amplitude $= 1$

$$\text{Period} = \frac{2\pi}{B} \qquad B = 2$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

Reflect about the x-axis because of the negative sign.



In general, for A and B positive numbers, the following are important properties of the functions $y = -A \sin Bx$ and $y = -A \cos Bx$:

- The amplitude of each of the functions is A.
- The period of each of the functions is $\frac{2\pi}{B}$.
- The graph of $y = -A \sin Bx$ (or $y = -A \cos Bx$) is obtained from the graph of $y = A \sin Bx$ (or $y = A \cos Bx$) by a reflection in the x-axis.
- The range of each of the functions is $-A \leq y \leq A$.

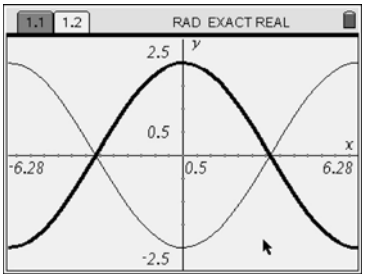
Using technology

Using the TI-Nspire:

1 Set the calculator to Radian mode.

2 Type **2 cos(0.5*x*)** into $f1(x)$ and press enter. (Make this a bold line for comparative reasons.)

3 Type **–2 cos(0.5*x*)** into $f2(x)$ and then press enter.

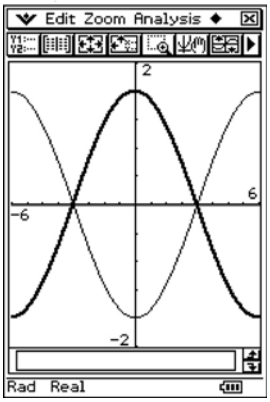


Using the ClassPad:

1 Set the calculator to Radian mode.

2 Type **2 cos(0.5*x*)** into $y1$, then press EXE. (Make this a bold line for comparative reasons.)

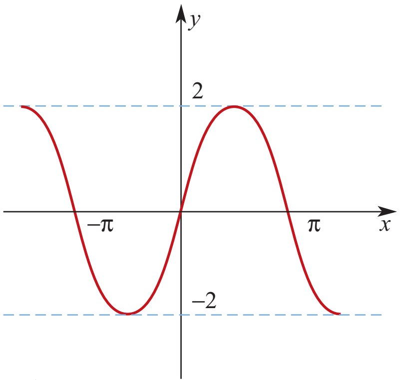
3 Type **–2 cos(0.5*x*)** into $y2$, then press EXE.



Example 7

Determine the equation of each of the following functions whose graphs are shown:

a



Solution

The graph passes through the y-axis at the origin, so is of the format $y = A\sin Bx$.

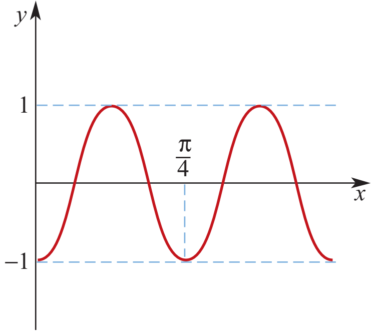
The amplitude is 2, so $A = 2$.

The period is 2π. This means that

$\frac{2\pi}{B} = 2\pi$, therefore $B = 1$.

Equation is $y = 2\sin x$.

b



Solution

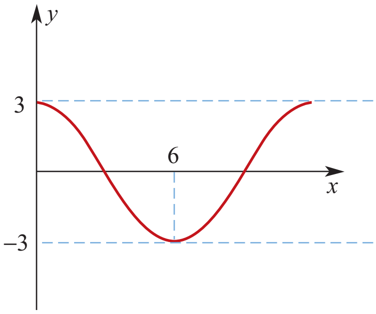
The graph passes through the y-axis at an amplitude point and is also inverted, so is of the format $y = -A\cos Bx$.

The amplitude is 1, so $A = 1$.

The period is $\frac{\pi}{4}$. This means that $\frac{2\pi}{B} = \frac{\pi}{4}$, therefore $B = 8$.

Equation is $y = -\cos 8x$.

c



Solution

The graph passes through the y-axis at an amplitude point, so is of the format $y = A\cos Bx$.

The amplitude is 3, so $A = 3$.

The period is 12. This means that $\frac{2\pi}{B} = 12$, therefore $B = \frac{\pi}{6}$.

Equation is $y = 3\cos\frac{\pi x}{6}$.

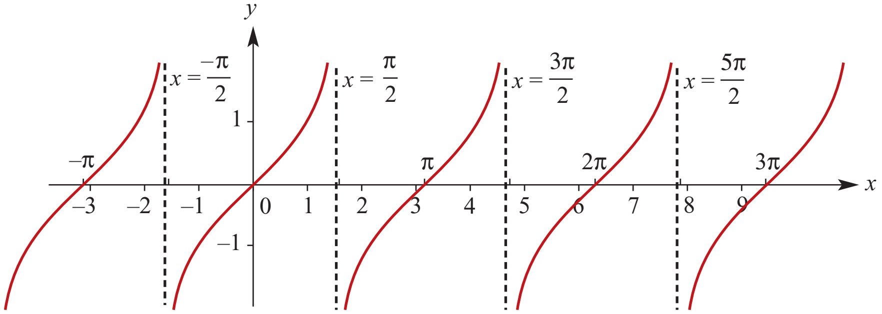
Note: Example 7c is a case where there are integer values rather than multiples of π along the horizontal axis. This is especially common for questions where the horizontal axis represents time, in which case the independent variable will be t rather than x.

The graph of $y = \tan x$

As discussed previously, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$. It can be seen from this that the vertical asymptotes of the graph of $y = \tan\theta$ occur where $\cos\theta = 0$; that is, for values of $\theta = \ldots -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$

A table of values for $y = \tan\theta$ is given below.

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	undefined	–1	0	1	undefined	–1	0	1	undefined	–1	0	1	undefined	–1	0



Note: $\theta = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $\dfrac{5\pi}{2}$ are asymptotes.

Observations from the graph

- The graph repeats itself every π units; that is, the period of $y = \tan x$ is π.
- Amplitude is not defined for $y = \tan x$.
- The range of $y = \tan x$ is **all real numbers** (compared to sin and cos, which were restricted to lying between certain values $-A$ and A, where A represented amplitude).
- The vertical asymptotes are located at odd multiples of $\dfrac{\pi}{2}$.

Exercise 7B

*Validate each of questions **1–6** using a calculator.*

1 For each of the following, state:

i the period

ii the amplitude

a $y = 2\sin x$ **b** $y = 3\sin 2x$ **c** $y = \dfrac{1}{2}\cos 3x$ **d** $y = 3\sin\dfrac{1}{2}x$

e $y = 4\cos 3x$ **f** $y = -\dfrac{1}{2}\sin 4x$ **g** $y = -2\cos\dfrac{1}{2}x$

Examples 2, 3 **2** Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 3\sin 2x$ **b** $y = 2\cos 3\theta$ **c** $y = 4\sin\dfrac{\theta}{2}$

d $y = \dfrac{1}{2}\cos 3x$ **e** $y = 4\sin 3x$ **f** $y = 5\cos 2x$

g $y = -3\cos\left(\dfrac{\theta}{2}\right)$ **h** $y = 2\cos(4\theta)$ **i** $y = -2\sin\left(\dfrac{\theta}{3}\right)$

Examples 4–6 **3** Sketch the graph of:

a $f(x) = \sin 2x$ for $-2\pi \le x \le 2\pi$ **b** $f(x) = 2\sin\dfrac{x}{3}$ for $-6\pi \le x \le 6\pi$

c $f(x) = 2\cos 3x$ for $0 \le x \le \pi$ **d** $f(x) = -2\sin 3x$ for $0 \le x \le 2\pi$

4 Sketch the graph of $y = \dfrac{5}{2}\cos\left(\dfrac{2x}{3}\right)$ for $0 \le x \le 2\pi$.

5 **a** On the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$, for $-2\pi \le x \le 2\pi$.

b By inspection from the graph, state the values of x for which $\sin x = \cos x$ in this domain.

6 Sketch, for $0 \le x \le 2\pi$, the graphs of:

a $y = \tan x$ **b** $y = -2\tan x$ **c** $y = \tan 2x$

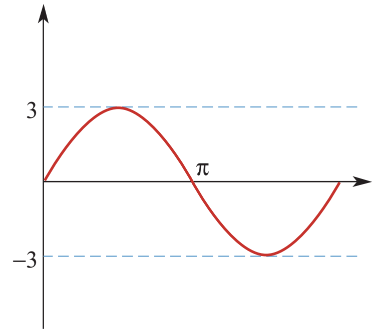
State the period of each.

7 **a** Using your calculator, sketch the graphs of $y = \tan x$ and $y = x$ for $-2\pi \le x \le 2\pi$.

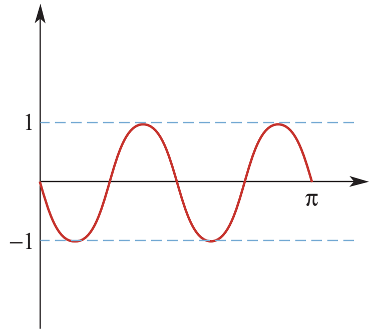
b Using your calculator, find all solutions to the equation $x = \tan x$, in the domain $-2\pi \le x \le 2\pi$.

Example 7 **8** Determine the equations of the following graphs:

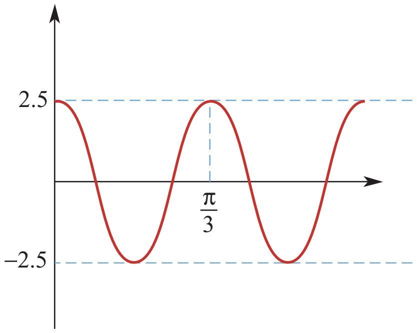
a



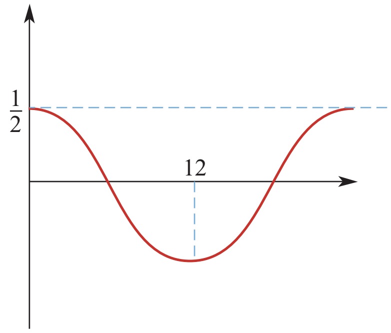
b



c



d

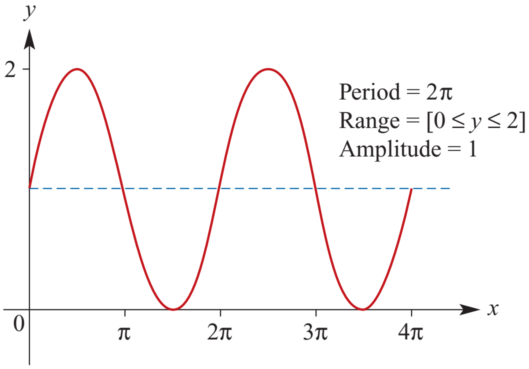


7.3 Translations in the horizontal and vertical directions

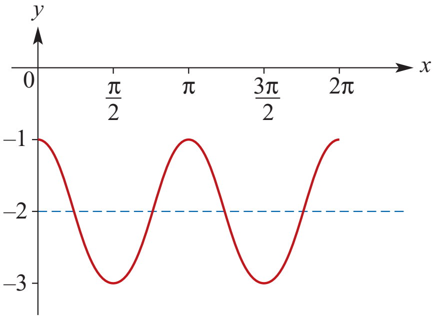
Vertical translations

The graph of $y = \sin x + 1$ is obtained from the graph of $y = \sin x$ by a translation of 1 unit upwards.

Note: The period and amplitude are unaffected by translations.

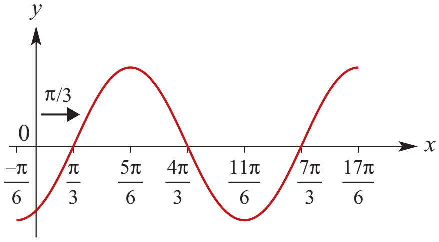


The graph of $y = \cos 2x - 2$ is obtained from the graph of $y = \cos 2x$ by a translation of 2 units downwards.

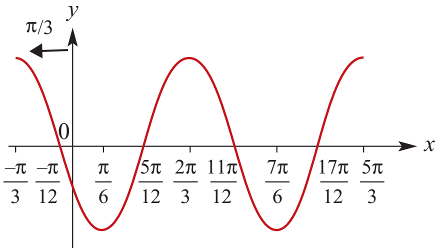


Horizontal translations (also called phase shift)

The graph of $y = \sin\left(x - \frac{\pi}{3}\right)$ is obtained from the graph of $y = \sin x$ by a translation of $\frac{\pi}{3}$ units to the right.



The graph of $y = \cos 2\left(x + \frac{\pi}{3}\right)$ is obtained from the graph of $y = \cos 2x$ by a translation of $\frac{\pi}{3}$ units to the left.



Using technology

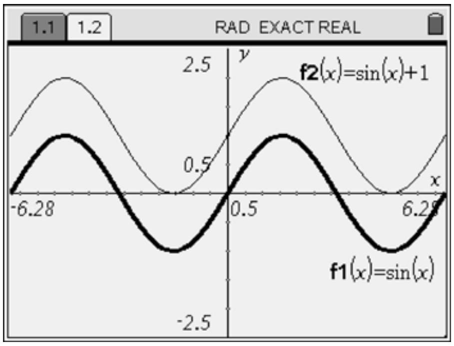
Using the TI-Nspire:

In the direction of the *y*-axis

1 Set the calculator to Radian mode.

2 Type **sin(*x*)** into $f1(x)$, then press enter. (Make this a bold line for comparative reasons.)

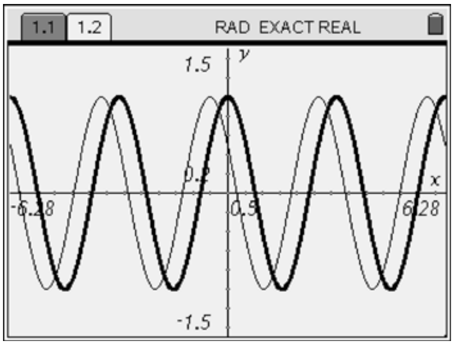
3 Type **sin(*x*) + 1** into $f2(x)$, then press enter.



In the direction of the *x*-axis

1 Type **cos(2*x*)** into $f1(x)$, then press enter. (Make this a bold line for comparative reasons.)

2 Type **cos** $\left(2x + \frac{\pi}{3}\right)$ into $f2(x)$ and then press enter.



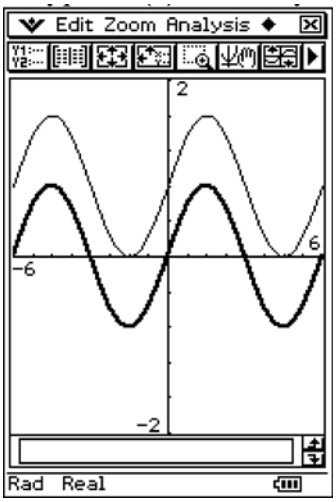
Using the ClassPad:

In the direction of the *y*-axis

1 Set the calculator to Radian mode.

2 Type **sin(*x*)** into $y1$, then press EXE. (Make this a bold line for comparative reasons.)

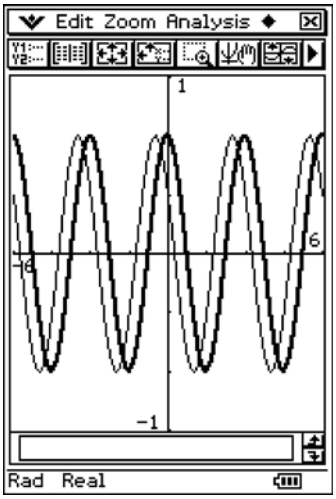
3 Type **sin(*x*) + 1** into $y2$, then press EXE.



In the direction of the *x*-axis

1 Type **cos(2*x*)** into $y1$, then press EXE. (Make this a bold line for comparative reasons.)

2 Type **cos** $\left(2x + \frac{\pi}{3}\right)$ into $y2$ and then press EXE.



Summary

In general, for the curves $y = A \sin B\,(x + C) + D$
$y = A \cos B\,(x + C) + D$

- The amplitude is A.
- The period is $\dfrac{2\pi}{B}$.
- The curve is translated up D units if $D > 0$; down D units if $D < 0$.
- The curve is translated left C units if $C > 0$; right C units if $C < 0$.
- The equilibrium line is $y = D$.
- The range is given by $D - A \le y \le D + A$.

Note: If the equation is given in the form $y = A \sin(Bx + E) + D$, it is necessary to factorise the brackets and extract the B so that it becomes the required format $y = A \sin B(x + C) + D$, where $C = \dfrac{E}{B}$.

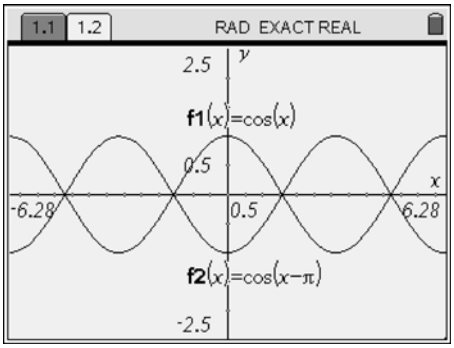
Using technology

Using the TI-Nspire:

x translation of $+\pi$

1 Type **cos(x)** into $f1(x)$, then press enter.

2 Type **cos($x - \pi$)** into $f2(x)$ and then press enter.



Dilation in the x direction by factor 2

1 Press menu and select *Hide/Show* from the Actions submenu.

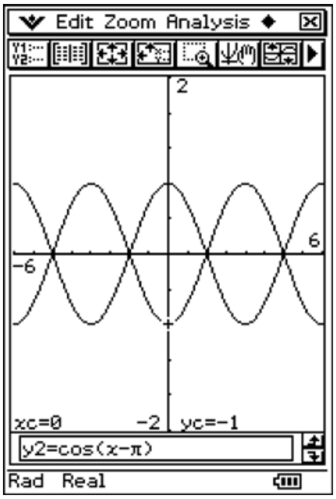
2 Move the cursor to the graph of $f1(x)$ and press enter to hide the graph.

Using the ClassPad:

x translation of $+\pi$

1 Type **cos(x)** into $y1$, then press EXE.

2 Type **cos($x - \pi$)** into $y2$, then press EXE.

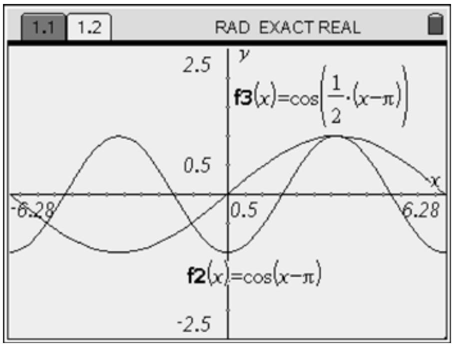


Dilation in the x direction by factor 2

1 Tap Y1: Y2: and then tap the check box to the left of $y1$ so that the tick is removed. (This ensures the graph of $y1$ will no longer be drawn.)

2 Type **cos $\left(\frac{1}{2}(x - \pi)\right)$** into $y3$ and then press EXE.

3 Type $\mathbf{cos}\left(\frac{1}{2}(x-\pi)\right)$ into $f3(x)$ and then press enter.

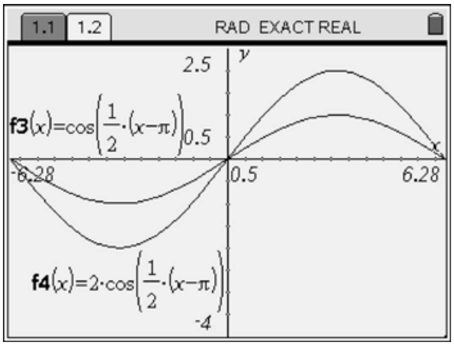


Dilation in the *y* direction by factor 2

1 Press menu and select *Hide/Show* from the Actions submenu.

2 Move the cursor to the graph of $f2(x)$ and press enter to hide the graph.

3 Type $\mathbf{2\ cos}\left(\frac{1}{2}(x-\pi)\right)$ into $f4(x)$, then press enter.

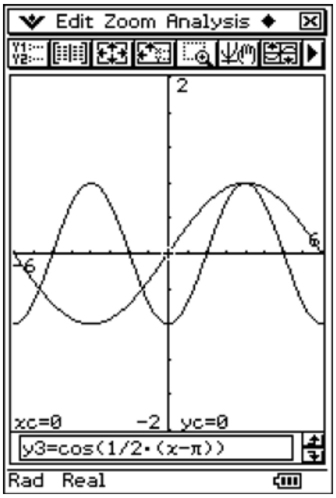


Reflection in the *x*-axis

1 Press menu and select *Hide/Show* from the Actions submenu.

2 Move the cursor to the graph of $f3(x)$ and press enter to hide the graph.

3 Tap 📈 to view the graphs.



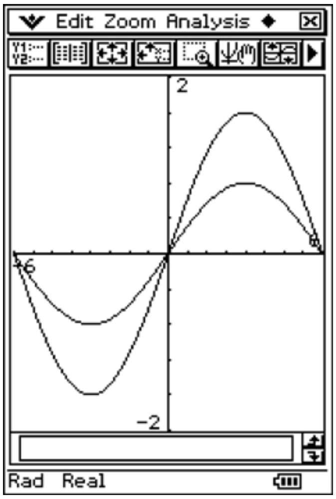
y3=cos(1/2·(x−π))

Dilation in the *y* direction by factor 2

1 Tap Y1: Y2: and, then tap the check box to the left of $y2$ so that the tick is removed. (This ensures the graph of $y2$ will no longer be drawn.)

2 Type $\mathbf{2\ cos}\left(\frac{1}{2}(x-\pi)\right)$ into $y4$ and then press EXE.

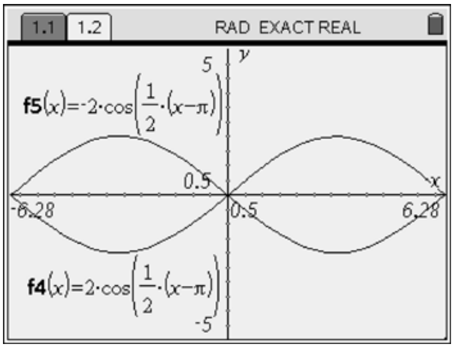
3 Tap 📈 to view the graphs.



Reflection in the *x*-axis

1 Tap Y1: Y2: and then tap the check box to the left of $y3$ so that the tick is removed. (This ensures the graph of $y3$ will no longer be drawn.)

3 Type $-2\cos\left(\frac{1}{2}(x-\pi)\right)$ into $f5(x)$ and then press (enter).

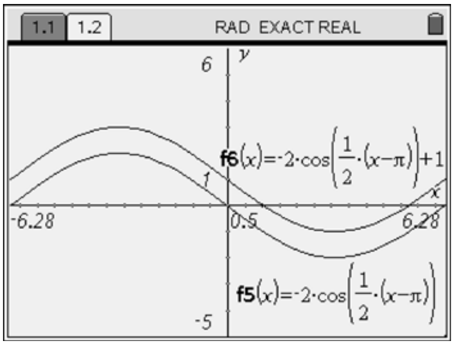


y translation of +1

1 Press (menu) and select *Hide/Show* from the Actions submenu.

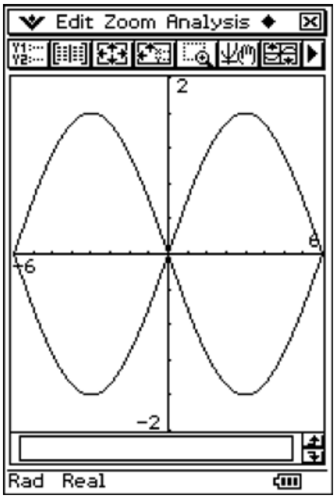
2 Move the cursor to the graph of $f4(x)$ and press (enter) to hide the graph.

3 Type $-2\cos\left(\frac{1}{2}(x-\pi)\right)$ into $f6(x)$, then press (enter).



2 Type $-2\cos\left(\frac{1}{2}(x-\pi)\right)$ into $y5$ and then press (EXE).

3 Tap ⊞ to view the graphs.

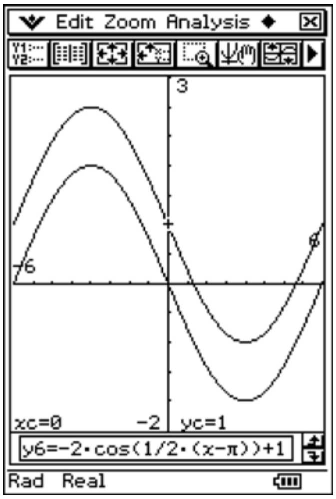


y translation of +1

1 Tap Y1:Y2: and then tap the check box to the left of $y4$ so that the tick is removed. (This ensures the graph of $y4$ will no longer be drawn.)

2 Type $-2\cos\left(\frac{1}{2}(x-\pi)\right)$ into $y6$ and then press (EXE).

3 Tap ⊞ to view the graphs.



xc=0 yc=1

y6=−2·cos(1/2·(x−π))+1

Example 8

On separate axes, sketch the graphs of:

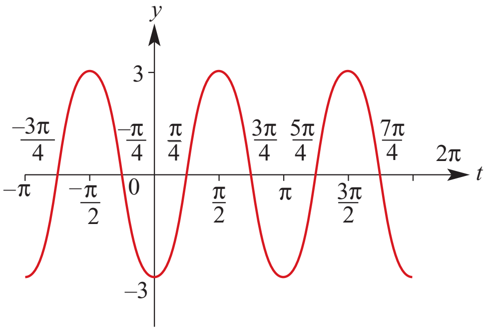
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ for $-\pi \le t \le 2\pi$ **b** $y = 3\cos 3\left(t + \frac{\pi}{3}\right)$ for $-\pi \le t \le \pi$

Solution

a $A = 3$ $\therefore$ Amplitude $= 3$

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi \qquad B = 2$$

$C = \frac{\pi}{4}$ $\therefore$ Translate to the right $\frac{\pi}{4}$.

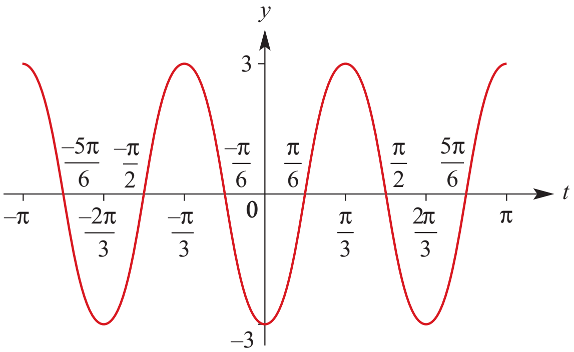


Note: This is also the graph of $y = -3\cos 2t$.

b $A = 3$ $\therefore$ Amplitude $= 3$

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3} \qquad B = 3$$

$C = -\frac{\pi}{3}$ $\therefore$ Translate to the left $\frac{\pi}{3}$.



Note: From Chapter 6 the dilation from the y-axis should occur before the shift left or right. In Example 8a students may benefit from first sketching $y = 3\sin 2x$ and then shifting it to the right. Similarly, first sketching $y = 3\cos 3x$ and then shifting it to the left may make it easier.

Exercise 7C

Example 8 **1** Sketch the graph of each of the following, showing at least one complete period or cycle. State the period, amplitude and the greatest and least values.

a $y = \sin x - 1$ **b** $y = 2\cos x + 1$ **c** $y = \sin 2x + 3$

d $y = 3\cos 2x - 2$ **e** $y = \sin \frac{\pi}{2}x + 2$

2 Sketch the graph of each of the following, showing one complete period or cycle. State the period, amplitude and the greatest and least values.

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin 2x$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin 2x$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

3 For the function $f(x) = \cos\left(x - \frac{\pi}{3}\right)$, where $0 \le x \le 2\pi$:

a Find $f(0), f(2\pi)$. **b** Sketch the graph of f.

4 For the function $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$, where $0 \le x \le 2\pi$:

a Find $f(0), f(2\pi)$. **b** Sketch the graph of f.

5 For the function $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$, where $-\pi \le x \le \pi$:

a Find $f(-\pi), f(\pi)$. **b** Sketch the graph of f.

6 For the function $y = 4\sin(2x - \pi)$, determine the amplitude, period and phase shift.

7.4 Solving trigonometric equations

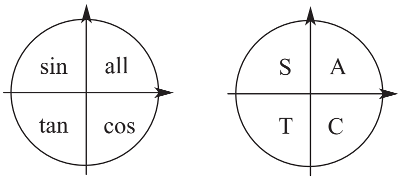
An equation such as $\sin x = \frac{1}{2}$ can have many solutions. In fact, there is an infinite number of values of x that can make the equation true. For example, on substituting each of $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ for x, we find that the equation is satisfied. There is an infinity of other values that will satisfy the equation as well.

Because of this, it is usually necessary to specify a domain before solving such an equation. All the examples in this section can be done by two methods:

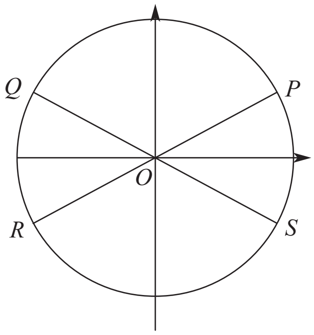
- Using the unit circle
- Using the calculator

The reader is reminded of three things from Chapter 2.

1 The CAST rule from Chapter 2 will help in solving trigonometric equations.



2 The sin, cos or tan of an angle in any quadrant will be equal to the sin, cos or tan of an angle in the first quadrant, except for a positive or negative sign, as determined by CAST. This means that the x and y coordinates of Q, R and S are equal to the coordinates of P, except for a positive or negative sign.



3 The acute angle between each of OQ, OR and OS and the x-axis is equal to the acute angle between OP and the x-axis.

Steps in solving trigonometric equations:

1 Find the acute or base value for the angle.

2 Establish in which quadrants the solutions will be by considering CAST.

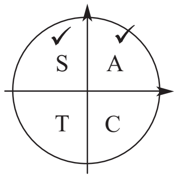
3 Find the solutions by adding or subtracting from 180°, 360°, 540° etc.

Example 9

Solve the equation $\sin x = \frac{1}{2}$ for $0 \leq x \leq 2\pi$.

Solution

Method 1



$$\sin x = \frac{1}{2}$$

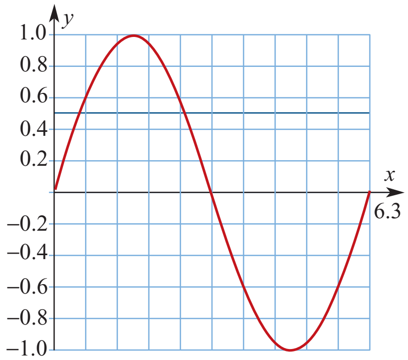
$$\text{base value for } x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$= 30$$

sin is positive in Q1 and Q2

$$\therefore\ x = 30,\ 180 - 30$$
$$= 30°,\ 150°$$
$$= \frac{\pi}{6}, \frac{5\pi}{6}$$

Note: Many students will find it easier to work in degrees and then convert their answers to radians.

Note: The next diagram shows the graphs of $y = \frac{1}{2}$ and $y = \sin x$ over the domain $0 \leq x \leq 2\pi$. The x coordinates of their points of intersection are the solutions of Example 9 above.

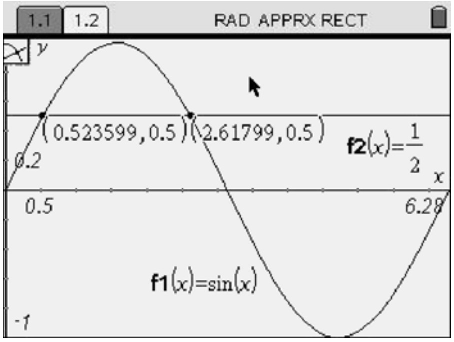


Method 2

Using technology

Using the TI-Nspire:

1. Enter **sin(*x*)** into $f1(x)$ and $\frac{1}{2}$ into $f2(x)$, then press enter.
2. Set the Window to:
   $X_{\min} = 0$, $X_{\max} = 2\pi$,
   $Y_{\min} = -1$, $Y_{\max} = 1$
3. Press menu and select *Intersection Point(s)* from the Points & Lines submenu.
4. Move the cursor to the point of intersection and press enter.



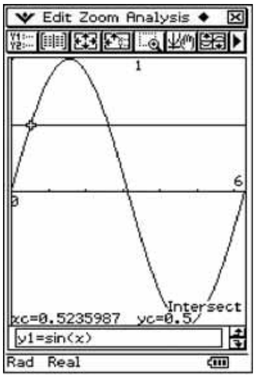
(0.523599,0.5) (2.61799,0.5)

$f2(x)=\frac{1}{2}$

$f1(x)=\sin(x)$

Using the ClassPad:

1. Enter **sin(*x*)** into $y1$ and $\frac{1}{2}$ into $y2$, then press enter.
2. Set the Window to:
   $X_{\min} = 0$, $X_{\max} = 2\pi$,
   $Y_{\min} = -1$, $Y_{\max} = 1$
3. Tap Analysis and select *Intersect* from the G-Solve submenu.
4. Press the right arrow key to display the coordinates of the other points of intersection.



xc=0.5235987 yc=0.5

y1=sin(x)

Note: This method will generate decimal approximations rather than exact answers. We will concentrate on the unit circle method for most of the examples in this section. It is left up to the student to verify each example using the calculator method.

Example 10

Solve the equation $2\cos x = 1$ for $0 \le x \le 2\pi$.

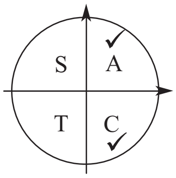
Solution

$$2\cos x = 1 \qquad (\div 2)$$

$$\cos x = \frac{1}{2}$$

$$\text{base value for } x = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= 60$$



cos is positive in Q1 and Q4

$$\therefore\ x = 60,\ 360 - 60$$

$$= 60°,\ 300°$$

$$= \frac{\pi}{3},\ \frac{5\pi}{3}$$

Example 11

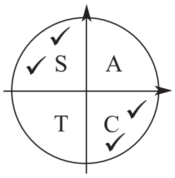
Solve $\tan x = -\frac{1}{\sqrt{3}}$, $0 \le x \le 4\pi$.

Solution

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$\text{base value for } x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 30$$



tan is negative in Q2, Q4, Q6 and Q8

$$\therefore\ x = 180 - 30,\ 360 - 30,\ 540 - 30,\ 720 - 30$$

$$= 150°,\ 130°,\ 510°,\ 690°$$

$$= \frac{5\pi}{6},\ \frac{11\pi}{6},\ \frac{17\pi}{6},\ \frac{23\pi}{6}$$

Note: The base value for x is still an acute angle and so it is necessary to find $\tan^{-1}$ of positive $\frac{1}{\sqrt{3}}$.

Special case (Boundary angles)

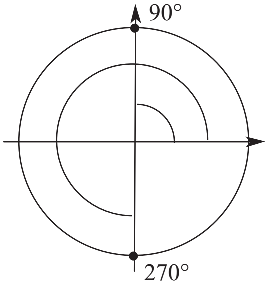
When solving $\sin x = 0$, $\cos x = 0$, $\sin x = \pm 1$, $\cos x = \pm 1$ and $\tan x = 0$ the solution is one or more of the boundary angles between the quadrants. Therefore, the solutions will be 0°, 90°, 180°, 270°, 360°, . . . or the equivalent in radians.

Example 12

Solve $\cos x = 0, 0^\circ \le x \le 360^\circ$.

Solution

$$\begin{aligned} \cos x &= 0 \qquad\qquad \therefore\ x \text{ coordinate} = 0 \\ x &= 90^\circ \text{ or } 270^\circ \end{aligned}$$

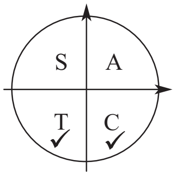


Example 13

Solve the equation $3\sin x + 2 = 0, 0^\circ \le x \le 360^\circ$.

Solution

$$\begin{aligned} 3\sin x + 2 &= 0 && (-\,2) \\ 3\sin x &= -2 && (\div\, 3) \\ \sin x &= -\frac{2}{3} \\ \text{base value for } x &= \sin^{-1}\left(\frac{2}{3}\right) \\ &\approx 41.81^\circ \end{aligned}$$



sin is negative in Q3 and Q4

$$\begin{aligned} \therefore\ x &\approx 180 + 41.81, 360 - 41.81 \\ &= 221.81^\circ, 318.19^\circ \end{aligned}$$

Exercise 7D

Examples 9–11

1 Solve the following equations for x in the domain $0^\circ \le x \le 360^\circ$:

a $\cos x = \frac{\sqrt{3}}{2}$ **b** $\sin x = -\frac{1}{2}$ **c** $\tan x = \sqrt{3}$

d $\sin x = \frac{1}{\sqrt{2}}$ **e** $\cos x = -\frac{\sqrt{3}}{2}$ **f** $\tan x = -1$

Example 12

2 Solve the following equations for x in the domain $0 \le x \le 2\pi$. Give your answer in exact form.

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\cos x = \frac{1}{2}$ **c** $\sin x = -\frac{1}{\sqrt{2}}$

d $\tan x = -\sqrt{3}$ **e** $\cos x = \frac{\sqrt{3}}{2}$ **f** $\tan x = 1$

3 Solve the following equations for x in the domain $0^\circ \le x \le 360^\circ$:

a $\cos x = -1$ **b** $\sin x = 1$ **c** $\tan x = 0$

d $\cos x = 1$ **e** $\sin x = 0$ **f** $\sin x = -1$

4 Use trigonometry and algebra to solve the following equations for θ in the specified domain. Give answers in exact form.

a $\sin\theta = -\frac{\sqrt{3}}{2}, \quad 0 \le \theta \le 2\pi$ **b** $\cos\theta = -\frac{1}{2}, \quad 0 \le \theta \le \pi$

c $\sqrt{3}\tan\theta = -1, \quad -180^\circ \le \theta \le 180^\circ$ **d** $2\sin\theta + 1 = 0, \quad 0 \le \theta \le 4\pi$

e $\sqrt{2}\cos x - 2 = 0, \quad -90^\circ \le x \le 270^\circ$ **f** $3 - \sqrt{3}\tan x = 0, \quad -2\pi \le x \le 2\pi$

Example 13 **5** Solve in the domain $0 \le x \le 2\pi$:

a $3\sin x = 1$ **b** $5\tan x = 2$ **c** $4\cos x + 1 = 0$

d $3 + 5\sin x = 2.1$ **e** $4.2\sin x - 2.1 = 0.5$ **f** $1.3 - 0.02\tan x = 2.1$

MAPS **6** Solve the following equations for x in the specified domain:

a $3\sin x = 2\cos x, 0^\circ \le x \le 360^\circ$

b $3\tan\left(x + \frac{\pi}{8}\right) - 3 = 0, -2\pi \le x \le 2\pi$

7.5 Further solutions of trigonometric equations

Example 14

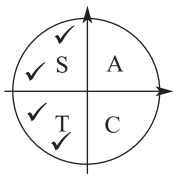
Solve $2\cos 2x = -1, 0 \le x \le 2\pi$.

Solution

$$2\cos 2x = -1 \qquad (\div 2)$$

$$\cos 2x = -\frac{1}{2} \qquad (\div 3)$$

$$\text{base value for } 2x = 2\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$



$$0 \le x \le 2\pi \quad \therefore\ 0 \le 2x \le 4\pi$$

cos is negative in Q2, Q3, Q6 and Q7

$$\therefore \quad 2x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Note: Again, it is not essential to work in radians, although the answer must be in radians. Students should note that $x \le 2\pi$, $2x$ is not, and therefore more solutions are possible.

Using technology

Now let us complete Example 14 using the calculator.

Solve $2\cos 2x = -1, 0 \le x \le 2\pi$.

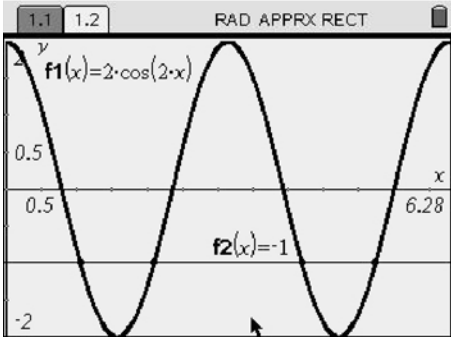
Using the TI-Nspire:

1 Enter **2 cos(2*x*)** into $f1(x)$ and **–1** into $f2(x)$, then press enter.

2 Set the Window to:

$X_{\text{min}} = 0, X_{\text{max}} = 2\pi,$

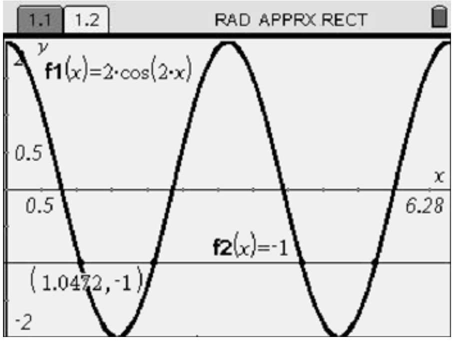
$Y_{\text{min}} = -2, Y_{\text{max}} = 2$



Note that there are four points of intersection.

3 Press menu and select *Intersection Point(s)* from the Points & Lines submenu.

4 Move the cursor to the point of intersection and press enter.



(1.0472,-1)

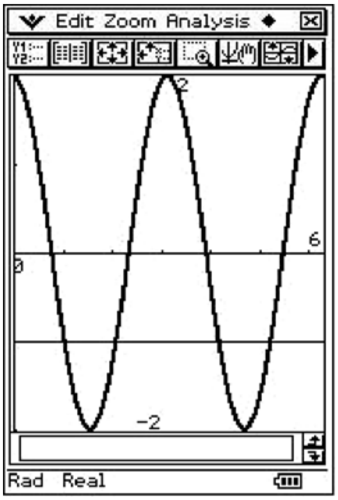
Using the ClassPad:

1 Enter **2 cos(2*x*)** into $y1$ and **–1** into $y2$, then press EXE.

2 Set the Window to:

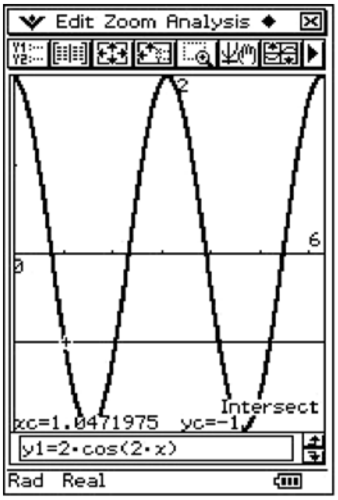
$X_{\text{min}} = 0, X_{\text{max}} = 2\pi,$

$Y_{\text{min}} = -2, Y_{\text{max}} = 2$



Note that there are four points of intersection.

3 Tap Analysis and select *Intersect* from the G-Solve submenu.



xc=1.0471975 yc=-1

y1=2·cos(2·x)

4 Press the right arrow key to display the coordinates of the other points of intersection.

This yields the first of the four solutions, i.e. $x = 1.047$, approximately.
This is equivalent to the first solution obtained using the unit circle; that is, $x = \frac{\pi}{3}$.

The remaining three solutions may be similarly found. Remember to position the cursor close to the required intersection point before pressing **ENTER**.

Note: Exact (i.e. surd) answers are generated by setting the calculator to EXACT mode.

Example 15

Solve the equation $2\cos^2 x - \cos x = 0$, for $0 \le x \le 2\pi$.

Solution

$$2\cos^2 x - \cos x = 0$$

$$2u^2 - u = 0 \qquad (u = \cos x)$$

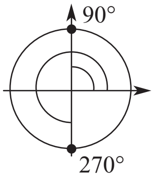
$$u(2u - 1) = 0$$

Null factor theorem

$u = 0$ or $2u - 1 = 0$

$\therefore \; u = 0$ or $u = \frac{1}{2}$

$\therefore \; \cos x = 0$ o $\cos x = \frac{1}{2}$

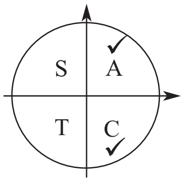


$\therefore \; x = 90° \text{ or } 270°$

base value for $x = \cos^{-1}\left(\frac{1}{2}\right)$

$= 60°$

cos is positive in Q1 and Q4



$\therefore \; x = 60, 360 - 60$

$= 60°, 300°$

$$\therefore \; x = 60°, 90°, 270°, 300°$$

$$= \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

Note: By convention $\cos^2 x = (\cos x)^2$. Also $\sin^2 x = (\sin x)^2$.

Example 16

Solve the equation $2\sin^2 x + 3\sin x = -1$, for $0 \le x \le 2\pi$.

Solution

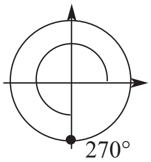
$$\begin{aligned} 2\sin^2 x + 3\sin x &= -1 \qquad (+\,1) \\ 2\sin^2 x + 3\sin x + 1 &= 0 \\ 2u^2 + 3u + 1 &= 0 \qquad (u = \sin x) \\ (u+1)(2u+1) &= 0 \end{aligned}$$

Null factor theorem

$$u + 1 = 0 \qquad \text{or} \qquad 2u + 1 = 0$$

$$\therefore\ u = -1 \qquad \text{or} \qquad u = -\frac{1}{2}$$

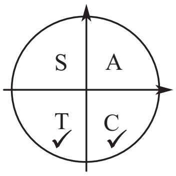
$$\therefore\ \sin x = -1 \qquad \text{or} \qquad \sin x = -\frac{1}{2}$$



$$\therefore\ x = 270^\circ$$

$$\begin{aligned} \text{base value for } x &= \sin^{-1}\left(\frac{1}{2}\right) \\ &= 30^\circ \end{aligned}$$

sin is negative in Q3 and Q4



$$\begin{aligned} \therefore\ x &= 180 + 30,\ 360 - 30 \\ &= 210^\circ,\ 330^\circ \end{aligned}$$

$$\begin{aligned} \therefore\ x &= 210^\circ, 270^\circ, 330^\circ \\ &= \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \end{aligned}$$

Special relations and identities

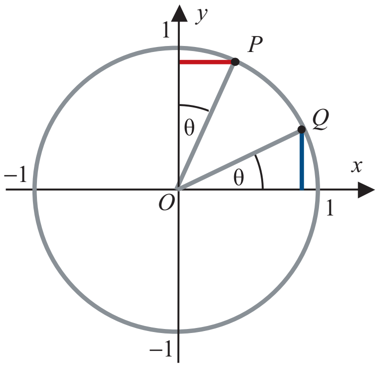
1 The complementary angle theorem

$$\boxed{\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta \end{aligned}}$$

Proof

The two triangles drawn in the diagram are congruent.

$$\therefore \quad \text{red line} = \text{blue line}$$
$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$



Note: These use may be written also as **sin x = cos(90° − x)** and **cos x = sin(90° − x)**.

Example 17

Find x, $0° \leq x \leq 90°$, if $\sin 57° = \cos x$.

Solution

$$\text{As } \sin x = \cos(90° - x)$$
$$90° - x = 57°$$
$$\therefore \quad x = 33°$$

2 The Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

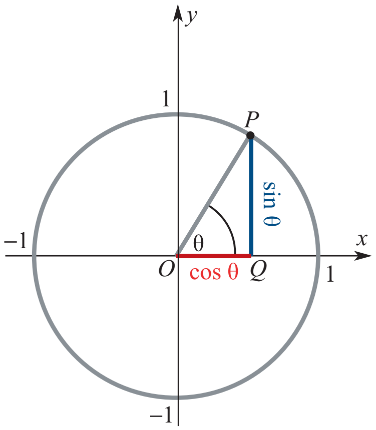
Proof

In $\triangle OPQ$

$PQ^2 + OQ^2 = OP^2$ (Pythagoras' theorem)

$\therefore \quad \sin^2 x + \cos^2 x = 1^2$

$\therefore \quad \sin^2 x + \cos^2 x = 1$ QED



Example 18

Given $\sin x = \frac{2}{3}$ and x is an angle in the first quadrant, determine:

a $\cos x$ **b** $\tan x$

Solution

a
$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \left(\frac{2}{3}\right)^2 + \cos^2 x &= 1 \\ \frac{4}{9} + \cos^2 x &= 1 \\ \cos^2 x &= \frac{5}{9} \\ \cos x &= \pm\frac{\sqrt{5}}{3} \end{aligned}$$

However, x is acute.

$$\therefore \cos x = \frac{\sqrt{5}}{3}$$

b
$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x} \\ &= \frac{2}{3} \div \frac{\sqrt{5}}{3} \quad \text{(using result from part } \mathbf{a}\text{)} \\ &= \frac{2}{3} \times \frac{3}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \end{aligned}$$

Exercise 7E

Example 14 **1** Solve the following equations for θ in the specified domain. Give answers in exact form.

a $2\cos 3x + 1 = 0, \quad 0^\circ \le x \le 360^\circ$

b $\tan 2\theta + 1 = 0, \quad 0 \le \theta \le 2\pi$

c $\sqrt{2}\sin 2\theta - 1 = 0, \quad 0 \le \theta \le 2\pi$

d $3\tan 3x = \sqrt{3}, \quad 0^\circ \le x \le 360^\circ$

e $4\sin 4x = 2\sqrt{3}, \quad 0 \le x \le \pi$

f $3\cos 2x - 0.6 = 2, \quad 0^\circ \le x \le 360^\circ$

Examples 15, 16 **2** Solve the following equations for θ in the specified domain. Give answers in exact form.

a $\sin^2\theta = \frac{1}{2}, \quad 0 \le \theta \le 2\pi$

b $\tan\theta(\sqrt{3}\tan\theta - 1) = 0, \quad 0 \le \theta \le \pi$

c $4\cos^2\theta - 1 = 0, \quad 0 \le \theta \le 2\pi$

d $\tan^2\theta = \tan\theta, \quad -\pi \le \theta \le \pi$

e $6\sin^2 x - x = 2, \quad 0^\circ \le x \le 360^\circ$

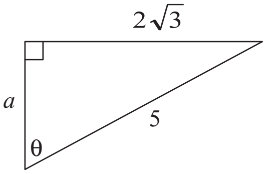
f $\cos x - 2\sin^2 x + 1 = 0, \quad 0 \le x \le 2\pi$

g $\sin^2 x - \cos^2 x = \frac{1}{2}, \quad 0 \le x \le 3\pi$

Example 17 **3** Complete the following:

a $\sin 35^\circ = \cos \ldots\ldots\ldots$ **b** $\cos 27^\circ = \sin \ldots\ldots\ldots$ **c** $\cos\frac{\pi}{6} = \sin \ldots\ldots\ldots$

Example 18 **4** Use the diagram to find the exact values of:



- **a** a
- **b** $\cos\theta$
- **c** $\tan\theta$

5 **a** If $\cos\theta = 0.6$ and $0^\circ \le \theta \le 90^\circ$, find:

  - **i** $\sin\theta$ **ii** $\tan\theta$

b Find the exact value of $\cos x$, if $\sin x = -\frac{12}{13}$ and $\pi \le x \le \frac{3\pi}{2}$.

c Find the exact value of $\tan x$, if $\cos x = \frac{3}{5}$ and $\sin x = -\frac{4}{5}$.

d Find all possible values of $\cos x$ if $\sin x = -\frac{2}{3}$.

6 Given x, y, z are acute angles, and $\sin x = \frac{2}{3}$, $\cos y = \frac{1}{2}$ and $\tan z = \frac{2}{\sqrt{3}}$, find exact values of the following. (Do not use a calculator.)

a $\tan x$	**b** $\cos(2\pi - y)$	**c** $\tan(\pi + z)$	**d** $\sin\left(\frac{\pi}{2} - y\right)$
e $\sin y$	**f** $\sin(\pi + x)$	**g** $\cos(\pi - y)$	**h** $\tan(2\pi + z)$
i $\sin(\pi - y)$	**j** $\cos(2\pi - z)$	**k** $\sin(3\pi + y)$	**l** $\tan\left(\frac{\pi}{2} - y\right)$

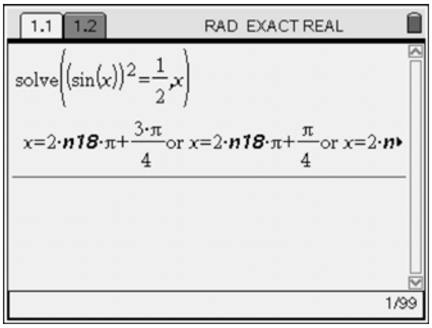
Using technology

Using the TI-Nspire:

Unrestricted domain

1 Set the calculator to Exact and Radian mode.

2 Type **solve** $\left(\mathbf{\sin(x)^\wedge 2 = \frac{1}{2}, x}\right)$ and then press enter.



$\text{solve}\left((\sin(x))^2=\frac{1}{2},x\right)$

$x=2\cdot n18\cdot\pi+\frac{3\cdot\pi}{4}$ or $x=2\cdot n18\cdot\pi+\frac{\pi}{4}$ or $x=2\cdot n$▸

1/99

Note: Given $\mathbf{n18 = \{0, 1, 2, 3, ...\}}$, a general solution is provided for all angles and quadrants.

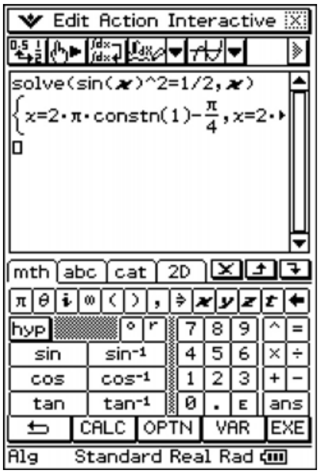
Using the ClassPad:

Unrestricted domain

1 Set the calculator to Standard and Radian mode.

2 Tap Keyboard.

3 Type **solve** $\left(\mathbf{\sin(x)^\wedge 2 = \frac{1}{2}, x}\right)$ and then press EXE.



solve(sin(x)^2=1/2,x)

$\{x=2\cdot\pi\cdot\text{constn}(1)-\frac{\pi}{4}, x=2\cdot$▸

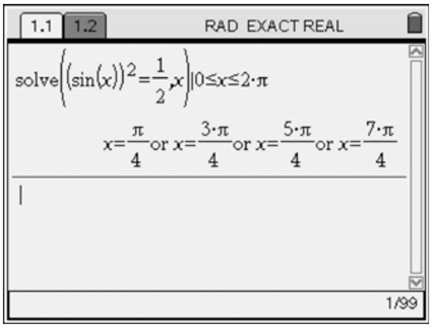
Note: Given **constn(1), constn(2), constn(3), constn(4)** $= \{\mathbf{0, 1, 2, 3, ...}\}$, a general solution is provided for all angles and quadrants.

Restricted domain

1 Type **solve** $\left(\mathbf{sin}(x)\mathbf{\hat{}2} = \frac{1}{2}, x\right)$.

2 Press |, then type $\mathbf{0} \le x \le \mathbf{2}\pi$.

3 Press enter.



$\text{solve}\left((\sin(x))^2=\frac{1}{2},x\right)|0\le x\le 2\cdot\pi$

$x=\frac{\pi}{4} \text{ or } x=\frac{3\cdot\pi}{4} \text{ or } x=\frac{5\cdot\pi}{4} \text{ or } x=\frac{7\cdot\pi}{4}$

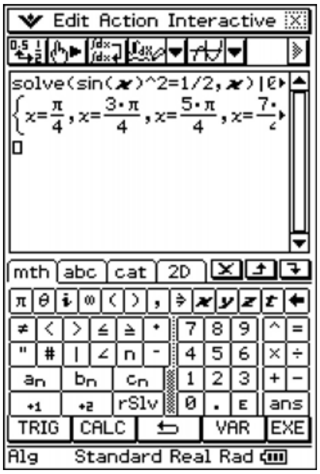
Restricted domain

1 Type **solve** $\left(\mathbf{sin}(x)\mathbf{\hat{}2} = \frac{1}{2}, x\right)$.

2 Tap the OPTN tab.

3 Tap |, then type $\mathbf{0} \le x \le \mathbf{2}\pi$.

4 Press EXE.



solve(sin(x)^2=1/2,x)|0

$\left\{x=\frac{\pi}{4}, x=\frac{3\cdot\pi}{4}, x=\frac{5\cdot\pi}{4}, x=\frac{7\cdot}{\ }\right.$

7.6 Applications of periodic functions

Example 19

Using technology

Using the calculator, solve the equation $2\cos\left(\frac{\pi t}{12}\right) = 1.5$ for $0 \le t \le 24$.

Solution

Using the TI-Nspire:

1 Set the calculator to Approximate and Radian mode.

2 Type **solve** $\left(\mathbf{2\cos}\left(\frac{\pi t}{\mathbf{12}}\right) = \mathbf{1.5}, t\right)$.

3 Press |, then type $\mathbf{0} \le t \le \mathbf{24}$.

Using the ClassPad:

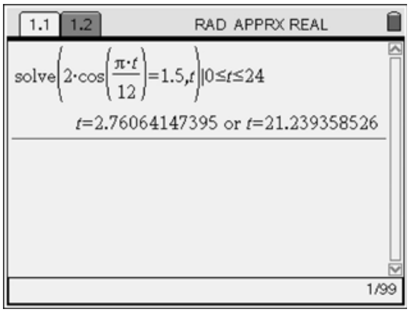
1 Set the calculator to Decimal and Radian mode.

2 Tap Keyboard.

3 Type **solve**

$\left(\mathbf{2\cos}\left(\frac{\pi t}{\mathbf{12}}\right) = \mathbf{1.5}, t\right)$.

4 Press enter.



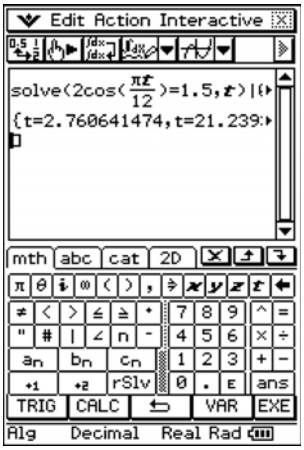
solve$\left(2\cdot\cos\left(\frac{\pi\cdot t}{12}\right)=1.5,t\right)|0\le t\le 24$

$t=2.76064147395$ or $t=21.239358526$

4 Tap the OPTN tab.

5 Tap |, then type **0** $\le$ ***t*** $\le$ **24**.

6 Press EXE.



solve$(2\cos(\frac{\pi t}{12})=1.5,t)|(▸$

$\{t=2.760641474,t=21.239▸$

Note: Tap ▸ to view more results.

Example 20

The height $h(t)$ metres of the tide above mean sea level on 1 January at a certain harbour is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

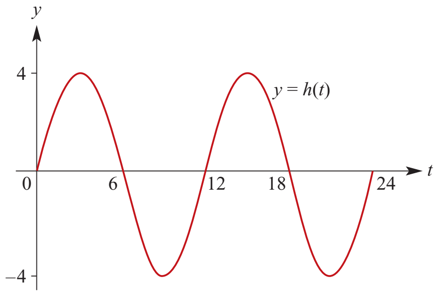
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can enter the harbour only when the tide is at least 1 metre above mean sea level. Between what times could the boat enter the harbour on 1 January?

Solution

a



$$\text{Period} = 2\pi \div \frac{\pi}{6} = 12$$

b $t = 3$ and 15 at the maximum turning points.

High tides occur at 0:300 hours (3 a.m.) and 15:00 hours (3 p.m.)

c High tide is 4 metres above the mean height.

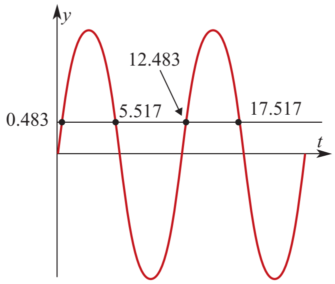
d $$\begin{aligned} h(8) &= 4\sin\left(\frac{\pi}{6}\times 8\right) \\ &\approx -3.4641\ldots \\ &\approx -3.5 \end{aligned}$$

The tide is approximately 3.5 metres below the mean height at 8 a.m.

e $h = 1$

$4\sin\left(\frac{\pi}{6}t\right) = 1$

Using the calculator with $y_1 = 4\sin\left(\frac{\pi}{6}x\right)$ and $y_2 = 1$, the points of intersection are: 0.483…, 5.517…, 12.483… and 17.517…



These are decimal hours, so multiplying the decimal parts each by 60 and rounding to the nearest minute gives 00:29, 05:31, 12:29 and 17:31, respectively.

Thus, the boat can enter the harbour between 00:29 and 05:31 and between 12:29 and 17:31.

Using technology

Using the TI-Nspire:

Using the solve command

1 Set the calculator to Approximate and Radian mode.

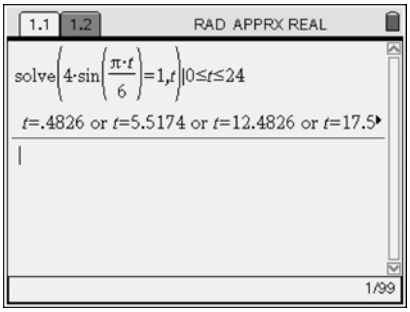
2 Change display digits in system settings to Fix 4.

3 Press menu and select *Solve* from the Actions submenu.

4 Type **solve** $\left(\mathbf{4\,sin}\left(\frac{\pi t}{6}\right) = \mathbf{1}, t\right)$.

5 Press | and then type $\mathbf{0} \le t \le \mathbf{24}$.

6 Press enter.



$\text{solve}\left(4\cdot\sin\left(\frac{\pi\cdot t}{6}\right)=1,t\right)|0\le t\le 24$

$t=.4826$ or $t=5.5174$ or $t=12.4826$ or $t=17.5$▸

Note: Tap ▸ to view more results.

Using the ClassPad:

Using the solve command

1 Set the calculator to Decimal and Radian mode.

2 Tap ▾, select Basic Format and change Number Format to Fix 4.

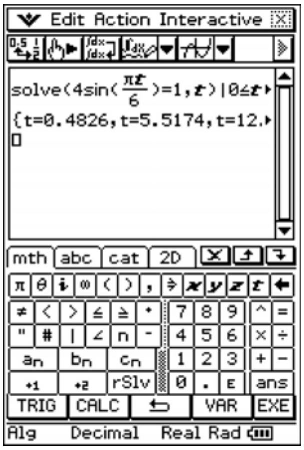
3 Tap Keyboard.

4 Type **solve** $\left(\mathbf{4\,sin}\left(\frac{\pi t}{6}\right) = \mathbf{1}, t\right)$.

5 Tap the OPTN tab.

6 Tap | and then type $\mathbf{0} \le t \le \mathbf{24}$.

7 Press EXE.



$\text{solve}(4\sin(\frac{\pi t}{6})=1,t)|0\le t$▸

{t=0.4826, t=5.5174, t=12.▸

Note: Tap ▸ to view more results.

Graph intersect method

1 Change display digits in system settings to Fix 3. (This ensures that all intersection points will be correct to 3 decimal places.)

2 Type **4 sin** $\left(\frac{\pi t}{6}\right)$ into $f1(x)$ then press enter.

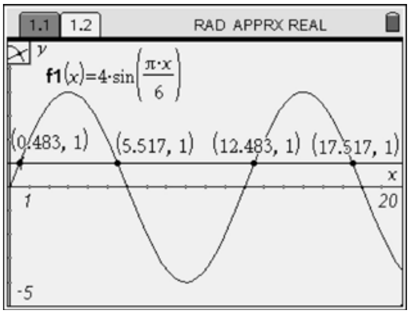
3 Type **1** into $f2(x)$ then press enter.

4 Set appropriate window settings.

5 Press menu and select *Intersection Point(s)* from the Points & Lines submenu.

6 Move the cursor to the graph of $f1(x)$ and then press enter.

7 Move the cursor to the graph of $f2(x)$ and then press enter. (All points of intersection will be displayed on the screen.)



f1(x)=4·sin(π·x/6)

(0.483, 1) (5.517, 1) (12.483, 1) (17.517, 1)

Graph intersect method

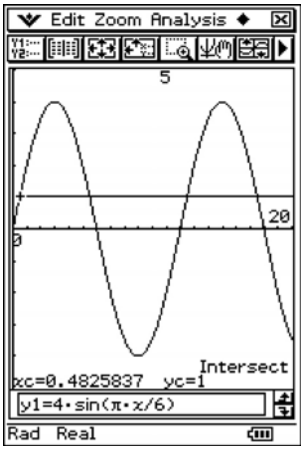
1 Type **4 sin** $\left(\frac{\pi t}{6}\right)$ into $y1$, then press EXE.

2 Type **1** into $y2$ and then press EXE.

3 Tap ▦ to view the graphs.

4 Tap Resize for a full-screen view and set an appropriate window.

5 Tap Analysis and select *Intersect* from the G-Solve submenu.



Intersect

xc=0.4825837 yc=1

y1=4·sin(π·x/6)

Note: Press the right and left arrows to display the other intersection points.

7.7 Modelling and problem solving

Exercise 7F

Examples 19, 20

1 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.

- **a** Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.
- **b** What is the depth of water after 5 hours?
- **c** At what time(s) in the 24-hour period is the water 12 metres deep?
- **d** Find the values of t for which $D(t) \geq 12$.
- **e** Boats that need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour or more. Find, correct to 1 decimal place, the largest value of w that satisfies this condition.

2 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right)$, where t is the number of months that have elapsed since January 1.

a Find d:

i on 21 June ($t \approx 5.7$) **ii** on 21 March ($t \approx 2.7$)

b When will there be 5 hours of daylight?

3 The depth of water at the entrance to a harbour t hours after high tide is D metres, where $D = p + q\cos(rt)$ for suitable constants p, q, r. At high tide the depth is 7 metres; at low tide, 6 hours later, the depth is 3 metres.

a Show that $r = 30$ and find the values of p and q, measuring in degrees.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 metres of water will be able to enter the harbour.

4 A particle moves on a straight line, Ox, and its distance x metres from O at time t (seconds) is given by $x = 3 + 2\sin 3t$.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 5 metres from O for $0 \leq t \leq 5$.

d Find the times at which it is 3 metres from O for $0 \leq t \leq 3$.

e Describe the motion of the particle.

5 The temperature $A°$ C inside a house at t hours after 4 a.m. is given by $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$ for $0 \leq t \leq 24$, and the temperature $B°$ C outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$ for $0 \leq t \leq 24$.

a Find the temperature inside the house at 8 a.m.

b Write an expression for $D = A - B$; that is, the difference between the inside and outside temperatures.

c Sketch the graph of D versus t for $0 \leq t \leq 24$.

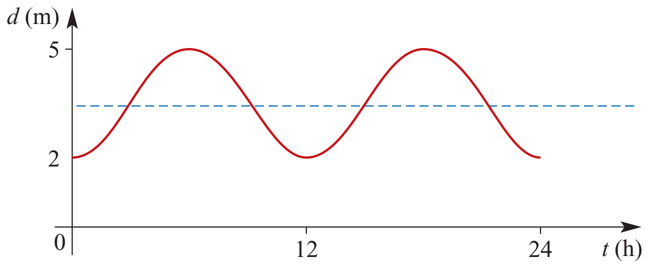
d Determine when the inside temperature is less than the outside temperature.

6 The high-water mark on a beach wall is modelled by the function $d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$, where t is the number of hours after midnight and d is the depth of the water in metres.

a What is the earliest time of day at which the water is at its highest?

b When is the water 2 metres up the wall?

7 The graph shows the distance $d(t)$ of the top of the hour hand of a large clock from the ceiling at time t hours.



a $d(t)$ is modelled by a sinusoidal function (i.e. of the format $d = \pm A \sin Bt + D$). Find the:

i amplitude **ii** period

iii rule for $d(t)$ **iv** length of the hour hand

b At what times is the distance less than 3.5 metres from the ceiling?

8 In a tidal river the time between high tide and low tide is 8 hours. The average depth of water at a point on the river is 4 metres; at high tide the depth is 5 metres.

a Sketch a graph of the depth of the water at the point over time if the relationship between time and depth is sinusoidal and there is a high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, how many hours before noon can it enter the point and by what time must it leave to avoid being stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point and by what time must it leave to avoid being stranded?

9 The population, N, of a particular species of ant varies with time. The population at time t weeks after 1 January, 1999 is given by $N = 3000 \sin \dfrac{(\pi (t - 10))}{26} + 4000$.

a For $N(t) = 3000 \sin \dfrac{(\pi (t - 10))}{26} + 4000$, state the:

i period **ii** amplitude **iii** range

b **i** State the values of $N(0)$ and $N(100)$.

ii Sketch the graph of $y = N(t)$ in the domain $0 \leq t \leq 100$.

c Find the values of t in this domain for which the population is:

i 7000 **ii** 1000

d Find the intervals of time during the first one hundred days for which the population of ants is greater than 5500.

e A second population $M(t)$ of ants also varies with time. The population has the following properties:

- Minimum population is 10 000 at $t = 20$.
- Maximum or minimum value between $t = 10$ and $t = 20$.
- Maximum population is 40 000 at $t = 10$.
- $M(t) = a \sin \dfrac{(\pi (t - c))}{b} + d$, where a, b, c and d are positive constants.

Find a set of possible values for a, b, c and d.

10 During winter an unexpected cyclone develops off the coast of Cairns. The water height (H_W metres) at the Cairns pier is the sum of the cyclone's swell (H_S metres) and the tide height (H_T metres); that is:

$$H_W = H_S + H_T$$

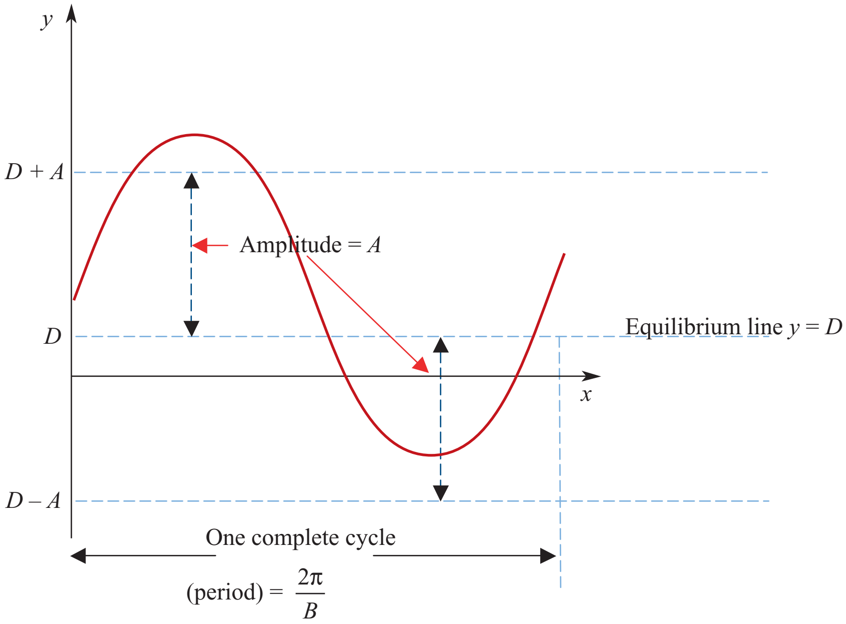
The tide height is modelled by the function $H_T = \sin\left(\frac{\pi t}{6}\right) + 2.1$, where t is the number of hours after midnight. The cyclone swell is modelled by the function $H_S = \sin\left(\frac{\pi t}{3}\right)$, where t is the number of hours after midnight.

A pier pylon in need of repair is marked at $H_W = 2.1$. The pier maintenance crew wish to repair the pier pylon *just below* the 2.1 metre mark at a suitable time after midnight. The work is expected to take up to 3 hours to complete.

Determine the most suitable time to repair the pylon (include an appropriate sketch). Fully justify and validate algebraically your solution, and discuss any assumptions.

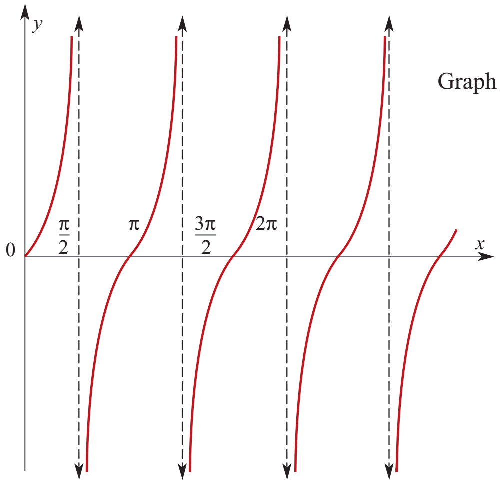
Chapter summary

- The graphs of $y = \pm A \sin B(x \pm C) \pm D$ and $y = \pm A \cos B(x \pm C) \pm D$, where A, B, C and D are positive numbers.
  - The domain is all real numbers, i.e. $-\infty < x < \infty$.
  - The range is $D - A \leq y \leq D + A$.
  - The (horizontal) equilibrium line is $y = D$.
  - The amplitude is A.
  - The period is $\frac{2\pi}{B}$.
  - The horizontal phase shift is:
    - C units to the right if $(x - C)$
    - C units to the left if $(x + C)$
  - The vertical displacement is:
    - D units up if $+D$
    - D units down if $-D$
  - The graph is inverted if A is preceded by a negative sign.



This diagram shows one full cycle of $y = A \sin Bx + D$. If the equation was $y = A \sin B(x + C) + D$ instead, then this graph would move C units to the left. If it was $y = A \sin B(x - C) + D$, then it moves C units to the right.

- The graph of $y = A \tan Bx$
  - The domain is all real numbers; i.e. $-\infty < x < \infty$, except where x is an odd multiple of $\frac{\pi}{2B}$. Vertical asymptotes occur at these values of x.
  - The range is all real numbers; i.e. $-\infty < y < \infty$.
  - The period is $\frac{\pi}{B}$.
  - The larger the value of A then the steeper the graph, but amplitude is undefined.



Graph of $y = \tan x$

- When solving trigonometric equations using the unit circle:
  1. Find the acute or base value for the angle.
  2. Establish in which quadrants the solutions will be by considering CAST.
  3. Find the solutions by adding or subtracting from 180°, 360°, 540° etc.
- Complementary angle formula

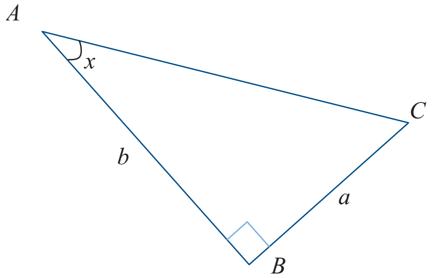
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \quad \text{or} \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

- Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

Multiple-choice questions

1 In the triangle ABC, $\cos x$ is equal to:



A $\dfrac{a}{\sqrt{a^2+b^2}}$ **B** $\dfrac{b}{\sqrt{a^2+b^2}}$

C $\dfrac{a}{b}$ **D** $\dfrac{b}{a}$

E $\dfrac{\sqrt{a^2+b^2}}{a}$

2 The period of the graph of $y = 2\sin(3x - \pi) + 4$ is:

A $\dfrac{2\pi}{3}$ **B** 2 **C** 3 **D** π **E** 2π

3 The amplitude of the graph of $y = -5\cos 5x + 3$ is:

A -5 **B** -2 **C** 2 **D** 5 **E** 8

4 The *range* of the graph of $y = 5\sin(2x - \pi) + 2$ is:

A 0 to 5 **B** 0 to 2 **C** 0 to 4 **D** -3 to 7 **E** 8

5 An angle of $\frac{3\pi}{11}$ radians when expressed in degrees (correct to 2 decimal places) is:

A 49 **B** 154.22 **C** 49.09 **D** 0.01 **E** 0.00

6 The solutions of $2\sin x + \sqrt{3} = 0$ in the domain $0 \leq x \leq 2\pi$ are:

A $\frac{\pi}{3}, \frac{2\pi}{3}$ **B** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **C** $\frac{\pi}{3}, \frac{4\pi}{3}$ **D** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **E** none of these

7 $\sin\left(\frac{\pi}{2} - x\right)$ is equal to:

A $-\sin x$ **B** $\tan x$ **C** $\cos\left(x - \frac{\pi}{2}\right)$ **D** $-\cos x$ **E** $\cos x$

8 $\tan(180° - \theta)$ is equal to:

A $\tan\theta$ **B** $\cos\theta$ **C** $-\sin\theta$ **D** $\frac{\sin\theta}{\cos\theta}$ **E** $-\tan\theta$

9 The period of the graph of $f(x) = 4\sin(3\pi x)$ is:

A 4 **B** $\frac{2}{3}$ **C** $\frac{2\pi}{3}$ **D** 3π **E** none of these

10 The graph of $y = 5\tan 3x$ has, as its first positive asymptote, the line $x =$

A $\frac{\pi}{2}$ **B** $\frac{\pi}{6}$ **C** $\frac{\pi}{3}$ **D** 5 **E** $\frac{6}{\pi}$

Short-response questions

1 Change each of the following to radian measure in terms of π:

a 330° **b** 810° **c** 1080° **d** 1035° **e** 135°
f 405° **g** 390° **h** 420° **i** 80°

2 Change each of the following to degree measure:

a $\frac{5\pi^c}{6}$ **b** $\frac{7\pi^c}{4}$ **c** $\frac{11\pi^c}{4}$ **d** $\frac{3\pi^c}{12}$ **e** $\frac{15\pi^c}{2}$
f $\frac{-3\pi^c}{4}$ **g** $-\frac{\pi^c}{4}$ **h** $-\frac{11\pi^c}{4}$ **i** $-\frac{23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin\left(\frac{11\pi}{4}\right)$ **b** $\cos\left(-\frac{7\pi}{4}\right)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\cos\left(-\frac{7\pi}{6}\right)$ **e** $\cos\left(\frac{13\pi}{6}\right)$
f $\sin\left(\frac{23\pi}{6}\right)$ **g** $\cos\left(-\frac{23}{3}\pi\right)$ **h** $\sin\left(-\frac{17}{4}\pi\right)$

4 State the amplitude and period of each of the following:

a $2\sin\frac{\theta}{2}$ **b** $-3\sin 4\theta$ **c** $\frac{1}{2}\sin 3\theta$
d $-3\cos 2x$ **e** $-4\sin\frac{x}{3}$ **f** $\frac{2}{3}\sin\frac{2x}{3}$

5 Sketch the graphs of each of the following (showing one cycle or period):

a $y = 2\sin 2(2x)$ **b** $y = -3\cos\left(\frac{x}{3}\right)$ **c** $y = -2\sin 3x$

d $y = 2\sin\frac{x}{3}$ **e** $y = \sin\left(x - \frac{\pi}{4}\right)$ **f** $y = \sin\left(x + \frac{2\pi}{3}\right)$

g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\cos\left(x + \frac{\pi}{6}\right)$

6 Solve each of the following equations for θ in the specified domain:

a $\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\pi \le \theta \le \pi$ **b** $\sin 2\theta = -\frac{\sqrt{3}}{2}, \quad -\pi \le \theta \le \pi$

c $\sin(2\theta) = -\frac{1}{2}, \quad 0 \le \theta \le 2\pi$ **d** $\sin\theta = -1, \quad 0 \le \theta \le 2\pi$

e $\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}, \quad 0 \le \theta \le 2\pi$

7 Sketch the graphs of each of the following for $-\pi \le x \le 2\pi$:

a $f(x) = 2\sin 2x + 1$ **b** $f(x) = 1 - 2\cos x$

c $f(x) = 3\cos\left(x + \frac{\pi}{3}\right)$ **d** $f(x) = 2 - \cos\left(x + \frac{\pi}{3}\right)$

e $f(x) = 1 - 2\sin 3x$

8 Solve:

a $2\sin^2 x = \sin x, \quad 0° \le x \le 360°$

b $4\cos^2 x = 3, \quad 0 \le x \le 2\pi$

c $2\cos^2 x = \cos x + 1, \quad 0° \le x \le 360°$

9 The depth, D metres, of sea water in a bay, t hours after midnight on a particular day, may be modelled by the function $D(t) = a + b\cos\left(\frac{2\pi t}{k}\right)$, where a, b and k are real numbers. The water is at a maximum depth of 15.4 metres at midnight and noon, and is at a minimum depth of 11.4 metres at 6:00 and 18:00 hours.

a Find the value of:

i a **ii** b **iii** k

b Find the times when the depth of the water is 13.4 metres.

c Find the values of t for which the depth of the bay is less than 14.4 metres.

10 The temperature (° C) in a small town in the mountains over a day is modelled by the function $T = 15 - 8\cos\left(\frac{\pi t}{12} + 6\right)$, where $0 \le t \le 24$ and where t is the time in hours after midnight.

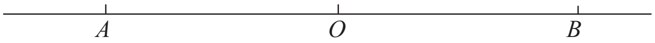
a What is the temperature at midnight, correct to 2 significant figures?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20° C?

d Sketch the graph for the temperatures over a day.

11 A particle oscillates back and forth, in a straight line, between points A and B about a point O. Its position, $x(t)$ metres, relative to O at time t seconds is given by the rule $x(t) = 3\sin(2\pi t - a)$. The position of the particle when $t = 1$ is $x = -1.5$ metres.



a If $0 \le a \le \frac{\pi}{2}$, find the value of a.

b Sketch the graph of $x(t)$ against t for $0 \le t \le 2$. Label maximum and minimum points, axes intercepts and endpoints with their coordinates.

c How far from O is point A?

d At what time does the particle first pass through A?

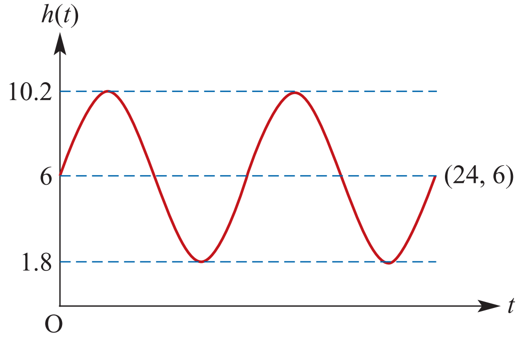
e How long is it before the particle returns to A?

f How long does it take for the particle to go from A to O?

g How far does the particle travel in:

  i the first 2 seconds of its motion? **ii** the first 2.5 seconds of its motion?

12 The depth of water, $h(t)$ metres, at a particular jetty in a harbour at time t hours after midnight is given by the rule $h(t) = p + q\sin\left(\frac{\pi t}{6}\right)$. The graph of $h(t)$ against t for $0 \le t \le 24$ is as shown.



The maximum depth is 10.2 metres and the minimum depth is 1.8 metres.

a Find the values of p and q.

b State the times at which the depth of water is a maximum for $0 \le t \le 24$.

c What is the average depth of the water in the time interval $0 \le t \le 24$?

d At what times in the time interval $0 \le t \le 24$ is the depth of the water 3.9 metres?

e For how long in the 24-hour period from midnight is the water more than 8.1 metres in depth?

CHAPTER

8

Exponential functions and logarithms

Objectives

- To **define** and **understand** the exponential function e^x.
- To **sketch** graphs of transformations of the exponential function e^x.
- To **sketch** graphs of transformations of ln(x).
- To **revise** rules of operatons of e^x and ln(x).
- To **solve** equations of applications of functions of e^x and ln(x) using exponential or logarithmic methods.
- To **apply** functions of e^x and ln(x) to modelling of growth and decay.
- To **understand** operations and applications of geometric sequences, series and infinite series.
- To **understand** and **use** equations involving compound interest.
- To **understand** how 'e' relates to compound interest.
- To **apply** the use of 'e' to compound interest models.
- To **solve, understand** and **use** equations modelling annuities and amortisation.

8.1 The exponential function, $f(x) = e^x$

In Chapter 3 the family of exponential functions $f(x) = a^x$, where a is a positive real number, was explored. One member of this family is of such importance in mathematics that it is known as *the* exponential function. This function has the rule $f(x) = e^x$, where e is Euler's number, named after the eighteenth century Swiss mathematician Leonhard Euler (pronounced 'Oiler').

Euler's number is defined as:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

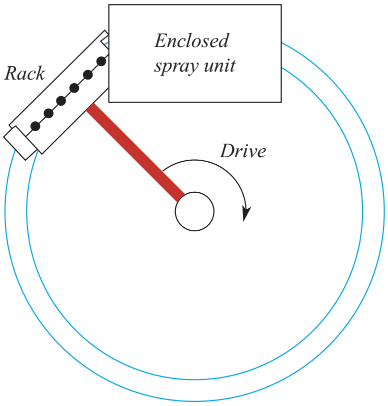
To see what the value of e might be we could try large values of n and a calculator to evaluate $\left(1 + \frac{1}{n}\right)^n$.

$$\begin{aligned}
&\text{Try} \quad n = 100 \qquad \text{then} \quad \left(1 + \frac{1}{100}\right)^{100} = (1.01)^{100} = 2.704\,8\ldots\\
&\phantom{\text{Try}} \quad n = 1000 \qquad\qquad (1.001)^{1000} = 2.716\,9\ldots\\
&\phantom{\text{Try}} \quad n = 10\,000 \qquad\qquad (1.000\,1)^{10\,000} = 2.718\,1\ldots\\
&\phantom{\text{Try}} \quad n = 100\,000 \qquad\qquad (1.000\,01)^{100\,000} = 2.718\,26\ldots\\
&\phantom{\text{Try}} \quad n = 1\,000\,000 \qquad\qquad (1.000\,001)^{1\,000\,000} = 2.718\,28\ldots
\end{aligned}$$

As n becomes larger it can be seen that $\left(1 + \frac{1}{n}\right)^n$ approaches a limiting value ($\approx 2.718\,28$). Like π, e is irrational: $e = 2.718\,281\,828\,459\,045\,235\,3\ldots$

Investigation into the production of glass marbles

A method of producing high-quality glass marbles has been proposed. A rack holding small silica 'cones' threaded on a wire will circulate around the track, as shown in the diagram. When the rack enters the spray unit it will be subjected to a fine spray of a liquid glass substance. It takes 1 minute to produce a marble.



A marble produced by a single passage around the unit will take 1 minute and the volume will be increased by 100%; that is, doubled. However, such a large increase in volume, at this slow speed, will tend to produce misshapen marbles. This suggests that the rack should be speeded up. We shall investigate what happens to the volume of the marble as the rack is speeded up and try to answer the question, 'Is there a maximum volume reached if the rack speeds up indefinitely?'

Let $V =$ volume of the marble at time t.

Also, let the original marble volume equal V_0.

For one passage per minute, $V = 2 \times V_0$.

Now assume that if the rack is speeded up to do 2 passages/minute then the growth in volume is 50% for each passage; that is,

$$V = \left(1\frac{1}{2}\right)\left(1\frac{1}{2}\right)V_0 = \left(1+\frac{1}{2}\right)^2 V_0 = 2.25V_0$$

and, similarly,

For 4 passages, $V = \left(1+\frac{1}{4}\right)^4 V_0 = 2.441\ldots V_0$

For 8 passages, $V = \left(1+\frac{1}{8}\right)^8 V_0 = 2.565\ldots V_0$

For 16 passages, $V = \left(1+\frac{1}{16}\right)^{16} V_0 = 2.637\ldots V_0$

For 64 passages, $V = \left(1+\frac{1}{64}\right)^{64} V_0 = 2.697\ldots V_0$

For n passages, $V = \left(1+\frac{1}{n}\right)^n V.$

As the rack speeds up, n is taken larger and larger, and it can be seen that $\left(1+\frac{1}{n}\right)^n$ approaches a limiting value; that is,

$$\begin{aligned} V &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n V_0 \\ &= eV_0 \end{aligned}$$

So, the maximum volume of the marble if the rack speeds up indefinitely is eV_0.

This investigation illustrates an occurrence of ‘e’. There are many important real-life situations described by exponential/logarithm functions of base ‘e’, such as:

- continuously compounding interest
- population growth
- radioactive decay
- catenary curves
- Newton’s law of cooling
- normal probability distribution function

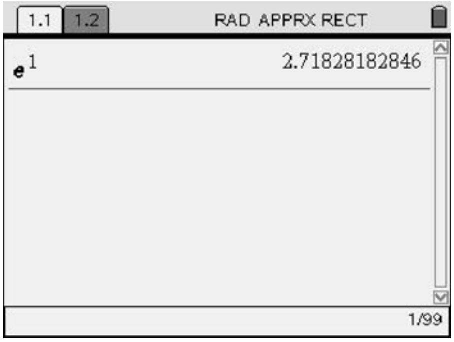
Therefore, most graphing and scientific calculators dedicate buttons for the exponential function e^x and the logarithmic function $\log_e$ or ln. Operations are executed in a manner similar to any other base.

Using technology

Using the TI-Nspire:

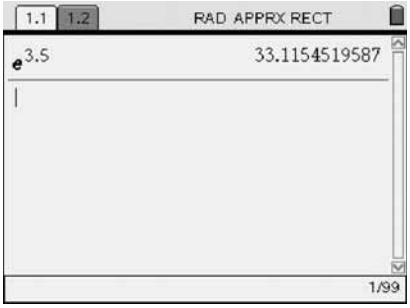
1 Set to Approximate mode.

2 Press ln e^x and place the number **1** in the power, then press enter.



e^1 2.71828182846

To evaluate $e^{3.5}$, press ln e^x and type **3.5** into the power, then press enter.



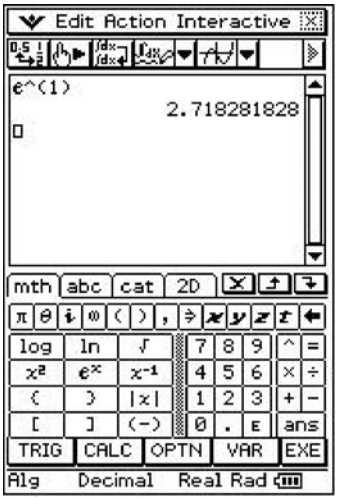
$e^{3.5}$ 33.1154519587

Using the ClassPad:

1 Set to Decimal mode.

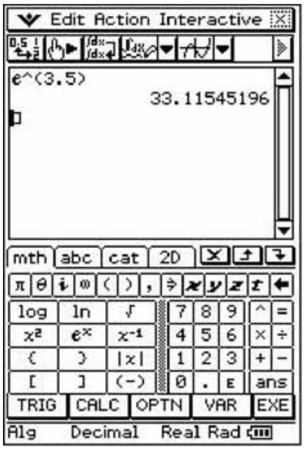
2 Press Keyboard and tap e^x.

3 Now type **1)** then press EXE.



e^(1)
2.718281828

To evaluate $e^{3.5}$, tap e^x and type **3.5)** then press EXE.

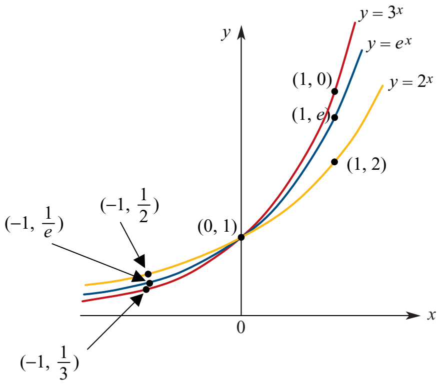


e^(3.5)
33.11545196

Graphing $y = e^x$

The graph of $y = e^x$ is as shown.

The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes. As $2 \le e \le 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

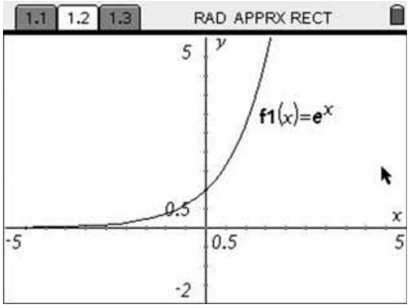


Using technology

Using the TI-Nspire:

1 Press ⌂ and choose the Graphs & Geometry application.

2 Input e^x into $f1(x)$ then press enter.

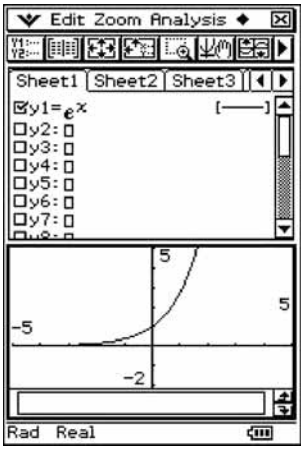


Using the ClassPad:

1 Enter into the Graph & Table application by tapping Graph&Tab.

2 Input e^x into $y1$ then press EXE.

3 To sketch the graph tap ⊞.



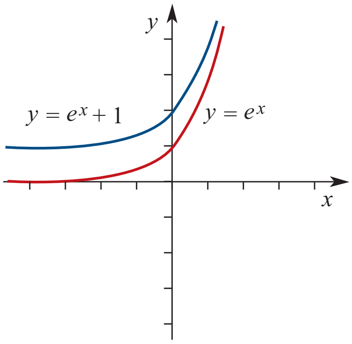
Transformations of $y = e^x$

Transformations for $y = e^x$ are as for $y = a^x$, where $a > 1$ (see Section 3.4). A brief summary is given below.

y Translation

The graph of $y = e^x + d$, where d is a constant, is $y = e^x$ translated d units in the positive y direction.

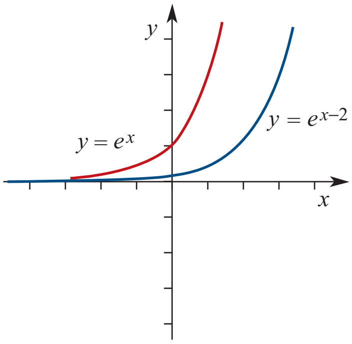
For example, $y = e^x + 1$ translates $y = e^x$ 1 unit in the positive y direction.



x Translation

The graph of $y = e^{x+c}$, where c is a constant, is $y = e^x$ translated c units in the negative x direction.

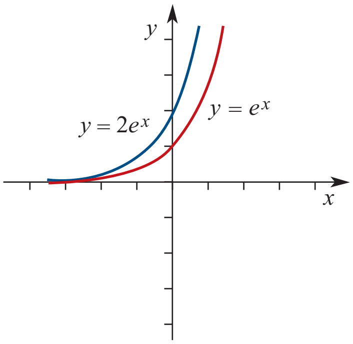
For example, $y = e^{x-2}$ translates $y = e^x$ 2 units in the positive x direction.



y Dilation

The graph of $y = ae^x$, where a is a constant, is $y = e^x$ dilated (i.e. stretched) a units in the y direction.

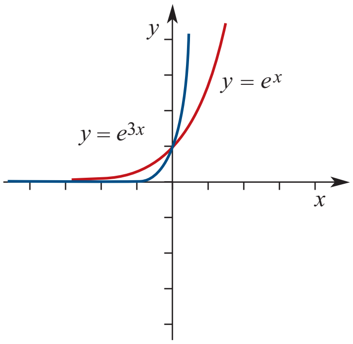
For example, $y = 2e^x$ dilates $y = e^x$ 2 units in the y direction.



x Dilation

The graph of $y = e^{bx}$, where b is a constant, is $y = e^x$ dilated (i.e. stretched) $\frac{1}{b}$ units in the x direction.

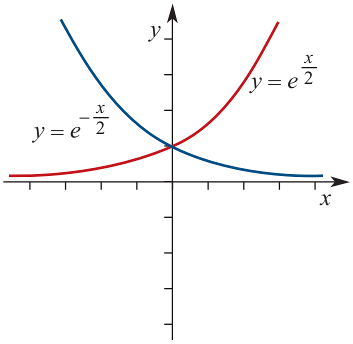
For example, $y = e^{3x}$ dilates $y = e^x$ $\frac{1}{3}$ units in the x direction.



Reflection in the y-axis

The graph of $y = e^{-bx}$, where b is a constant, is $y = e^{bx}$ reflected in the y-axis.

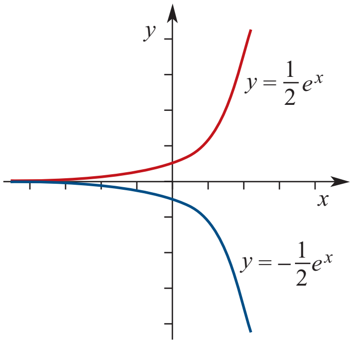
For example, $y = e^{-\frac{x}{2}}$ is a reflection of $y = e^{\frac{x}{2}}$ in the y-axis.



Reflection in the x-axis

The graph of $y = -ae^x$, where a is a constant, is $y = ae^x$ reflected in the x-axis.

For example, $y = -\frac{1}{2}e^x$ is a reflection of $y = \frac{1}{2}e^x$ in the x-axis.

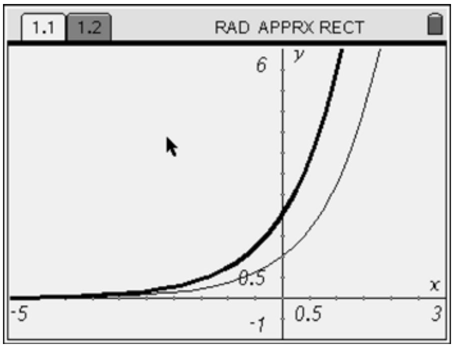


Using technology

Using the TI-Nspire:

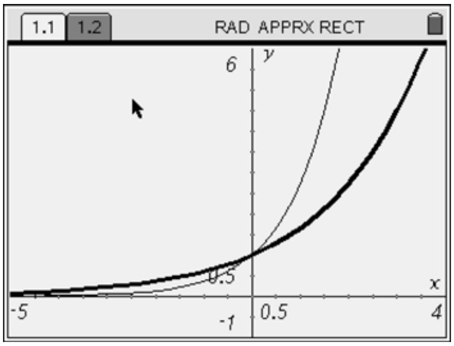
a

1. Select the Graphs & Geometry application.
2. Type **e^(x)** into $f1(x)$ then press enter.
3. Type **2e^(x)** into $f2(x)$ then press enter.
   Make this a bold line.



b

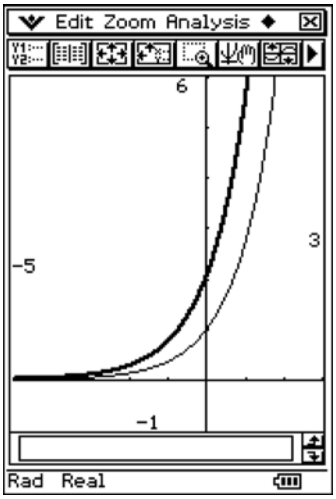
1. Type **e^(x)** into $f1(x)$ then press enter.
2. Type **e^($x/2$)** into $f2(x)$ then press enter.
   Make this a bold line.



Using the ClassPad:

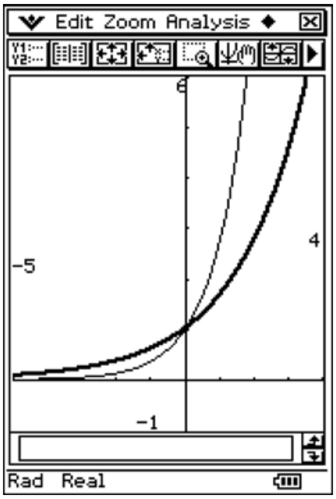
a

1. Select the Graphs and Tables application by tapping on Graph&Tab.
2. Type **e^(x)** into $y1$ then press EXE.
3. Type **2e^(x)** into $y2$ then press EXE.
   Make this a bold line.
4. Tap to sketch the two functions.



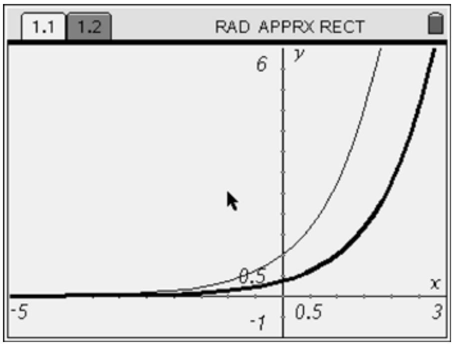
b

1. Type **e^(x)** into $y1$ then press EXE.
2. Type **e^($x/2$)** into $y2$ then press EXE.
   Make this a bold line.
3. Tap to sketch the two functions.



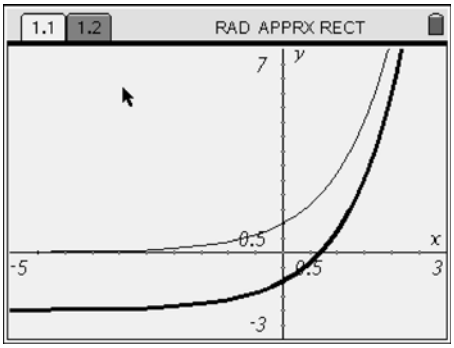
c

1. Type **e^(*x*)** into $f1(x)$ then press enter.
2. Type **e^(*x* − 1)** into $f2(x)$ then press enter. Make this a bold line.



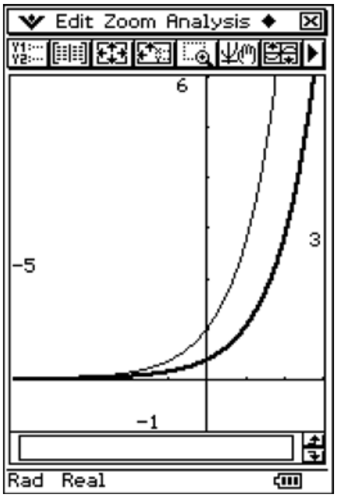
d

1. Type **e^(*x*)** into $f1(x)$ then press enter.
2. Type **e^(*x*) − 2** into $f2(x)$ then press enter. Make this a bold line.



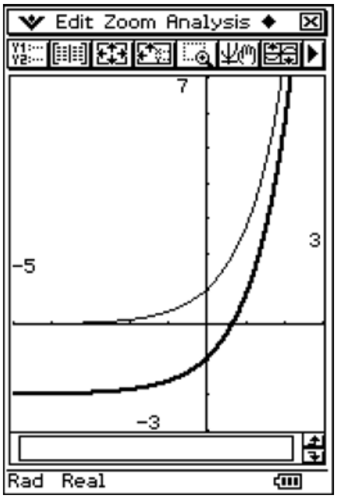
c

1. Type **e^(*x*)** into $y1$ then press EXE.
2. Type **e^(*x* − 1)** into $y2$ then press EXE. Make this a bold line.
3. Tap 📈 to sketch the two functions.



d

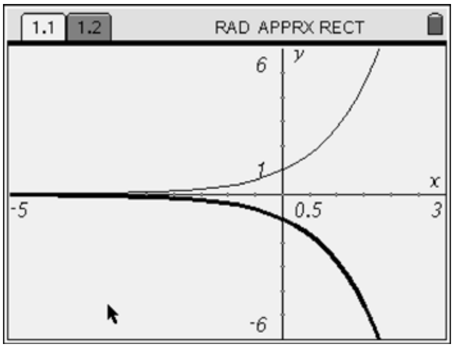
1. Type **e^(*x*)** into $y1$ then press EXE.
2. Type **e^(*x*) − 2** into $y2$ then press EXE. Make this a bold line.
3. Tap 📈 to sketch the two functions.



e

1 Type ***e*^(*x*)** into $f1(x)$ then press enter.

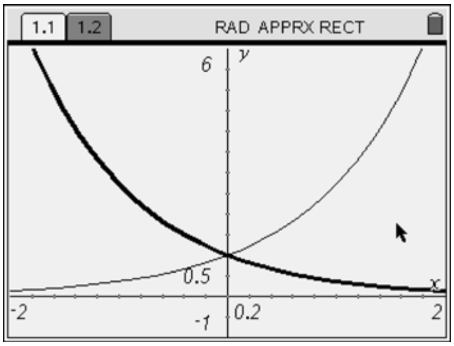
2 Type **−*e*^(*x*)** into $f2(x)$ then press enter. Make this a bold line.



f

1 Type ***e*^(*x*)** into $f1(x)$ then press enter.

2 Type ***e*^(−*x*)** into $f2(x)$ then press enter. Make this a bold line.

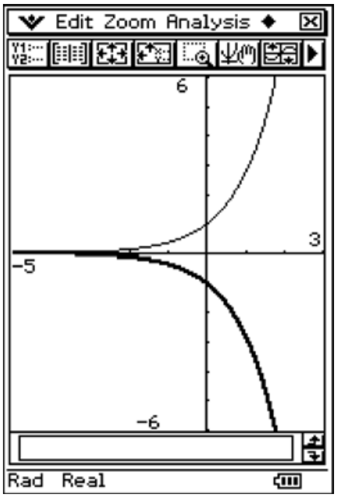


e

1 Type ***e*^(*x*)** into $y1$ then press EXE.

2 Type **−*e*^(*x*)** into $y2$ then press EXE. Make this a bold line.

3 Tap to sketch the two functions.

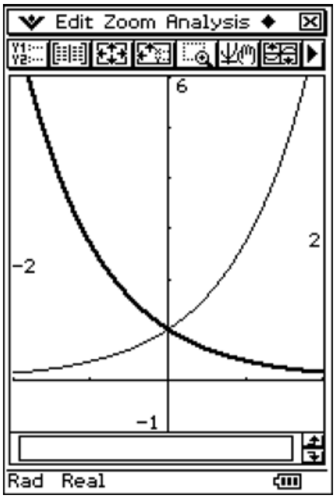


f

1 Type ***e*^(*x*)** into $y1$ then press EXE.

2 Type ***e*^(−*x*)** into $y2$ then press EXE. Make this a bold line.

3 Tap to sketch the two functions.



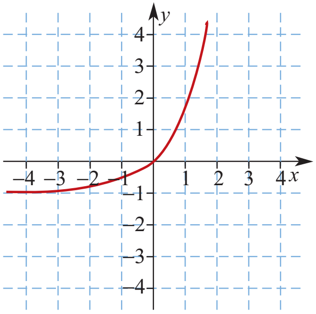
Example 1

For each graph:

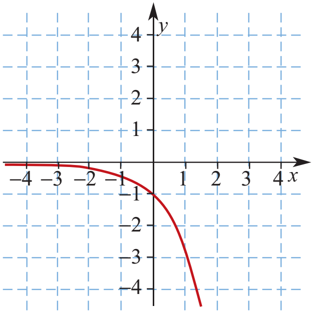
i determine the exponential function

ii identify the transformation of $f(x) = e^x$

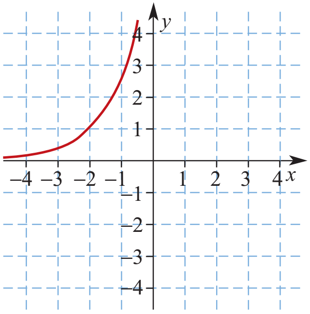
a



b



c



Solution

a $e^x - 1$, y translation of -1

b $-e^x$, reflection in the x-axis

c e^{x+2}, x translation of -2

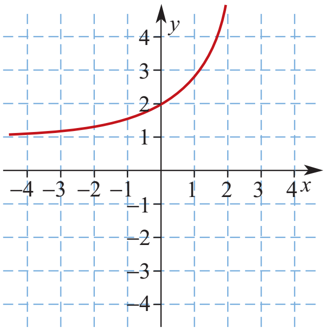
Example 2

Use a combination of transformations of $f(x) = e^x$ to sketch the function $g(x) = e^{2x} + 1$.

Solution

$e^x \rightarrow e^{2x}$ is an x dilation of factor $\frac{1}{2}$, $e^{2x} \rightarrow e^{2x} + 1$ is a y translation of 1.

$\therefore f(x) = e^x$ dilated by factor $\frac{1}{2}$ in the x direction followed by a y translation of 1 is transformed to $g(x) = e^{2x} + 1$.



Example 3

Use a graphics calculator to plot:

a **i** $f(x) = e^x$ **ii** $y = -2f(x)$ **iii** $y = f(1 - x)$

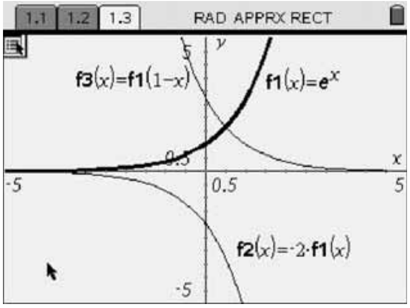
b Identify the sequence for transformations of **i** to **ii**, and for **i** to **iii**.

Solution

Using the TI-Nspire:

a i, ii and iii

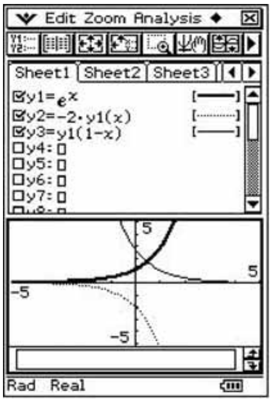
1. Type $\boldsymbol{e^x}$ into $f1(x)$ and then press enter.
2. Type $\boldsymbol{-2f1(x)}$ into $f2(x)$ and then press enter.
3. Type $\boldsymbol{f1(1-x)}$ into $f3(x)$ and then press enter.



Using the ClassPad:

a i, ii and iii

1. Type $\boldsymbol{e^x}$ into $y1$ then press EXE.
2. Type $\boldsymbol{-2y1(x)}$ into $y2$ then press EXE.
2. Type $\boldsymbol{y1(1-x)}$ into $y3$ then press EXE.
4. To sketch the graphs tap ⊞.



☑y1=e^x
☑y2=−2·y1(x)
☑y3=y1(1−x)
☐y4:□
☐y5:□
☐y6:□
☐y7:□

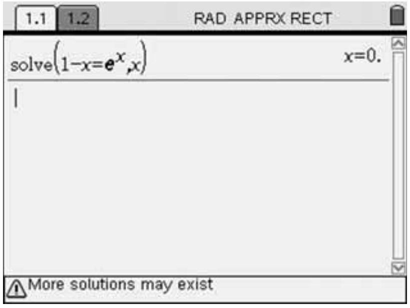
Example 4

Use a graphics calculator to solve the equation $1 - x = e^x$. Validate your answer.

Solution

Using the TI-Nspire:

1. In the Calculator application press menu and select the *solve* command from the Algebra submenu.
2. Type $\boldsymbol{1 - x = e^x, x)}$ and then press enter.

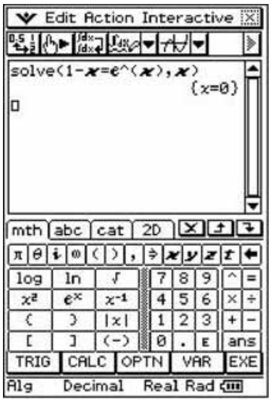


solve(1−x=e^x,x) x=0.

More solutions may exist

Using the ClassPad:

1. In the Main application tap Action and select the *solve* command from the Advanced submenu.
2. Type $\boldsymbol{1 - x = e^x, x)}$ and then press EXE.



solve(1−x=e^(x),x)
{x=0}

Exercise 8A

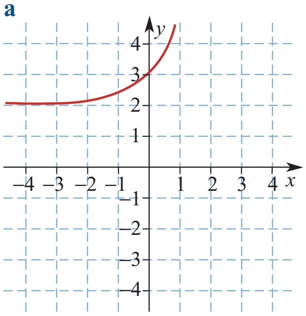
Examples 1, 2 1 Sketch the graph of each of the following and identify the transformations of the exponential function e^x:

a $f(x) = e^x + 1$ **b** $f(x) = 1 - e^x$ **c** $f(x) = 1 - e^{-x}$
d $f(x) = e^{-2x}$ **e** $f(x) = e^{x-1} - 2$ **f** $f(x) = 2e^x$
g $h(x) = 2(1 + e^x)$ **h** $h(x) = 2(1 - e^{-x})$ **i** $g(x) = 2e^{-x} + 1$
j $h(x) = 2e^{x-1}$ **k** $f(x) = 3e^{x+1} - 2$ **l** $h(x) = 2 - 3e^x$

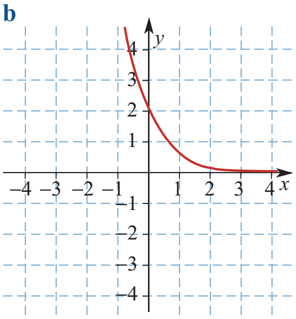
Examples 1, 2 2 For each graph:

i determine the exponential function **ii** identify the transformation of $f(x) = e^x$

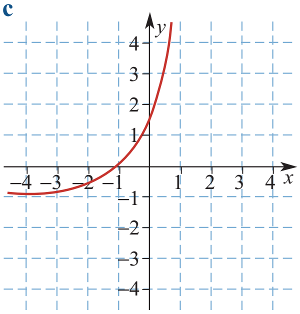
a



b



c



Example 3 3 **a** Using a graphics calculator, plot the graph of $y = f(x)$, where $f(x) = e^x$.

b Using the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f\left(\frac{x}{3}\right)$ **iii** $y = f(-x)$

Example 4 4 Solve each of the following equations using a calculator. Give answers correct to 3 decimal places.

a $e^x = x + 2$ **b** $e^{-x} = x + 2$ **c** $x^2 = e^x$ **d** $x^3 = e^x$

8.2 The logarithm function, $f(x) = \ln(x)$

Consider the statement $e^2 = 7.3891$ (to 4 decimal places).

This may be written in an alternative form

$$\log_e 7.3891 = 2$$

which is read as ‘the logarithm of 7.3891 to the base e is equal to 2’.

In general, if x is a real number, then the statements $e^x = n$ and $\log_e n = x$ are equivalent.

$$e^x = y \text{ is equivalent to } \log_e y = x$$

Further examples are:

- $e^3 = 20.0855$ (to 4 decimal places) is equivalent to $\log_e 20.0855 = 3$
- $e^0 = 1$ is equivalent to $\log_e 1 = 0$

As discussed previously, 'e' is an important and frequently used base in the analysis of many naturally occurring phenomena.

Consequently, $\log_e x$ is commonly shortened to $\ln x$; that is, $\log_e x \equiv \ln x$.

In this form:

$$e^x = y \text{ is equivalent to } \ln y = x$$

Hence,

- $e^3 = 20.0855$ (to 4 decimal places) is equivalent to $\ln 20.0855 = 3$
- $e^0 = 1$ is equivalent to $\ln 1 = 0$

Example 5

Without the aid of a calculator, evaluate the following:

a $\ln e^{1.5}$ **b** $e^{\ln 4}$

Solution

a Let $\ln e^{1.5} = x$

$$e^x = x^{1.5}$$

Equating indices

$$x = 1.5$$

b Let $e^{\ln 4} = x$

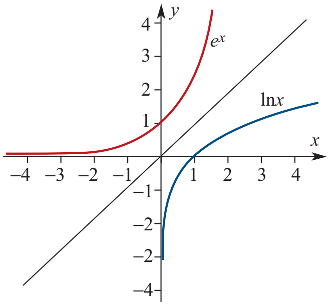
$$\ln 4 = \ln x$$

$$\therefore x = 4$$

Note: In general, $y = \log_e x$ is the inverse of the function $y = e^x$, where $x > 0$.

Note that $\log_e e^x = x$ for all x, and $e^{\log_e x} = x$ for positive values of x, as they are *inverse functions*, which are reflections in the line $y = x$.

Example 6 will demonstrate this concept.



Note: The inverse of $f(x)$ is written as $f^{-1}(x)$.

$\therefore$ If $f(x) = e^x$, then $f^{-1}(x) = \ln(x)$ and if $g(x) = \ln(x)$, then $g^{-1}(x) = e^x$.

Example 6

Find the inverse of:

a $y = e^{2x}$ **b** $y = \ln(2x)$

Solution

a $y = e^{2x}$

Interchanging x and y gives

$$x = e^{2y}$$

Therefore, $2y = \ln x$

and $$y = \frac{1}{2}\ln x$$

b $y = \ln(2x)$

Interchanging x and y gives

$$x = \ln(2y)$$

Therefore, $e^x = 2y$

and $$y = \frac{1}{2} \times e^x$$

Example 7

Find the inverse of each of the following:

a $f(x) = e^x + 3$ **b** $f(x) = \ln(x - 2)$ **c** $f(x) = 5 \times e^x + 3$

Solution

a Let $y = e^x + 3$

Interchanging x and y gives

$$x = e^y + 3$$

$$\therefore \quad x - 3 = e^y$$

and $y = \ln(x - 3)$

$\therefore f^{-1}(x) = \ln(x - 3)$

where domain f^{-1} is $x > 3$

b Let $y = \ln(x - 2)$

Interchanging x and y gives

$$x = \ln(y - 2)$$

$$\therefore \quad e^x = y - 2$$

and $y = e^x + 2$

$\therefore \quad f^{-1}(x) = e^x + 2$

where domain f^{-1} is all real numbers

c Let $y = 5 \times e^x + 3$

Interchanging x and y gives

$$x = 5 \times e^y + 3$$

$$\therefore \quad \frac{x - 3}{5} = e^y$$

and $y = \ln\left(\frac{x - 3}{5}\right)$

$$\therefore \quad f^{-1}(x) = \ln\left(\frac{x - 3}{5}\right)$$

where domain f^{-1} is $x > 3$

Note: Transformations of the graph of $\ln(x)$ are similar to that for e^x. This is shown in Example 8.

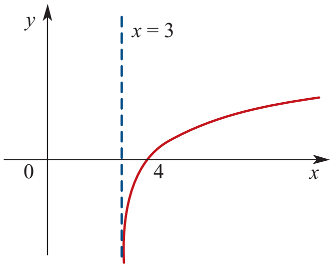
Example 8

Sketch the graphs of each of the following. Give the maximal domain, the equation of the asymptote and the axes intercepts.

a $f(x) = \ln(x - 3)$ **b** $f(x) = \ln(x + 2)$ **c** $f(x) = \ln(3x)$

Solution

a $f(x) = \ln(x - 3)$



For the function to be defined

$$x - 3 > 0, \text{ i.e. } x > 3$$

The maximal domain is $x > 3$.

For the y-axis intercept

$$f(0) = \ln(0 - 3)$$
$$= \ln(-3)$$

no solution

For the x-axis intercept

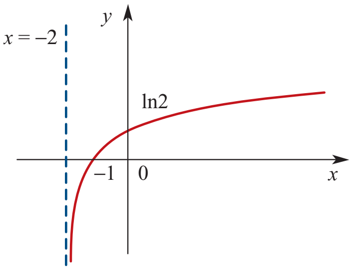
$$\ln_2(x - 3) = 0 \text{ implies } x - 3 = 1$$

i.e. $x = 4$

The asymptote

$x \to 3^+, y \to -\infty$

b $f(x) = \ln(x + 2)$



For the function to be defined

$$x + 2 > 0, \text{ i.e. } x > -2$$

The maximal domain is $x > -2$.

For the y-axis intercept

$$f(0) = \ln(0 + 2)$$
$$= \ln(2)$$

For the x-axis intercept

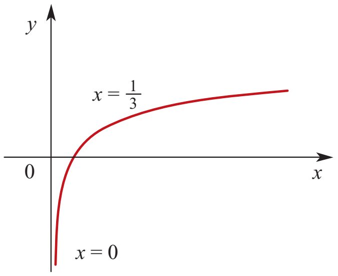
$$\ln(x + 2) = 0 \text{ implies } x + 2 = e^0$$

i.e. $x = -1$

The asymptote

$$x \to -2^+, y \to -\infty$$

c $f(x) = \ln(3x)$



For the function to be defined

$$3x > 0, \text{ i.e. } x > 0$$

The maximal domain is $x > 0$.

For the y-axis intercept

$$f(0) = \ln(3 \times 0)$$
$$= \ln(0)$$

no solution

For the x-axis intercept

$$\ln(3x) = 0 \text{ implies } 3x = e^0$$

i.e. $x = \frac{1}{3}$

The asymptote

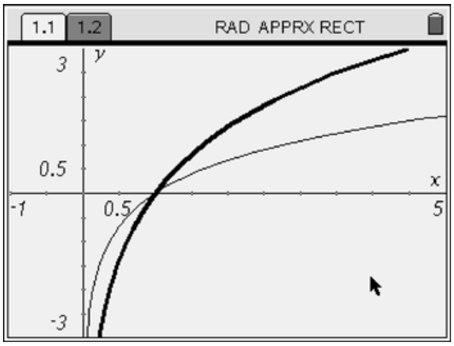
$$x \to 0^+, y \to -\infty$$

Using technology

Using the TI-Nspire:

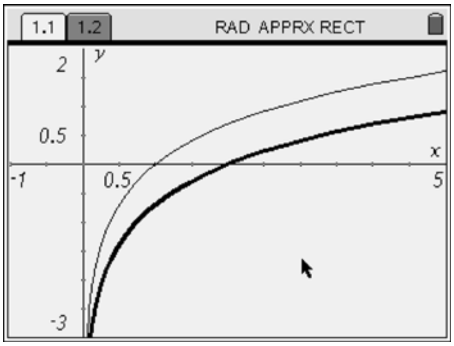
a

1. Select the Graphs & Geometry application.
2. Type **ln(*x*)** into $f1(x)$ then press enter.
3. Type **2 ln(*x*)** into $f2(x)$ then press enter. Make this a bold line.



b

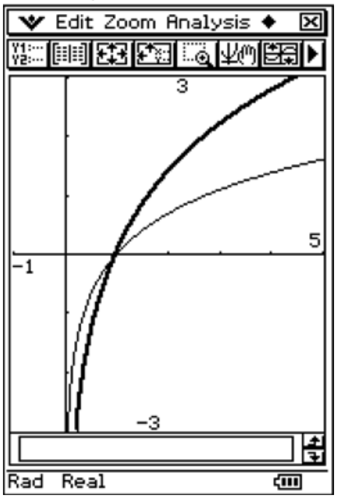
1. Type **ln(*x*)** into $f1(x)$ then press enter.
2. Type **ln(*x*/2)** into $f2(x)$ then press enter. Make this a bold line.



Using the ClassPad:

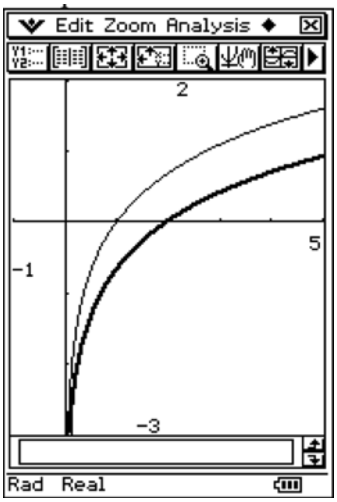
a

1. Select the Graphs and Tables application by tapping on Graph&Tab... .
2. Type **ln(*x*)** into $y1$ then press EXE.
3. Type **2 ln(*x*)** into $y2$ then press EXE. Make this a bold line.
4. Tap to sketch the two functions.



b

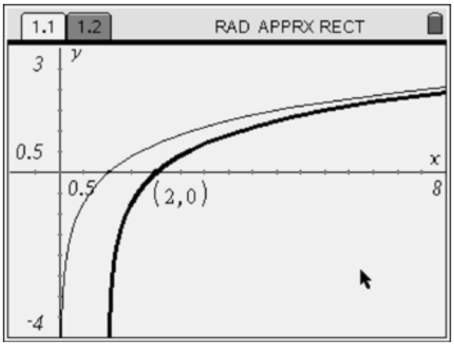
1. Type **ln(*x*)** into $y1$ then press EXE.
2. Type **ln(*x*/2)** into $y2$ then press EXE. Make this a bold line.
3. Tap to sketch the two functions.



c

1 Type **ln(x)** into $f1(x)$ then press enter.

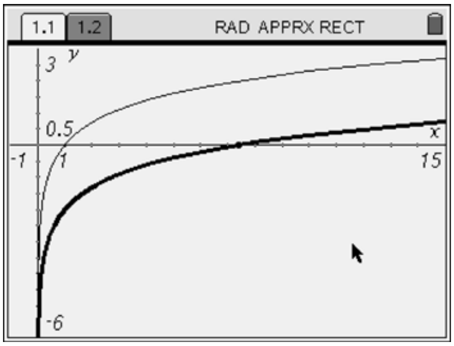
2 Type **ln(x − 1)** into $f2(x)$ then press enter. Make this a bold line.



d

1 Type **ln(x)** into $f1(x)$ then press enter.

2 Type **ln(x) − 2** into $f2(x)$ then press enter. Make this a bold line.

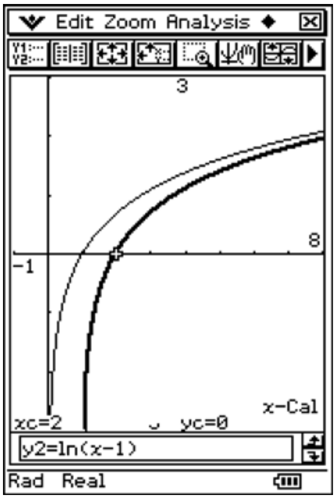


c

1 Type **ln(x)** into $y1$ then press EXE.

2 Type **ln(x − 1)** into $y2$ then press EXE. Make this a bold line.

3 Tap to sketch the two functions.



xc=2 yc=0

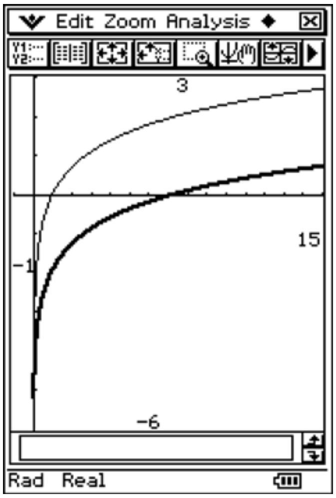
y2=ln(x-1)

d

1 Type **ln(x)** into $y1$ then press EXE.

2 Type **ln(x) − 2** into $y2$ then press EXE. Make this a bold line.

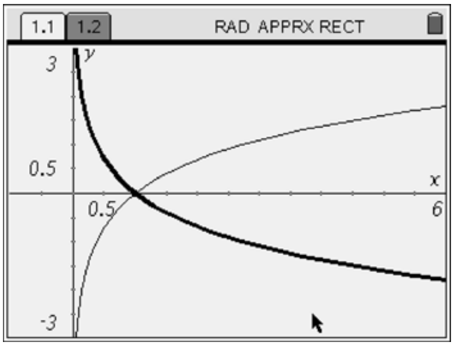
3 Tap to sketch the two functions.



e

1 Type **ln(x)** into $f1(x)$ then press enter.

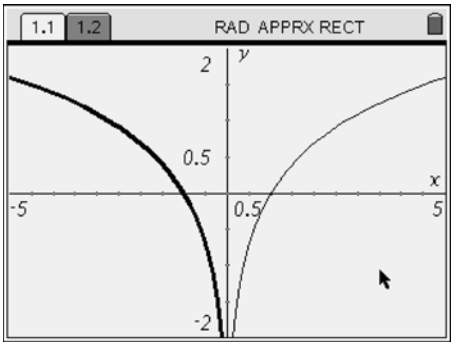
2 Type **−ln(x)** into $f2(x)$ then press enter. Make this a bold line.



f

1 Type **ln(x)** into $f1(x)$ then press enter.

2 Type **ln(−x)** into $f2(x)$ then press enter. Make this a bold line.

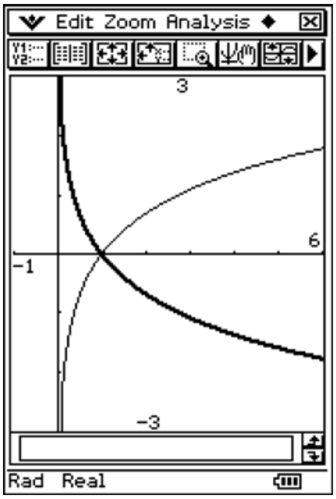


e

1 Type **ln(x)** into $y1$ then press EXE.

2 Type **−ln(x)** into $y2$ then press EXE. Make this a bold line.

3 Tap to sketch the two functions.

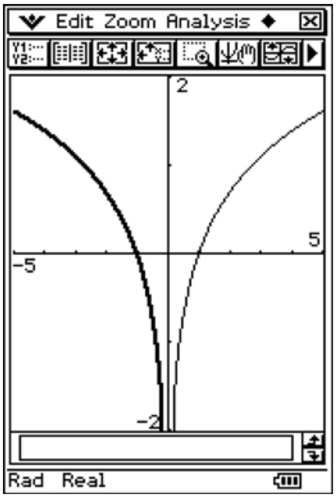


f

1 Type **ln(x)** into $y1$ then press EXE.

2 Type **ln(−x)** into $y2$ then press EXE. Make this a bold line.

3 Tap to sketch the two functions.



Exercise 8B

Example 4 **1** Use a graphics calculator to solve each of the following equations, correct to 2 decimal places:

a $e^{-x} = x$ **b** $\ln(x) + x = 0$

Example 5 **2** Sketch the graph of each of the following and state the domain and range for each:

a $y = \ln 2x$ **b** $y = 2\ln x$ **c** $y = \ln\left(\frac{x}{2}\right)$

d $y = 2\ln 3x$ **e** $y = -\ln x$ **f** $y = \ln(-x)$

Example 6 **3** Determine the inverse of each of the following:

a $y = e^{0.5x}$ **b** $y = 3\ln x$ **c** $y = e^{3x}$ **d** $y = 2\ln 3x$

Example 7 **4** Determine $f^{-1}(x)$ for each of the following:

a $f(x) = e^x + 2$ **b** $f(x) = \ln(x - 3)$ **c** $f(x) = 4 \times e^x + 2$

d $f(x) = e^x - 2$ **e** $f(x) = \ln(3x)$ **f** $f(x) = \ln\left(\frac{x}{3}\right)$

g $f(x) = \ln(x + 3)$ **h** $f(x) = 5 \times e^x - 2$

Example 8 **5** Sketch the graphs of each of the following, without using a graphics calculator. Give the maximal domain, the equation of the asymptote and axes intercepts. Now validate your answers using a graphics calculator.

a $f(x) = \ln(x - 3)$ **b** $f(x) = \ln(x + 3)$ **c** $f(x) = \ln(2x)$

d $f(x) = -\ln(2 - x)$ **e** $f(x) = 1 + \ln\left(\frac{x}{3}\right)$ **f** $f(x) = \ln(-2x)$

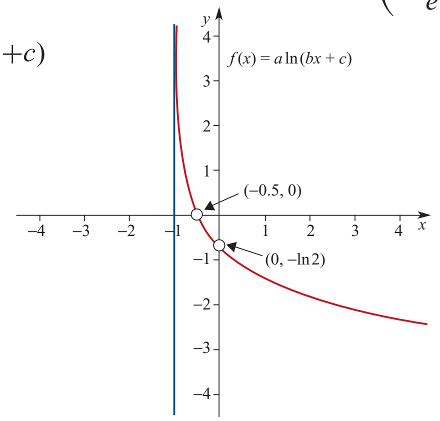
6 Use a graphics calculator to plot the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$ for $-e \le x \le e, x \ne 0$. What do you conclude?

7 On the same set of axes, plot the graph of $y = \ln(\sqrt{x})$ and $y = \frac{1}{2}\ln(x)$ for $0 > x \ge e$. What do you conclude?

8 Use a graphics calculator to plot the graphs of $y = \ln(2x) + \ln(3x)$ and $y = \ln(6x^2)$. What do you conclude?

9 Find a and k such that the graph of $y = ae^{kx}$ passes through the points $(0, e)$ and $\left(2, \frac{1}{e}\right)$.

10 Evaluate a, b and c for the plot $f(x) = a\ln(bx+c)$ given.



8.3 Exponential growth and decay

Exponential and logarithmic functions are used to model many physical occurrences. In next year's course it will be shown that if a quantity increases or decreases at a rate that is, at any time, proportional to the quantity present, then the quantity present at time t is given by the **law of exponential change**.

Let A be the quantity at time t. Then,

$A = A_0e^{kt}$, where A_0 is a constant.
Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

Physical situations in which this is applicable include:

- the growth of cells
- population growth
- continuously compounded interest
- radioactive decay
- Newton's law of cooling

Example 9

Given that $y = Ae^{bt}$ and $y = 6$ when $t = 1$ and $y = 8$ when $t = 2$, find the values of b and A.

Solution

When $t = 1, y = 6$

Thus $6 = Ae^b \ldots\ldots (1)$

When $t = 2, y = 8$

Thus $8 = Ae^{2b} \ldots\ldots (2)$

Dividing (2) by (1) gives

$$\frac{4}{3} = e^b$$

$$\therefore b = \log_e \frac{4}{3}$$

Substituting into (1):

$$6 = Ae^{\log_e \frac{4}{3}}$$

$$\therefore 6 = \frac{4}{3}A$$

$$\therefore A = \frac{18}{4} = \frac{9}{2}$$

Hence, $y = \frac{9}{2}e^{(\log_e \frac{4}{3})t}$ $\quad \left(= \frac{9}{2}\left(\frac{4}{3}\right)^t\right)$

$$y \approx \frac{9}{2}e^{0.288t}$$

Example 10

Rewrite the equation $P = Ae^{kt}$ with t as the subject.

Solution

$$P = Ae^{kt}$$

Taking logarithms to the base e of both sides:

$$\begin{aligned} \log_e P &= \log_e (Ae^{kt}) \\ \therefore\ \log_e P &= \log_e (e^{kt}) + \log_e A \\ \therefore\ t &= \frac{1}{k}(\log_e P - \log_e A) \\ \therefore\ t &= \frac{1}{k}\log_e\left(\frac{P}{A}\right) \end{aligned}$$

Example 11

Rewrite the equation $y = (2\log_e x) + 3$ with x as the subject.

Solution

$$y = 2\log_e x + 3$$

Therefore, $\dfrac{y-3}{2} = \log_e x$

and $x = e^{\frac{y-3}{2}}$

Example 12

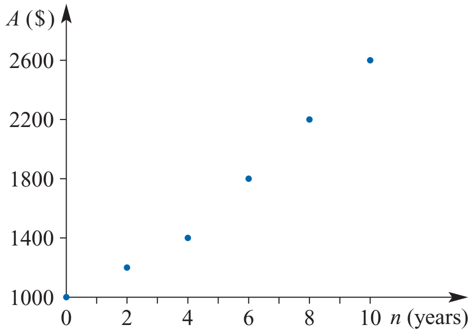
Investing money

A bank pays 10% interest, compounded annually. An amount of \$1000 is invested. Draw a graph showing the value of this investment over 10 years.

Solution

Set out in tabular form.

End of year		Amount, A
	\$1000	\$1000.00
1	$1000 \times 110\% = 1000\left(\frac{110}{100}\right) = 1000\left(1+\frac{10}{100}\right)$	\$1100.00
2	$(1000 \times 110\%)\,110\% = 1000\left(1+\frac{10}{100}\right)\left(1+\frac{10}{100}\right) = 1000\left(1+\frac{10}{100}\right)^2$	\$1210.00
3	$(1000 \times 110\%)\,110\% \times 110\% = 1000\left(1+\frac{10}{100}\right)^3$	\$1331.00
4	$= 1000\left(1+\frac{10}{100}\right)^4$	\$1464.10
↓		
10	$= 1000\left(1+\frac{10}{100}\right)^{10}$	\$2593.74



Note: In general, if P = original investment

A = amount the investment grows to after n years.

r = compound interest rate r% per annum

n = number of years invested

then $$\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$$

Example 13

An amount of $10 000 dollars is invested at 7.5% per annum (p.a.) compound interest. Evaluate the amount the investment grows to after 5 years.

Solution

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$P = 10\,000, \quad r = 7.5 \quad n = 5$$

$$A = 10\,000\left(1 + \frac{7.5}{100}\right)^5$$

$$= 10\,000 \times 1.075^5$$

$$\approx 14\,356$$

The investment grows to $14 356.

Example 14

The population of a town was 8000 at the beginning of 2003 and 15 000 at the end of 2010. Assume that the growth is exponential.

a Find the population at the end of 2012.

b In what year will the population be double that of 2010?

Solution

a Let P be the population at time t years (measured from 1 January 1992).

Then $P = 8000e^{kt}$

At the end of 1999, $t = 8$ and $P = 15\,000$.

$$\begin{aligned} \therefore\ 15\,000 &= 8000e^{8k} \\ \therefore\ \frac{15}{8} &= e^{8k} \\ \therefore\ k &= \frac{1}{8}\log_e \frac{15}{8} \\ &\approx 0.079 \end{aligned}$$

The rate of increase is 7.9% p.a.

Note: The approximation 0.079 was not used in the calculations that follow. The value for k was held in the calculator.

When $t = 10$

$$\begin{aligned} P &= 8000e^{10k} \\ &\approx 17\,552.6049 \\ &\approx 17\,550 \end{aligned}$$

The population is approximately 17 550.

b When does $P = 30\,000$? Consider the equation:

$$\begin{aligned} 30\,000 &= 8000.e^{kt} \\ \therefore\ \frac{30\,000}{8000} &= e^{kt} \\ \therefore\ \frac{15}{4} &= e^{kt} \\ \therefore\ 3.75 &= e^{kt} \\ \therefore\ t &= \frac{1}{k}\log_e 3.75 \\ &\approx 16.82 \end{aligned}$$

The population reaches 30 000 approximately 16.82 years after the beginning of 1992; that is, during the year 2008.

Exercise 8C

Example 9 **1** The graph of the function with equation $y = ae^{-x} + b$ is shown below. The values of a and b, respectively, are:

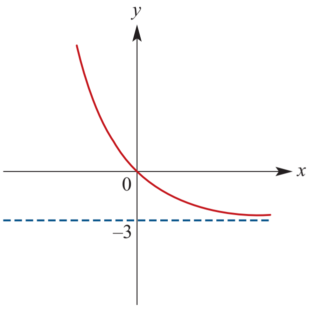
A 3, −3

B −3, 3

C −3, −3

D 0, −3

E −3, 0



2 Find the value of x for which $4e^{3x} = 287$, giving the answer to 3 significant figures.

3 The graph of the function with equation $f(x) = e^{2x} - 3ke^{x} + 5$ intersects the axes at $(0, 0)$ and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

4 Find the values of a and b such that the graph of $y = ae^{bx}$ goes through the points $(3, 10)$ and $(6, 50)$.

5 Find the values of a and b such that the graph of $y = ae^{-bx}$ goes through the points $(3, 50)$ and $(6, 10)$.

Examples 10,11

6 For formula, make the pronumeral in brackets the subject:

a $y = 2\log_e(x) + 5 \quad (x)$
b $P = Ae^{-6x} \quad (x)$
c $y = ax^n \quad (n)$
d $y = 5 \times 10^x \quad (x)$
e $y = 5 - 3\log_e(2x) \quad (x)$
f $y = 6x^{2n} \quad (n)$
g $y = \log_e(2x - 1) \quad (x)$
h $y = 5(1 - e^{-x}) \quad (x)$

Example 12

7 **a** A bank pays 25% interest, compounded annually. An amount of \$4000 is invested. Draw a graph showing the value of this investment over 6 years.

b An amount of \$5000 is invested at 19% p.a., compounded annually. Draw a graph showing the value of this investment over 8 years.

c An amount of \$1200 is invested at 10% p.a., compounded annually. Draw a graph showing the value of this investment over 5 years.

Example 13

8 **a** A bank pays 7.5% interest, compounded annually. An amount of \$8000 is invested for 5 years. Find the value of the investment at the end of the 5-year period.

b An amount of \$2500 is invested for 10 years at 7.2% p.a., compounded annually. What is the value of the investment at the end of the 10-year period?

c An amount of \$3500 is invested for 12 years at 6% p.a., compounded annually. What is the value of the investment at the end of the 12-year period?

Example 14

9 The value, \$$V$, of a particular car can be modelled by the equation $V = ke^{-\lambda t}$, where t years is the age of the car.
The car's original price was \$22 497, and after 1 year it is valued at \$18 000.

a State the value of k and calculate λ, giving your answer to 2 decimal places.
b Find the value of the car when it is 3 years old.

10 Due to a nearby growing major industrial development, the value \$$M$ of a particular property in a certain area during the period 2009 to 2015 can be modelled by the equation $M = Ae^{-pt}$, where t is the time, in years, after 1 January 2009.
The value of the property on 1 January 2009 is \$650 000 and its value on 1 January 2010 is \$610 000.

a State the value of A and calculate the value of p, correct to 2 significant figures.
b What is the value of the property in 2014? Give your answer to the nearest 1000.

11 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter of the tree in cm, t years after the beginning of this period. The diameter after 1 year is 52 cm and after 3 years, 80 cm. Calculate the values of the constants d_0 and m.

12 The number of bacteria in a certain culture at time t weeks is given by the rule $N = N_0 e^{kt}$. If $N = 101$ when $t = 2$ and $N = 203$ when $t = 4$, calculate the values of N_0 and k.

13 The number of people, N, who have a particular disease at time t years is given by $N = N_0 e^{kt}$.

a If the initial number is 20 000 and the number decreases by 20% each year, find the value of:

i N_0 **ii** k

b How long does it take for there to be 5000 people who have the disease?

14 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula $M = M_0 e^{-kt}$, where M is the mass of polonium-210 left after t days, and M_0 and k are constants. At time $t = 0$, $M = 10$ grams and at $t = 140$, $M = 5$ grams.

a Find the values of M_0 and k.

b What will be the mass of the polonium-210 after 70 days?

c After how many days is the mass remaining 2 grams?

15 A liquid cools from its original temperature of 90° C to a temperature T° C in x minutes. Given that $T = 90(0.98)^x$ find the value of:

a T when $x = 10$ **b** x when $T = 27$

16 The barometric pressure, P cm of mercury, at a height h km above sea level is given by $P = 75(10^{-0.15h})$. Find:

a P when $h = 0$ **b** P when $h = 10$ **c** h when $P = 60$

17 In a chemical reaction the amount (x grams) of a substance that has reacted is given by $x = 8(1 - e^{-0.2t})$, where t is the time from the beginning of the reaction, in minutes.

a Sketch the graph of x against t.

b Find the amount of substance that has reacted after:

i 0 minutes **ii** 2 minutes **iii** 10 minutes

c Find the time when there is exactly 7 grams of the substance that has reacted.

8.4 Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**.

For example, 2, 6, 18, 54, . . . is a geometric sequence where the next term is the previous term multiplied by 3.

A geometric sequence can be defined by a recursive (i.e. repeating) equation of the form

$t_n = rt_{n-1}$, where r is a constant multiplier called the **ratio**.

i.e. the next term is r × the previous term.

If the first term of a geometric sequence $t_1 = a$, then the geometric sequence of successive terms is:

$$\begin{aligned} t_1 &= a \\ t_2 &= ar \\ t_3 &= ar^2 \\ &\vdots \\ &\vdots \\ t_n &= ar^{n-1} \end{aligned}$$

i.e. the nth term of the sequence is $t_n = ar^{n-1}$.

Note: $r = \dfrac{t_n}{t_{n-1}}$, where r is the common ratio.

Example 15

Calculate the tenth term of the sequence 2, 2.2, 2.42, . . .

Solution

$$\begin{aligned} a = 2, \quad r &= \frac{2.2}{2} = 1.1 \\ t_n &= ar^{n-1} \\ t_{10} &= 2 \times 1.1^{(10-1)} \\ &= 2 \times 1.1^9 \\ &= 4.7159 \text{ (to 4 decimal places)} \end{aligned}$$

Using technology

Using the TI-Nspire:

1. Press ⌂ then enter on the Lists & Spreadsheets application.
2. Enter the values from 1 to 20 into column A.
3. Give column A the name **n**.

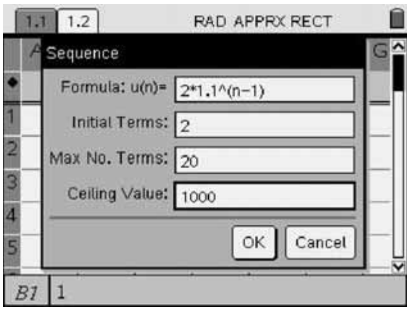
Using the ClassPad:

1. Press EXE on the Sequence application Sequence.
2. Tap the Recursive tab.
3. Type $\mathbf{2 \times 1.1\hat{}\,} \boldsymbol{n}$ into a_{n+1} then press EXE.

4 Give column B the name **term** and then place the cursor in cell B1.

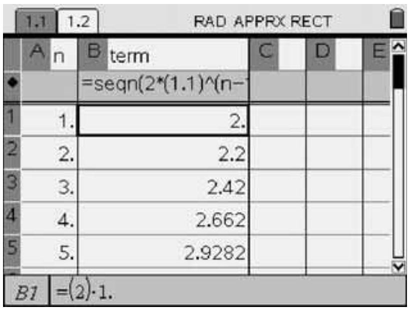
5 Press (menu) and select *Generate Sequence* from the Data submenu.

6 Enter the following information:



Formula: u(n)= 2*1.1^(n–1)
Initial Terms: 2
Max No. Terms: 20
Ceiling Value: 1000
B1 1

7 Press (enter) on OK.
The first 20 terms of the sequence are displayed in column B.



	A n	B term	C	D	E
◆		=seqn(2*(1.1)^(n–			
1	1.	2.			
2	2.	2.2			
3	3.	2.42			
4	4.	2.662			
5	5.	2.9282			

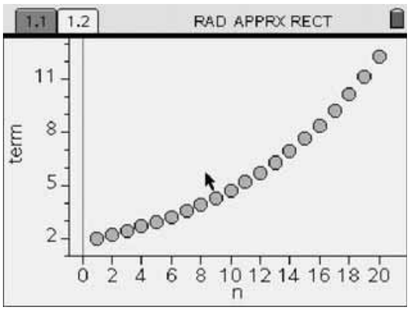
B1 =(2)·1.

To sketch a graph of the sequence, do the following:

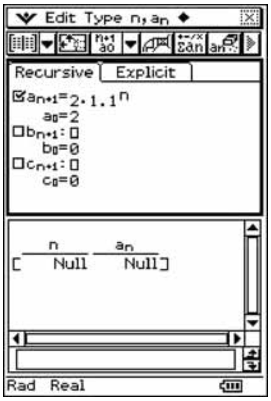
1 Highlight columns A and B.

2 Press (menu) and select *Quick Graph* from the Data menu.

3 For a full-screen view of the graph press: (ctrl), (tab), (ctrl), (K), (ctrl), (clear), (ctrl), (⌂)
Scroll to *Page Layout*→*Select Layout*→*1: Layout 1*.



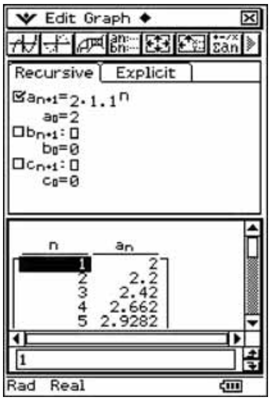
4 Type **2** into a_0 then press (EXE).



☑$a_{n+1}=2\cdot1.1^n$
$a_0=2$
☐b_{n+1}: ☐
$b_0=0$
☐c_{n+1}: ☐
$c_0=0$

n	an
Null	Null

5 Now tap ▦ to generate the sequence in a table.



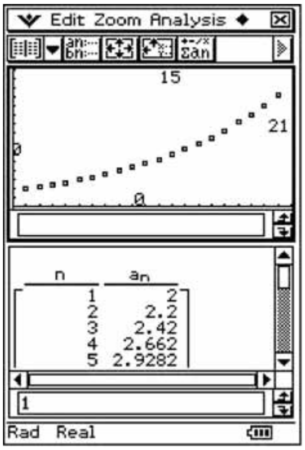
☑$a_{n+1}=2\cdot1.1^n$
$a_0=2$
☐b_{n+1}: ☐
$b_0=0$
☐c_{n+1}: ☐
$c_0=0$

n	an
1	2
2	2.2
3	2.42
4	2.662
5	2.9282

1

6 To sketch a graph of the first 20 terms in the sequence tap ⊹.

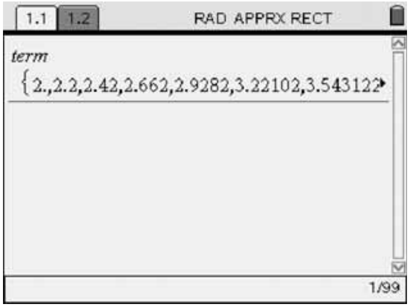
7 Change the window settings if necessary by tapping ⊡. Set
*x*min: 0
max: 21
*y*min: 0
max: 15



n	an
1	2
2	2.2
3	2.42
4	2.662
5	2.9282

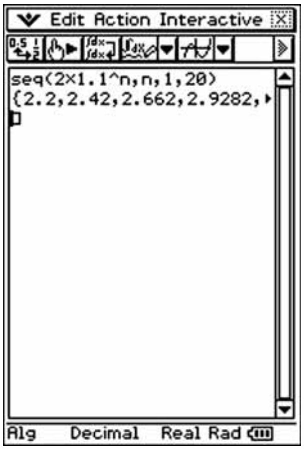
1

The sequence may be displayed in the Calculator application by typing **term** then pressing enter.



```
term
{2.,2.2,2.42,2.662,2.9282,3.22102,3.543122▸
```

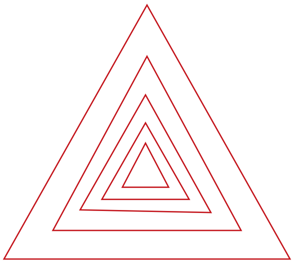
The sequence may be displayed in the Main application by typing **seq (2 × 1.1^n, n,1, 20)** and then pressing EXE.



```
seq(2×1.1^n,n,1,20)
{2.2,2.42,2.662,2.9282,▸
```

Example 16

Georgina draws a pattern consisting of a number of similar equilateral triangles. The first triangle has sides of length 4 cm and the side length of each successive triangle is 1.5 times the side length of the previous one.



a How long is the side length of the fifth triangle?

b Which triangle has a side length of $45\frac{9}{16}$ cm?

Solution

a

$$a = 4, r = \frac{3}{2}$$

$$t_n = ar^{n-1}$$

$$t_5 = ar^4 = 4 \times \left(\frac{3}{2}\right)^4 = 20\frac{1}{4}$$

Hence, the fifth triangle has a side length of $20\frac{1}{4}$ cm.

b

$$a = 4, r = \frac{3}{2}, t_n = 45\frac{9}{16} = \frac{729}{16}$$

$$\begin{aligned} t_n &= ar^{n-1} \\ \therefore \quad \frac{729}{16} &= 4 \times \left(\frac{3}{2}\right)^{n-1} \\ \frac{729}{64} &= \left(\frac{3}{2}\right)^{n-1} \\ \therefore \quad \left(\frac{3}{2}\right)^6 &= \left(\frac{3}{2}\right)^{n-1} \end{aligned}$$

Equating indices $n - 1 = 6$

$$\therefore n = 7$$

Hence, the seventh triangle will have a side length of $45\frac{9}{16}$ cm.

Alternative method

$$\begin{aligned}\frac{729}{64} &= \left(\frac{3}{2}\right)^{n-1}\\ \ln\frac{729}{64} &= \ln\left(\frac{3}{2}\right)^{n-1}\\ \ln\frac{729}{64} &= (n-1)\ln\left(\frac{3}{2}\right)\\ 2.4328 &= (n-1)\times 0.4055\\ n &= 7\end{aligned}$$

An application of geometric sequences is **compound interest**. As seen in the previous section, compound interest is interest calculated at regular intervals on the total of the amount originally invested and the amount accumulated in the previous years.

So \$1000 invested at 10% p.a. would grow to

$\$1000 + 10\%(\$1000) = \$1000(1+0.1) = \$1000 \times 1.1 = \$1100$ at the end of the first year.

At the end of the second year, it will have grown to

$$\begin{aligned}&\$1100 + 10\%(\$1100) = \$1100(1+0.1) = \$1100 \times 1.1\\ &= \$1000 \times 1.1 \times 1.1 = \$1000 \times 1.1^2 = \$1210\end{aligned}$$

Hence:

Years invested	\$	Total amount (\$)
0	1000×1.1^0	1000
1	1000×1.1^1	1100
2	1000×1.1^2	1210
3	1000×1.1^3	1331
4	1000×1.1^4	1464
5	1000×1.1^5	1610
n	1000×1.1^n	1000×1.1^n

The value of the investment at the end of each year forms the **geometric sequence**.

$$1000, 1000 \times 1.1^1, 1000 \times 1.1^2, 1000 \times 1.1^3, 1000 \times 1.1^4, 1000 \times 1.1^5 \;...........\; 1000 \times 1.1^n$$

Such that $a = 1000$, $r = 1.1$; i.e. $r = 100\% + 10\%$

Example 17

Hamish invests \$2500 at 7% p.a., compounded annually. Find:

a the value of his investment after 5 years

b how long it takes until his investment is worth \$10 000

Solution

$a = 2500, r = 1.07$

a
$$\begin{aligned} t_6 &= ar^5 \\ &= 2500(1.07)^5 \\ &= 3506.38 \end{aligned}$$

The value of his investment after 5 years is \$3506.38.

b
$$\begin{aligned} t_n = ar^{n-1} &= 10\,000 \\ 2500(1.07)^{n-1} &= 10\,000 \\ (1.07)^{n-1} &= 4 \\ \ln(1.07)^{n-1} &= \ln 4 \\ (n-1)\ln(1.07) &= \ln 4 \\ n - 1 &= \frac{\ln(4)}{\ln(1.07)} \\ n &= 21.489 \end{aligned}$$

By the *end* of the 22nd year, his investment will be worth in excess of \$10 000.

Note: The number of years can also be found by:

- trial and error or through using the table facility of a calculator
- using a calculator to find the intersection of $y = 4$ and $y = (1.07)^{x-1}$
- using the calculator ‘Solve’ function.

Example 18

The third term of a geometric sequence is 10 and the sixth term is 80. Find r and the first term.

Solution

$$\begin{aligned} t^3 &= ar^2 = 10 \ldots\ldots (1) \\ t_6 &= ar^5 = 80 \ldots\ldots (2) \end{aligned}$$

Divide (2) by (1):

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$

$$\therefore \quad r^3 = 8$$

$$\therefore \quad r = 2$$

Substitute into (1) to find a.

$$a \times 4 = 10$$
$$\therefore \quad a = \frac{5}{2}$$

The first term is $\frac{5}{2}$.

The **geometric mean** of two numbers a and c (of the same sign) is $\pm\sqrt{ac}$.
Note that if three numbers a, b and c are consecutive terms of a geometric sequence,

Then $$\frac{b}{a} = \frac{c}{b}$$
$$\therefore \quad b^2 = ac$$
$$\therefore \quad b = \pm\sqrt{ac}$$

Example 19

Insert a geometric mean between 3 and 12.

Solution

Let the mean be m.

$\therefore$ 3, m, 12 are consecutive terms of a geometric sequence.

$$\therefore \quad \frac{m}{3} = \frac{12}{m}$$
$$m^2 = 3 \times 12$$
$$m = \pm\sqrt{36}$$
$$m = \pm 6$$

The geometric mean is -6 or 6. There are two possible sequences 3, -6, 12 or 3, 6, 12.

Exercise 8D

1 For a geometric sequence $t_n = ar^{n-1}$, find the first four terms given that:

a $a = 3, r = 2$ **b** $a = 3, r = -2$

c $a = 10\,000, r = 0.1$ **d** $a = r = 3$

Example 15

2 Find the specified term in each of the following geometric sequences:

a $\frac{15}{7}, \frac{5}{7}, \frac{5}{21}, \ldots$ find t_6 **b** $1, -\frac{1}{4}, \frac{1}{16}, \ldots$ find t_5

c $\sqrt{2}, 2, 2\sqrt{2}, \ldots$ find t_{10} **d** $a^x, a^{x+1}, a^{x+2}, \ldots$ find t_6

3 Find the rule for the geometric sequence whose first few terms are:

a $3, 2, \frac{4}{3}$ **b** $2, -4, 8, -16$ **c** $2, 2\sqrt{5}, 10$

4 For a geometric sequence the first term is 25 and the fifth term is $\frac{16}{25}$. Find the common ratio.

5 A geometric sequence has first term $\frac{1}{4}$ and common ratio 2. Which term of the sequence is 64?

6 If t_n is the nth term of these geometric sequences, find n in each case.

a $2, 6, 18, \ldots \quad t_n = 486$

b $5, 10, 20, \ldots \quad t_n = 1280$

c $768, 384, 192, \ldots \quad t_n = 3$

d $\frac{8}{9}, \frac{4}{3}, 2, \ldots \quad t_n = \frac{27}{4}$

e $-\frac{4}{3}, \frac{2}{3}, -\frac{1}{3}, \ldots \quad t_n = \frac{1}{96}$

*Use a graphics calculator to validate questions **7–13**.*

Example 16

7 An art collector has a painting that is increasing in value by 8% each year. If the painting is currently valued at \$2500,

a how much will it be worth in 10 years?

b how many years before its value exceeds \$100 000?

8 An algal bloom is growing in a lake. The area it covers triples each day. If it initially covers an area of 10 m^2:

a How many square metres will it cover after 1 week?

b If the lake has a total area of 200 000 m^2, how long before the entire lake is covered?

9 A ball is dropped from a height of 2 metres and continues to bounce so that it rebounds to $\frac{3}{4}$ of the height from which it previously falls. Find the height it rises to on the fifth bounce.

10 The Tour de Moravia is a cycling event that lasts for 15 days. On the first day the cyclists must ride 120 km and each successive day they ride 90% of the distance of the previous day.

a How far do they ride on the eighth day? **b** On which day do they ride 30.5 km?

11 A child negotiates a new pocket money deal with her unsuspecting parents, whereby she receives 1 cent on the first day of the month, 2 cents on the second, 4 cents on the third, 8 cents on the fourth, and so on until the end of the month. How much would the child receive on the 30th day of the month? (Give your answer to the nearest thousand dollars.)

12 The number of fish in the breeding tanks of a fish farm follow a geometric progression. The third tank contains 96 fish and the sixth tank contains 768.

a How many fish are in the first tank? **b** How many fish are in the tenth tank?

Example 17 **13** An amount of \$5000 is invested at 6% p.a., compounded annually.

a Find the value of the investment after 6 years.

b Find how long it will take for the original investment to double in value.

Example 18 **14** The twelfth term of a geometric sequence is 2 and the fifteenth term is 54. Find the seventh term.

15 A geometric sequence has $t_2 = \frac{1}{2\sqrt{2}}$ and $t_4 = \sqrt{2}$. Find t_8.

*Use a graphics calculator to validate questions **16** and **17**.*

16 The first three terms of a geometric sequence are 4, 8 and 16. Find the term that first exceeds 2000.

17 The first three terms of a geometric sequence are 3, 9 and 27. Find the term in the sequence that first exceeds 500.

18 What amount would need to be invested at 8.5% p.a., compounded annually, to yield a return of \$8000 after 12 years?

19 What annual interest rate would be required to triple the value of an investment of \$200 in 10 years?

20 The number of ‘type A’ apple bugs present in an orchard is estimated to be 40 960 and is reducing in number by 50% each week. At the same time it is estimated that there are 40 ‘type B’ apple bugs whose number is doubling each week.

a Use algebra to calculate after how many weeks there will be the same number of each type of bug.

b Use a graphics calculator to plot the functions and determine the intersection to verify your answer.

Example 19 **21** Find the geometric means of:

a 5 and 720 **b** 1 and 6.25

c $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ **d** x^2y^3 and x^6y^{11}

8.5 Geometric series

The sum of the terms in a geometric sequence, S_n, is called a **geometric series**. An exact expression for S_n, the sum of n terms, of a geometric sequence can be found.
Given the general geometric sequence

$$a, ar, ar^2, \dots\dots\dots + ar^{n-1}$$

Then the sum or addition, S_n, of all the terms of the sequence is given by:

$$S_n = a + ar + ar^2 + ar^3 \dots + ar^{n-1}$$

As this series could contain a large number of terms, it is useful to derive a formula for S_n.

$$S_n = a + ar + ar^2 + ar^3 \dots + ar^{n-1} \quad \dots\dots\dots\dots(1)$$

Multiplying both sides by r:

$$rS_n = \qquad ar + ar^2 + ar^3 \dots + ar^{n-1} + ar^n \dots\dots(2)$$

Consider the vertically aligned terms to subtract (1) from (2):

$$\therefore \quad rS_n - S_n = ar^n - a$$
$$\therefore \quad S_n(r-1) = a(r^n - 1)$$
$$\therefore \quad S_n = \frac{a(r^n-1)}{r-1}, \qquad r < -1 \quad \text{or} \quad r > 1$$

For values of r such that $-1 < r < 1$, it is often more convenient to use the alternative formula

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad -1 < r < 1$$

which is obtained by subtracting (2) from (1) above.

Example 20

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$

Solution

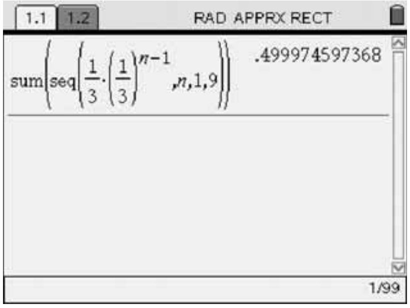
$$a = \frac{1}{3}, r = \frac{1}{3}, n = 9$$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(\left(\frac{1}{3}\right)^9 - 1\right)}{\frac{1}{3} - 1}$$
$$= \frac{-1}{2}\left(\left(\frac{1}{3}\right)^9 - 1\right)$$
$$\approx \frac{1}{2}(0.999949)$$
$$= 0.499975$$

Using technology

Using the TI-Nspire:
To calculate the sum of a sequence, combine the **sum** and **seq** commands.

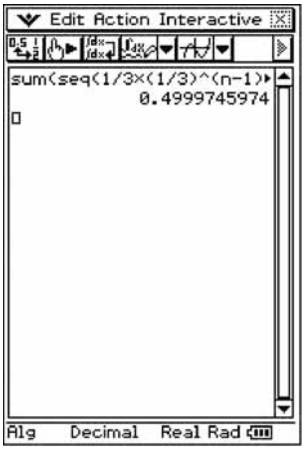
Thus, in the Calculator application type the following:
sum (seq (1/3 × (1/3)^(n – 1), n, 1, 9))
then press enter.



$\text{sum}\left(\text{seq}\left(\frac{1}{3}\cdot\left(\frac{1}{3}\right)^{n-1},n,1,9\right)\right)$ .499974597368

Using the ClassPad:
To calculate the sum of a sequence, combine the **sum** and **seq** commands.

Thus, in the Main application type the following:
sum (seq (1/3 × (1/3)^(n – 1), n, 1, 9))
then press EXE.



```
sum(seq(1/3×(1/3)^(n-1)▸
                0.4999745974
```

Example 21

For the geometric sequence 1, 3, 9, . . ., find how many terms must be added together to obtain a sum of 1093.

Solution

$$a = 1, r = 3, S_n = 1093$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{1(3^n - 1)}{3 - 1} = 1093$$

$$\therefore \quad 3^n - 1 = 1093 \times 2$$

$$\therefore \quad 3^n = 2187$$

Taking ln of both sides

$$\ln 3^n = \ln 2187$$

$$n \ln 3 = \ln 2187$$

$$n \times 1.0986 = 7.6903$$

$$n = 7$$

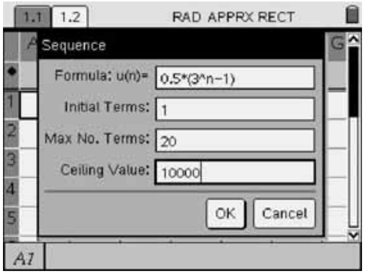
Seven terms are required to give a sum of 1093.

Using technology

Using the TI-Nspire:

1 Open a new Lists & Spreadsheet application.

2 With the cursor in cell A1, press menu and generate the sequence below.



Sequence

Formula: u(n)= 0.5*(3^n-1)

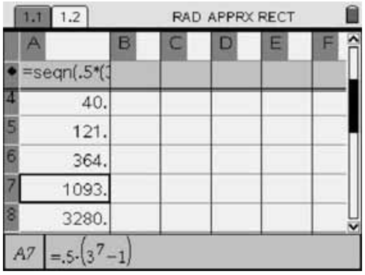
Initial Terms: 1

Max No. Terms: 20

Ceiling Value: 10000

The sequence in now displayed in column A.

3 Scroll through the sequence and look for the number **1093**.

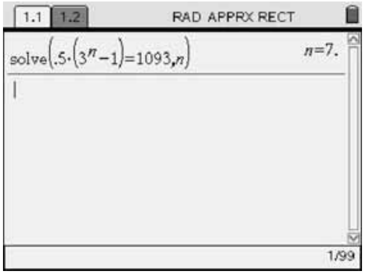


	A
◆	=seqn(.5*(3
4	40.
5	121.
6	364.
7	1093.
8	3280.

A7 $=.5\cdot(3^7-1)$

Thus, the seventh term is 1093.

Alternatively, the *solve* command may be used. Type **solve (0.5 × (3^*n* – 1) = 1093, *n*)**



$\text{solve}(.5\cdot(3^n-1)=1093,n)$ $\quad n=7.$

Using the ClassPad:

1 Press EXE on the Sequence application Sequence.

2 Tap the Recursive tab.

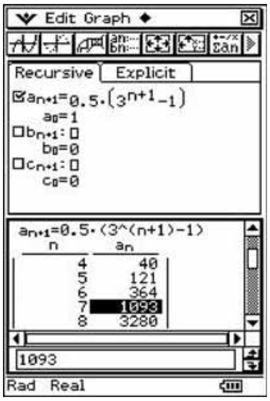
3 Type **0.5 × (3^(*n* + 1) –1)** into a_{n+1} and then press EXE.

Note: The reason for typing ($n + 1$) instead of n is simply to do with how the ClassPad generates the table of values of a sequence.

4 Type **1** into a_0 then press EXE.

5 Now tap ▦ to generate the sequence in a table.

6 Scroll through the sequence and look for the number **1093**.



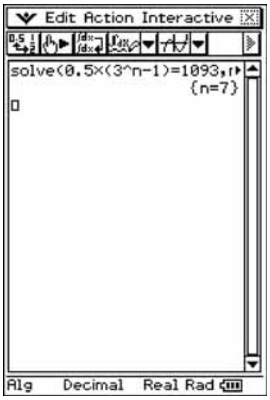
☑ $a_{n+1}=0.5\cdot(3^{n+1}-1)$
$a_0=1$
☐ b_{n+1}: □
$b_0=0$
☐ c_{n+1}: □
$c_0=0$

$a_{n+1}=0.5\cdot(3\text{^}(n+1)-1)$

n	a_n
4	40
5	121
6	364
7	1093
8	3280

1093

Alternatively, the *solve* command may be used. Type **solve (0.5 × (3^*n* – 1) = 1093, *n*)**



```
solve(0.5×(3^n-1)=1093,n
                    {n=7}
```

Trial and error or using the table facility of a graphics calculator will also give the required result.

Example 22

In the 15-day Tour de Moravia the cyclists ride 120 km on the first day and thereafter on each successive day they ride 90% of the distance of the previous day.

a How far do the cyclists ride in total, to the nearest kilometre?

b After how many days will they have ridden half the total 15-day distance?

Solution

a $a = 1230, r = 0.9$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{120(1 - (0.9)^{15})}{1 - 0.9}$$

$$= 952.93$$

$$\approx 953 \text{ km}$$

b $a = 120, r = 0.9, S_n = 476.5$ km

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{120(1 - (0.9)^n)}{1 - 0.9} = 476.5$$

$$\therefore \quad 1 - (0.9)^n = \frac{476.5 \times 0.1}{120} = 0.3971$$

$$\therefore \quad (0.9)^n = 1.0.3971$$

$$\therefore \quad (0.9)^n = 0.6029$$

Taking ln of both sides, $\ln (0.9)^n = \ln(0.6029)$

$$n \ln(0.9) = \ln(0.6029)$$

$$n = \frac{\ln(0.6029)}{\ln(0.9)}$$

$$n = 4.8023$$

$\therefore$ On the fifth day they pass the half-way mark.

Note: The fifth day is $4 \leq n < 5$.

Exercise 8E

*Use a calculator to validate questions **1–9**.*

Example 20

1 Find the sum specified for each of the following geometric series:

a $5 + 10 + 20 + \ldots,$ find S_{10}

b $1 - 3 + 9 - \ldots,$ find S_6

c $-\frac{4}{3} + \frac{2}{3} - \frac{1}{3} + \ldots,$ find S_9

2 Find:

a $2 - 6 + 18 - \ldots + 1458$

b $-4 + 8 - 16 + \ldots - 1024$

c $6250 + 1250 + 250 + \ldots + 2$

3 Gerry owns a milking cow. On the first day he milks the cow, it produces 600 mL of milk. On each successive day, the amount of milk increases by 10%.

a How much milk does the cow produce on the seventh day?

b How much milk does it produce in the first week?

Example 21 **4** **a** How many terms of the geometric sequence, where $t_1 = 1$, $t_2 = 2$, $t_3 = 4, \ldots$, must be taken for $S_n = 255$?

b Let $S_n = 1 + 2 + 4 + \ldots + 2^{n-1}$. Find $\{n : S_n > 1\,000\,000\}$.

Example 22 **5** An insurance salesman makes $15 000 commission on sales in his first year. Each year, he increases his sales by 5%.

a How much commission would he make in his fifth year?

b How much commission would he make in total over 5 years?

6 On Monday, William spends 20 minutes practising the piano. On Tuesday, he spends 25 minutes practising and on each successive day increases the time he spends practising, in the same ratio.

a For how many hours does he practise on Friday?

b How many hours in total does he practise from Monday to Friday?

c On which day of the following week will his total time spent practising pass 15 hours?

7 A ball dropped from a height of 15 metres rebounds from the ground to a height of 10 metres. With each successive rebound, it rises two-thirds of the height of the previous rebound. What total distance will it have travelled when it strikes the ground for the tenth time?

8 Andrew invests $1000 at 20% p.a. simple interest for 10 years. Bianca invests her $1000 at 12.5% p.a. compound interest for 10 years. At the end of 10 years, whose investment is worth more?

9 For the geometric sequence, with nth term t_n,

a $t_3 = 20$, $t_6 = 160$, find S_5 **b** $t_3 = \sqrt{2}$, $t_8 = 8$, find S_8

10 Find $1 - x^2 + x^4 - x^6 + \ldots + x^{2m}$ (m is even).

8.6 Extension: Infinite geometric series

If the common ratio of a geometric sequence has a magnitude less than 1, i.e. $-1 < r < 1$, then each successive term of the sequence is closer to zero.

Consider a geometric sequence with a common ratio r of magnitude less than 1.

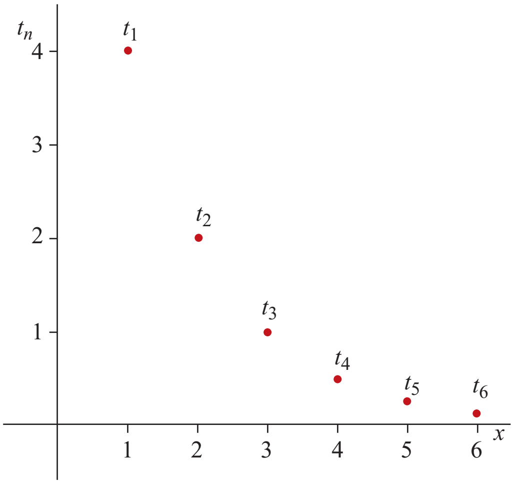
$$\text{i.e.} \qquad |r| < 1, -1 < r < 1$$

An example is $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots\ldots, r = \frac{1}{2}$, $a = 4$

$$t_n = 2^{(3-n)}$$

The value of successive terms is becoming smaller, eventually approaching zero.

$$\begin{aligned} \text{e.g.} \qquad t_{15} &= 2^{(3-15)} \\ &= 2^{-12} \\ &\approx 0.0002 \end{aligned}$$



Plotting t_1 to t_6 illustrates $t_n \to 0$ from above as $n \to \infty$.

The geometric series, formed by summing the terms, will eventually be adding successive terms of value very close to zero as n becomes very large. The series will approach a limiting value and is said to be **convergent**. Consider the series formed from the previous sequence.

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots\ldots \text{ to a large number of terms.}$$

$$\begin{aligned} a = 4, r = \frac{1}{2} \qquad S_n &= \frac{a\,(1 - r^n)}{(1 - r)} \\ &= \frac{4\,(1 - 0.5^n)}{(1 - 0.5)} \\ &= 8\,(1 - 0.5^n) \\ \text{As } n \to \infty \qquad S_n &\to 8(1 - 0) \\ \therefore \quad S_n &\to 8 \end{aligned}$$

The series is **convergent** approaching the limiting value 8.
i.e. $S_\infty = 8$

Using technology

Using the TI-Nspire:

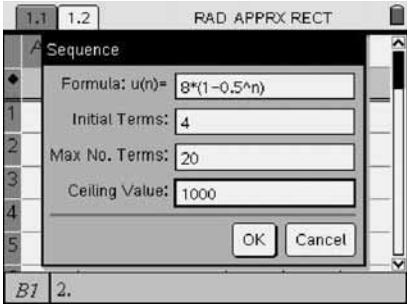
1. Press ⌂ then enter on the Lists & Spreadsheets application.
2. Enter the values from 1 to 20 into column A.
3. Give column A the name **n**.
4. Give column B the name **term** and then place the cursor in cell B1.

Using the ClassPad:

1. Press EXE on the Sequence application Sequence.
2. Tap the Recursive tab.
3. Type **8 × (1 – 0.5^*n*)** into a_{n+1} and then press EXE.
4. Type **4** into a_0 then press EXE.

5 Press menu and select *Generate Sequence* from the Data submenu.

6 Enter the following information:



Sequence

Formula: u(n)= 8*(1−0.5^n)

Initial Terms: 4

Max No. Terms: 20

Ceiling Value: 1000

7 Press enter on OK.

The first 20 terms of the sequence are displayed in column B.

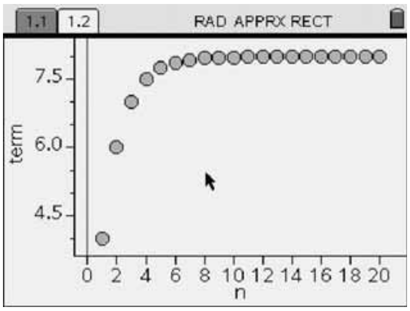
To sketch a graph of the sequence, do the following:

1 Highlight column A and B.

2 Press menu and select *Quick Graph* from the Data menu.

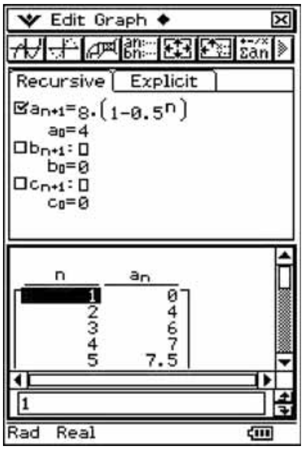
3 For a full-screen view of the graph press: ctrl, tab, ctrl, K, ctrl, clear, ctrl, ⌂

Scroll to *Page Layout*→*Select Layout*→ *1: Layout 1*.



By examining the sequence in column B, it is clear that as n becomes larger the sequence approaches 8.

5 Now tap ▦ to generate the sequence in a table.



☑$a_{n+1}=8\cdot(1-0.5^n)$
$a_0=4$
☐b_{n+1}: □
$b_0=0$
☐c_{n+1}: □
$c_0=0$

n	a_n
1	0
2	4
3	6
4	7
5	7.5

1

6 To sketch a graph of the first 20 terms in the sequence tap ⊹.

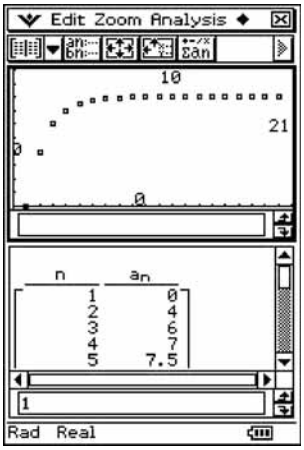
7 Change the window settings if necessary by tapping ⊞. Set

*x*min: 0

max: 21

*y*min: 0

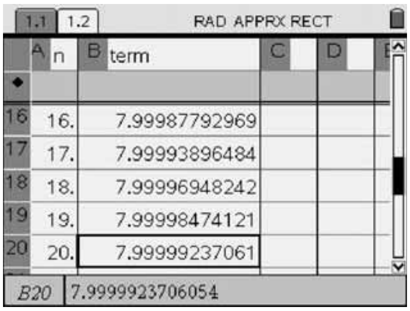
max: 10



n	a_n
1	0
2	4
3	6
4	7
5	7.5

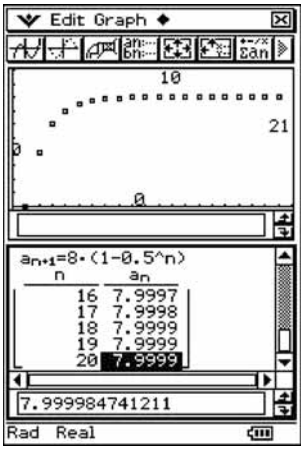
1

By examining the sequence in column B, it is clear that as n becomes larger the sequence approaches 8.



	A n	B term	C	D
◆				
16	16.	7.99987792969		
17	17.	7.99993896484		
18	18.	7.99996948242		
19	19.	7.99998474121		
20	20.	7.99999237061		

B20 7.9999923706054



$a_{n+1}=8\cdot(1-0.5\wedge n)$

n	a_n
16	7.9997
17	7.9998
18	7.9999
19	7.9999
20	7.9999

7.999984741211

Rad Real

When the terms of the sequence are added, since when n is very large each successive term being added is getting closer to zero, the corresponding series

$$a + ar + ar^2 + \ldots + ar^{n-1} \quad \text{will approach a limiting value}$$

i.e. as $n \to \infty$, $S_n \to$ a limiting value.

Such a series is called **convergent**.

In Example 20 from the previous section, it was found that for the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots \text{ the sum of the first 9 terms, } S_9, \text{ was } 0.499975$$

For the same sequence, $S_{20} = 0.4999999999 \approx 0.5$

So even for a relatively small value of n (i.e. 20), the sum approaches the limiting value of 0.5 very quickly.

Given

$$S_n = a + ar + ar^2 \ldots + ar^{n-1}$$

$$\therefore \quad S_n = \frac{a\,(1 - r^n)}{1 - r}$$

when $|r| < 1$ and $n \to \infty$, then $r^n \to 0$

$$\therefore \quad S_n \to \frac{a(1-0)}{1-r}$$

$$\therefore \quad S_n \to \frac{a}{1-r}$$

It follows then that the limit as $n \to \infty$, of S_n is $\dfrac{a}{1-r}$.

i.e. $\quad S_\infty = \dfrac{a}{1-r}$

This is also referred to as **the sum to infinity** of the series.

Alternatively, given that $S_n = \dfrac{a(1-r^n)}{1-r}$

$$\Rightarrow S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

as $n \to \infty$, $r^n \to 0$ and hence $\dfrac{ar^n}{1-r} \to 0$.

It follows then that the limit as $n \to \infty$ of S_n is $\dfrac{a}{1-r}$.

So, $$S_\infty = \frac{a}{1-r}$$

This is also referred to as **the sum to infinity** of the series.

Example 23

Find the sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Solution

$$r = \frac{1}{2}, a = 1$$

$$\therefore \quad S_\infty = \frac{1}{1-\frac{1}{2}} = 2$$

Example 24

A square has a side length of 40 cm. A copy of the square is made so that the area of the copy is 80% of the original. The process is repeated, whereby each time the area of the new square is 80% of the previous one. If this process continues indefinitely, find the total area of all the squares.

Solution

Area of first square is $40^2 = 1600\text{ cm}^2$

$a = 1600, r = 0.8$

$$S_\infty = \frac{a}{1-r}$$

$$\therefore \quad S_\infty = \frac{1600}{1-0.8} = 8000\text{ cm}^2$$

Example 25

Express the recurring decimal $0.\dot{3}\dot{2}$ as a ratio of two integers.

Solution

$0.\dot{3}\dot{2} = 0.32 + 0.0032 + 0.000\,032 + \ldots$

$\therefore \quad a = 0.32, r = 0.01$

and $$S_\infty = \frac{0.32}{0.99} = \frac{32}{99}$$

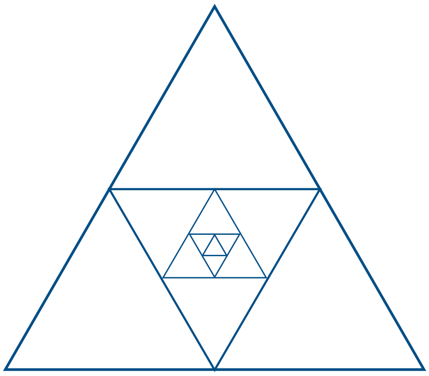
i.e. $$0.\dot{3}\dot{2} = \frac{32}{99}$$

Exercise 8F

Example 23 **1** Evaluate:

a $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots$ **b** $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

Example 24 **2** An equilateral triangle has perimeter p cm. The midpoints of the sides are joined to form another triangle, and this process is repeated. Find the perimeter and area of the nth triangle, and find the limits as $n \to \infty$ of the sums of perimeters and areas of the first n triangles.



3 A rocket is launched into the air so that it reaches a height of 200 metres in the first second. With each subsequent second it gains 6% less height. Calculate how high the rocket will climb.

4 A patient has an infection that, if it exceeds a certain level, will kill him. He is given a drug that will inhibit the spread of the infection. The drug acts in such a way that the level of infection only increases by 65% of the previous day's level. On the first day, the level of infection is measured at 450.
The critical level of infection is 1280. Will the infection kill him?

5 A man can walk 3 km in the first hour of a journey, but for each succeeding hour walks half the distance covered in the preceding hour. Can he complete a journey of 6 km? Where does this problem cease to be realistic?

6 A frog standing 10 m from the edge of a pond sets out to jump towards it. Its first jump is 2 m, its second jump is $1\frac{1}{2}$ m, its third jump is $1\frac{1}{8}$ m and so on. Show that the frog will never reach the edge of the pond.

7 A computer-generated virus acts in such a way that it initially blocks out one-third of the area of the screen of an infected computer. On each successive day, it blocks out $\frac{1}{3}$ of the area it blocked the previous day. If the virus continues to act unchecked indefinitely, what percentage of the user's screen will eventually be blocked out?

8 A stone is thrown so that it skips across the surface of a lake. If each skip is 30% less than the previous skip, how long should the first skip be if the total distance travelled by the stone is 40 metres?

9 A ball dropped from a height of 15 metres rebounds from the ground to a height of 10 metres. With each successive rebound, it rises two-thirds of the height of the previous rebound. If it continues to bounce indefinitely, what is the total distance it will travel?

Example 25

10 Express each of the following recurring decimals as the ratio of two integers:

a $0.\dot{4}$ **b** $0.0\dot{3}$ **c** $10.\dot{3}$ **d** $0.0\dot{3}\dot{5}$ **e** $0.\dot{9}$ **f** $4.\dot{1}$

11 The sum of the first four terms of a geometric series is 30 and the sum to infinity is 32. Determine the first two terms.

12 Evaluate the third term of a geometric sequence that has a common ratio of $-\frac{1}{4}$ and a sum to infinity of 8.

13 Calculate the common ratio of a geometric sequence that has a first term of 5 and a sum to infinity of 15.

8.7 Compound interest

As seen in sections 8.3 and 8.4, compound interest is generated when interest on the principal is credited to the account at the end of each period. This new total amount becomes the principal for the next period upon which interest is duly credited. Thus, the interest amounts increase over successive periods and are **compounding** as interest is paid on previous interest amounts.

In general,

If $P =$ original investment

$A =$ amount the investment grows to after n years

$r =$ compound interest rate r% per annum

$n =$ number of years invested

Then $A = P\left(1 + \frac{r}{100}\right)^n$.

Consider \$5000 invested at 7% p.a. compound interest for 4 years.

$$P = 5000,\ x = 7$$
$$A = P\left(1 + \frac{r}{100}\right)^n$$

After 1 year, $n = 1$

$$\begin{aligned} A_1 &= 5000\left(1 + \frac{7}{100}\right)^1 \\ &= 5000 \times 1.07^1 \end{aligned}$$

After 2 years, $n = 2$

$$\begin{aligned} A_2 &= 5000\left(1 + \frac{7}{100}\right)^2 \\ &= 5000 \times 1.07^2 \end{aligned}$$

After 3 years, $n = 3$

$$A_3 = 5000 \times 1.07^3$$

After 4 years, $n = 4$

$$A_4 = 5000 \times 1.07^4$$

Thus, $A_n = 5000 \times 1.07^n$, where $n = 0, 1, 2, 3, 4$
is an **exponential function** with a growth factor of 1.07.

The sequence of amounts

$$5000, 5000 \times 1.07^1, 5000 \times 1.07^2, 5000 \times 1.07^3, 5000 \times 1.07^4$$

is a **geometric sequence** with $a = 5000$ and $r = 1.07$.

Example 26

Anthony invested an amount, \$$P$, at 5% p.a. compound interest, which grew to \$2000 over 10 years. Evaluate P.

Solution

$$\begin{aligned} A &= 2000,\ r = 5,\ n = 10 \\ A &= P\left(1 + \frac{r}{100}\right)^n \\ 2000 &= P\left(1 + \frac{5}{100}\right)^{10} \\ 2000 &= P \times 1.05^{10} \\ P &= \frac{2000}{1.05^{10}} \\ \therefore P &= 1227.83 \end{aligned}$$

Anthony initially invested approximately \$1228 (to the nearest dollar).

Example 27

Cassandra invests her savings of \$3000 for her 21st birthday in 5 years' time. She hopes her investment will grow to \$4000. Calculate the compound interest rate required per annum.

Solution

$$\begin{aligned} A &= 4000,\ P = 3000,\ n = 5 \\ A &= P\left(1 + \frac{r}{100}\right)^n \\ 4000 &= 3000\,(1 + 0.01r)^5 \\ \sqrt[5]{\frac{4000}{3000}} &= 1 + 0.01r \\ 1.0592 &= 1 + 0.01r \\ r &= 5.92 \end{aligned}$$

Cassandra's required interest rate is 5.92% (to 2 decimal places).

Example 28

Nick's bank offers him 7.75% p.a. compound interest if he invests today. After how many years will his investment have doubled?

Solution

Let \$$P$ be Nick's original investment, therefore \$$A = 2$\$$P$ and $r = 7.75$.

$$\begin{aligned} A &= P\left(1+\frac{r}{100}\right)^n \\ 2P &= P\left(1+\frac{7.75}{100}\right)^n \\ 2 &= (1.0775)^n \end{aligned}$$

Taking ln of both sides:

$$\begin{aligned} \ln 2 &= n \times \ln 1.0775 \\ n &= 9.29 \text{ (to 2 decimal places)} \end{aligned}$$

Nick's investment will more than double after 10 years.

The 'rule' of 72

This is a simple calculation to approximate the number of years (n) for an investment to double at r% p.a. compound interest.

'Rule' of 72	$n \times r \approx 72$

Consider Nick's investment in Example 28.

$$\begin{aligned} n \times 7.75 &\approx 72 \\ \therefore \quad n &\approx 9.29 \text{ (to 2 decimal places)} \end{aligned}$$

He would *estimate* that his investment will double during the tenth year.

Note: The 'rule' of 72 is increasingly inaccurate as r increases.
e.g. Given $r\% = 20\%$ $\quad n \times 20 \approx 72,\ n \approx 3.6$.
Check that this represents an error of $\approx 5\%$.

Generally, in today's financial markets, compound interest is accrued and paid more frequently than once per year. In these cases:

- r% p.a. is the **nominal** rate of interest, as it is generally the nominated (quoted) rate.
- If the nominal interest rate is $r\% = 6\%$ p.a., compounding monthly for 15 years, then $\frac{6}{12}\% = 0.5\%$ interest (on the current account balance) is payable at the *end* of each month for 15 years.
- The period during which the money is resting while accruing interest is called the **rest period**. (For the example given in the previous point, the rest period would be monthly.)

- The total number of rests (interest accruals) = Number of rests/year × Number of years.
- For the previous example, the number of rests = $12 \times 15 = 180$.

$$A = P\left(1 + \frac{r}{(100\,m)}\right)^{mn}$$

where

r% is the nominal rate of interest

m is the number of rests per year

n is the number of years

P is the initial principal

A is the final amount.

Example 29

On her fifth birthday, Lucy's grandmother invests $1000 at a nominal 8% p.a., compounding monthly. How much is the investment worth on Lucy's 21st birthday?

Solution

$$P = 1000,\ r = \frac{8}{12},\ n = 16 \times 12 = 192$$

$$A = 1000\left(1 + \frac{\frac{8}{12}}{100}\right)^{192}$$

$$= 1000\left(1 + \frac{2}{300}\right)^{192}$$

$$= 3581.39$$

The investment grows to $3581, to the nearest dollar.

Example 30

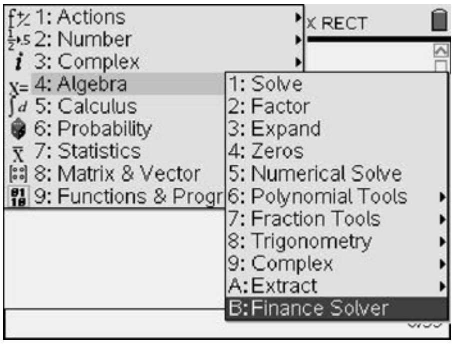
Use a graphics calculator application program to calculate the nominal annual interest rate, compounding weekly, that is necessary for a $6000 investment to amount to $20 000 for a 15-year term.

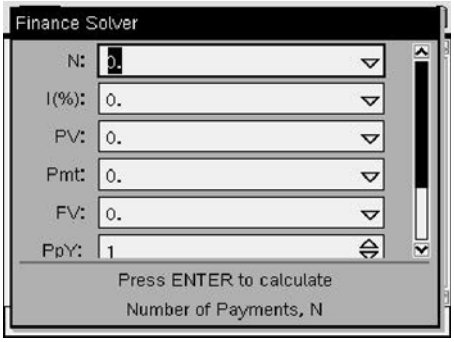
Using technology

Operations involving compound interest, loans and annuities can be executed using the **TVM Solver**. TVM is an acronym for **T**ime **V**alue **M**oney.

Using the TI-Nspire:

To access the TMV Solver, press menu and press enter on *Finance Solver* from the Algebra submenu.

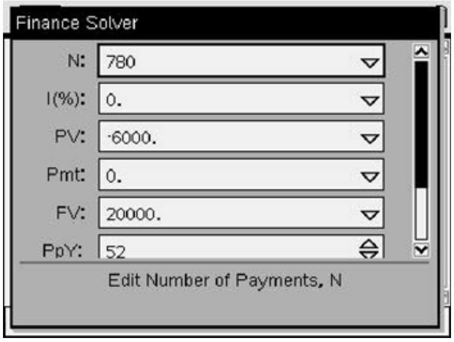




N: 0.
I(%): 0.
PV: 0.
Pmt: 0.
FV: 0.
PpY: 1

Press ENTER to calculate
Number of Payments, N

Enter variable values, as shown below.

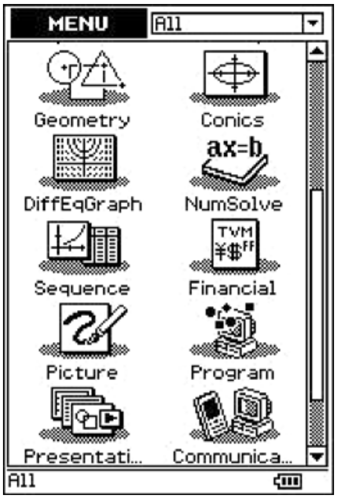


N: 780
I(%): 0.
PV: ⁻6000.
Pmt: 0.
FV: 20000.
PpY: 52

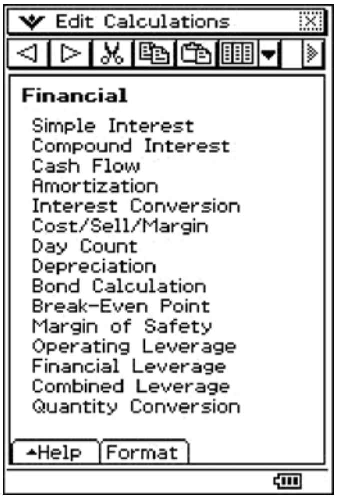
Edit Number of Payments, N

Using the ClassPad:

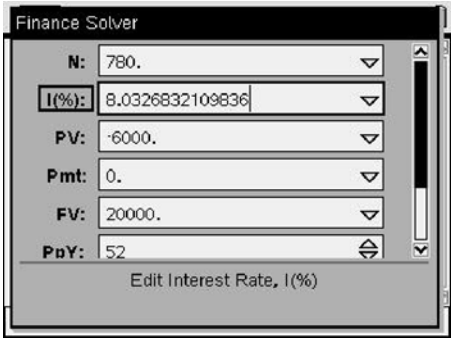
To access the financial programs, tap on the Financial application.



You now have to select the appropriate program.



Set the cursor position to I(%) and press enter to evaluate.



N: 780.

I(%): 8.0326832109836

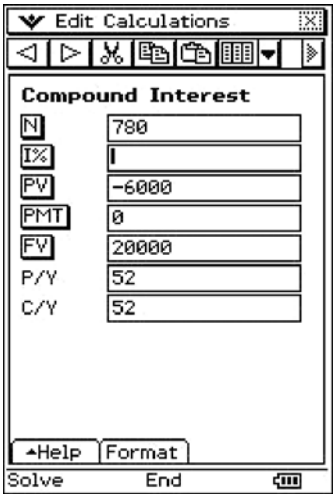
PV: ⁻6000.

Pmt: 0.

FV: 20000.

PpY: 52

For this example, tap Compound Interest and enter the variables, as shown.



Compound Interest

N 780

I%

PV -6000

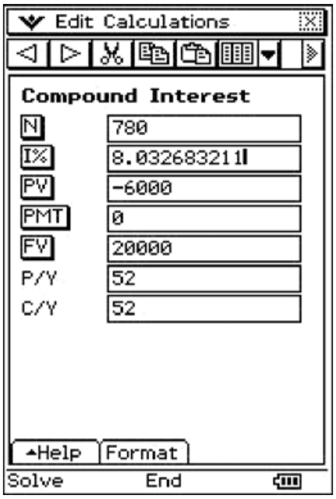
PMT 0

FV 20000

P/Y 52

C/Y 52

Set the cursor to I% then tap the word 'Solve' in the bottom left-hand corner.



Compound Interest

N 780

I% 8.032683211

PV -6000

PMT 0

FV 20000

P/Y 52

C/Y 52

Annual interest rate is 8.03%.

The variables shown may be interpreted as:

N ≡ the total number of repayments

I% ≡ r%, the nominal annual interest rate

PV ≡ P, loan/credit (+), investment/debit (−)

PMT ≡ the value of regular payments to (+) or from (−) your account

FV ≡ A

P/Y ≡ the number of payments per year ≡ C/Y

C/Y ≡ the number of compounding interest periods per year

PMT : END BEGIN ≡ Use the default END, as all interest and PMT will be considered only when at the end of the rest period.

Effective rate of interest

Consider \$1 invested at a nominal rate of r% p.a. compound interest. Let this interest compound m times in 1 year.

If \$$A$ is the amount, in 1 year the invested \$1 grows to:

$$\begin{aligned} A &= 1 \times \left(1 + \frac{r}{(100\,m)}\right)^m \\ &= \left(1 + \frac{r}{(100\,m)}\right)^m \end{aligned} \qquad \ldots\ldots\ldots(1)$$

What rate of s% p.a., compounding only once after a period of 1 year, would return \$$A$?

s% is the **effective rate** of the investment.

$$\begin{aligned} A &= 1 \times \left(1 + \frac{s}{100}\right)^1 \\ &= \left(1 + \frac{s}{100}\right) \end{aligned} \qquad \ldots\ldots\ldots(2)$$

Equating (1) and (2):

$$\begin{aligned} \left(1 + \frac{s}{100}\right) &= \left(1 + \frac{r}{(100\,m)}\right)^m \\ \therefore \frac{s}{100} &= \left(1 + \frac{r}{(100\,m)}\right)^m - 1 \end{aligned}$$

Therefore,

$s\% = 100\left[\left(1 + \dfrac{r}{(100\,m)}\right)^m - 1\right]$% is the equivalent **effective rate** of the investment.

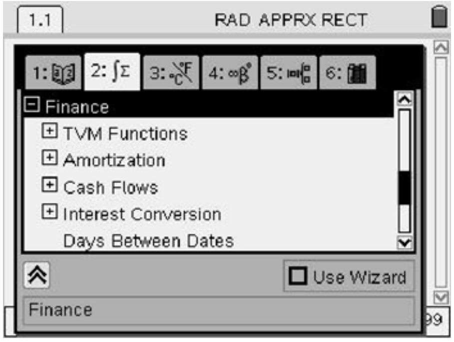
Conversion to the effective rate enables a comparison of rates with different r% and m values.

Using technology

Using the TI-Nspire:

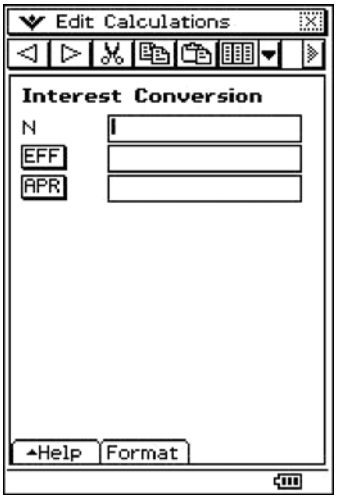
1 In the calculator application press ∞β° to enter into the catalog.

2 Press 2, scroll down to Finance and press enter.



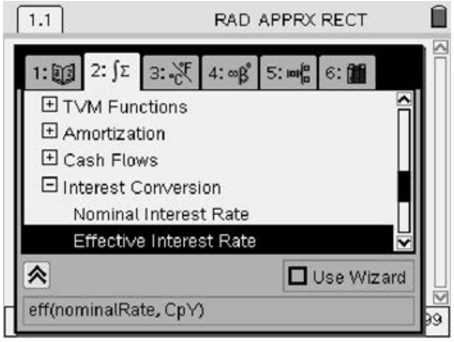
Using the ClassPad:

1 Once in the Financial application tap Interest Conversion.

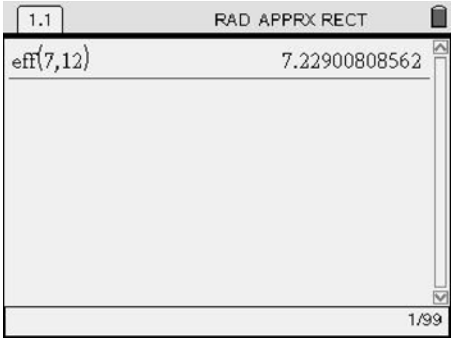


3 Press enter on Interest Conversion to expand the menu.

4 Press enter on Effective Interest Rate.

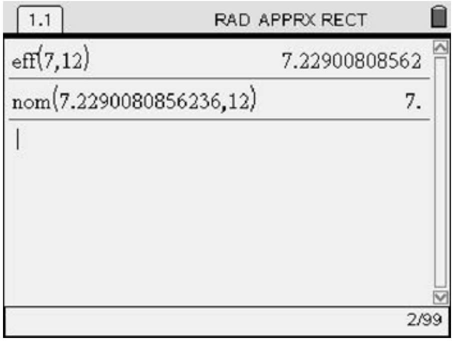


5 Now type **7, 12)** then press enter.



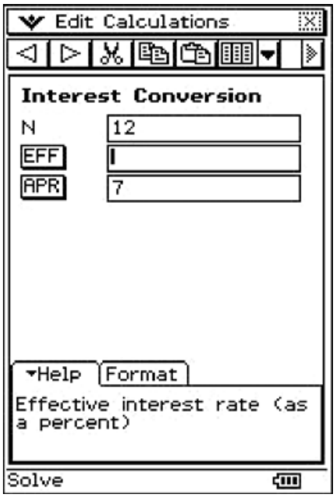
```
eff(7,12)                      7.22900808562
```

6 For the inverse of the above, select Nominal Interest Rate, as seen above, then type **ans, 12)** and press enter.



```
eff(7,12)                      7.22900808562
nom(7.2290080856236,12)                   7.
```

2 Enter variable values as shown.



Interest Conversion

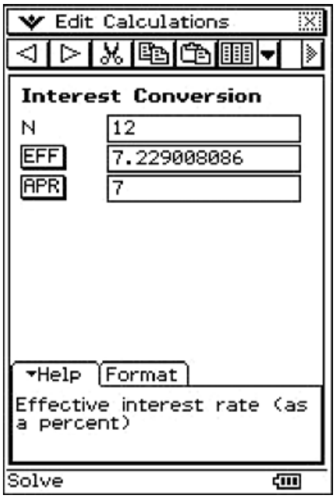
N 12

EFF

APR 7

Effective interest rate (as a percent)

3 Set the cursor to EFF then tap the word ‘Solve’ in the bottom left-hand corner.



Interest Conversion

N 12

EFF 7.229008086

APR 7

Effective interest rate (as a percent)

4 For the inverse of the above, put **N = 12** and **EFF = 7.229008086**. Set the cursor to APR then tap Solve in the bottom left-hand corner. This should yield a result of APR = 7.

Example 31

Richard is offered two different rates, A and B. Which is the better rate?

	Nominal rate %	Rests/year
A	7.8	weekly
B	8.0	monthly

Solution

Calculate the effective rate for each of A and B.

A

$$r = 7.8, m = 52$$

$$s\% = 100\left[\left(1+\frac{r}{(100m)}\right)^{m} - 1\right]\%$$

$$s_{\mathrm{A}}\% = 100\left[\left(1+\frac{7.8}{(100 \times 52)}\right)^{52} - 1\right]\%$$

$$\approx 8.11\%$$

B

$$r = 8.0, m = 12$$

$$s\% = 100\left[\left(1+\frac{r}{(100m)}\right)^{m} - 1\right]\%$$

$$s_{\mathrm{B}}\% = 100\left[\left(1+\frac{8}{(100 \times 12)}\right)^{12} - 1\right]\%$$

$$\approx 8.30\%$$

B is the better investment as its effective rate is greater.

Exercise 8G

1 Evaluate the amount (A) an investment grows to when:

- **a** \$1000 is invested for 25 years at 10% p.a. compound interest, compounded annually
- **b** \$5000 is invested for 10 years at 8.5% p.a. compound interest, compounded annually
- **c** \$250 is invested for 50 years at 6.5% p.a. compound interest, compounded annually

Example 26

2 Calculate the principal amount (P) invested given:

- **a** \$$P$ invested for 5 years at 10% p.a. compound interest amounts to \$1610.51
- **b** \$$P$ invested for 30 years at 9% p.a. compound interest amounts to \$6633.84
- **c** \$$P$ invested for 12 years at 7.9% p.a. compound interest amounts to \$1610.51

Example 27

3 Evaluate the nominal rate of interest (r%) given:

- **a** \$4000 invested for 10 years, compounding annually, returns a balance of \$7868.61
- **b** \$2500 invested for 50 years, compounding annually, returns a balance of \$82 757.06
- **c** \$100 invested for 7 years, compounding annually, returns a balance of \$476.84

Example 28

4 Calculate the number of years required for:

- **a** \$1000 invested at 10% p.a. compound interest, compounded annually, to mature to \$4177.25

b \$750 invested at 8.25% p.a. compound interest, compounded annually, to mature to \$3661.17

c \$25 000 invested at 7.99% p.a. compound interest, compounded annually, to mature to \$50 000

5 A compound interest investment made over a period of 9 years has doubled in value.

a Use the 'rule' of 72 to estimate the nominal rate.

b Evaluate the nominal rate to the fourth decimal place.

c Calculate the percentage error of the 'rule' of 72 estimated in part **a**.

6 An amount invested at 25% p.a. compound interest, compounding annually, has doubled in value.

a Use the 'rule' of 72 to estimate the number of years invested.

b Evaluate the number of years invested, to 2 decimal places.

c Calculate the percentage error of the 'rule' of 72 estimated in part **a**.

Examples 29, 30

7 **i** Algebraically evaluate each variable within the table below, except for parts **g** and **h**.

ii Validate each answer, using the graphics calculator TVM Solver.

	\$$A$	\$$P$	r% (Nominal rate)	m (Periods/year)	n (Years)
a	A	4000	10	12	5
b	A	7500	6.5	4	10
c	20 000	P	10.5	2	15
d	500	P	7.75	52	8
e	1000	750	r	1	5
f	8000	6000	r	12	4
g	11 402	5000	8.25	m	10
h	13 997	2500	6.95	m	25
i	543 0209	100 000	7.99	365	n
j	14 067	1500	12.5	12	n

Example 31

8 Evaluate the effective annual rate of interest for compound interest at a nominal rate of 5% p.a., compounding monthly.

Example 31

9 Evaluate the effective annual rate of interest for compound interest at a nominal rate of 6.95% p.a., compounding weekly.

MAPS

10 Which is the better effective rate?

a A nominal rate of 8.25% p.a., compounding weekly.

b A nominal rate of 8.20% p.a., compounding daily.

11 An amount of \$100 is invested at a nominal rate of 8.5% p.a., compounding weekly. At the same time \$105 is invested at a nominal rate of 8% p.a., compounding weekly. During which year from the commencement of the investments will they have accumulated to the same amounts? Validate your answer using a graphics calculator plot.

Example 30 **12** An investor states that her investment is effectively 10%. Determine the nominal rate, given that the interest is compounded monthly.

MAPS **13** After how many years does an investment, earning interest at a nominal rate of 12% p.a., compounding daily, amount to twice the accrued value of the same initial amount invested at a nominal rate of 8% p.a., compounding quarterly? Solve algebraically and validate using a graphics calculator plot.

8.8 Compound interest and ‘*e*’

Continuously compounding interest

Consider a nominal interest rate of 5% p.a. interest, compounding annually. The growth factor will be 100% + 5% = 105% = 1.05.

i.e. $$\left(1+\frac{5}{100}\right)^1 = 1.05$$

If the interest is compounded twice per year, then the growth factor will be

$$\left(1+\frac{5}{(100\times 2)}\right)^2 = 1.050\,625$$

If interest compounds m times in 1 year, then for

$m = 10$

$$\left(1+\frac{5}{(100\times 10)}\right)^{10} = 1.051\,140\,132$$

$m = 100$

$$\left(1+\frac{5}{(100\times 100)}\right)^{100} = 1.051\,257\,96$$

$m = 1000$

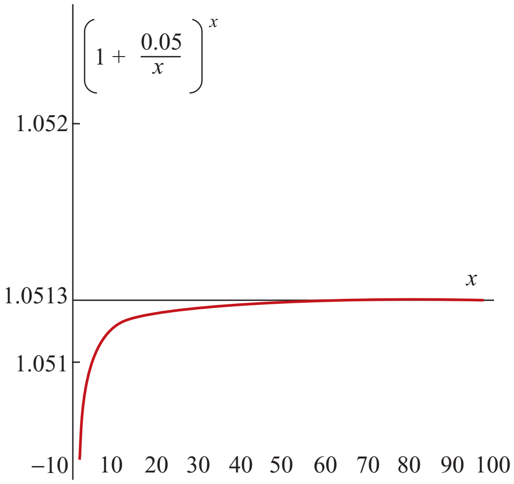
$$\left(1+\frac{5}{(100\times 1000)}\right)^{1000} = 1.051\,269\,782$$

⋮

⋮

$m = 10^{10}$

$$\left(1+\frac{5}{(100\times 10^{10})}\right)^{10^{10}} = 1.051\,271\,096$$



The growth factor is converging to the limit around 1.05127 (to 5 decimal places) as m becomes infinitely large and therefore compounding **continuously**.

Hence, a **continuously** compounding investment of $\$P$ at a nominal interest rate of 5% p.a. amounts to $\$A$ over n years.

$$A \approx P \times 1.051\,271\,096^n$$

In general, $$A = P \times \left(1+\frac{s}{100}\right)^n$$

However, $\ln(1.051\,271\,096\ldots.) = 0.05$

$$\begin{aligned} \therefore\ e^{0.05} &= 1.051\,271\,096\ldots \\ \therefore\quad A &= P \times \left(e^{0.05}\right)^n \\ A &= P \times e^{0.05n} \end{aligned}$$

The effective interest rate

$$\begin{aligned} s\% &= 100 \times (1.051\,271\,096 - 1)\,\% \\ &= 5.127\,109\,6\%\text{ p.a.} \end{aligned}$$

In general, if the nominal rate is r% p.a., then a continuously compounding investment is given by:

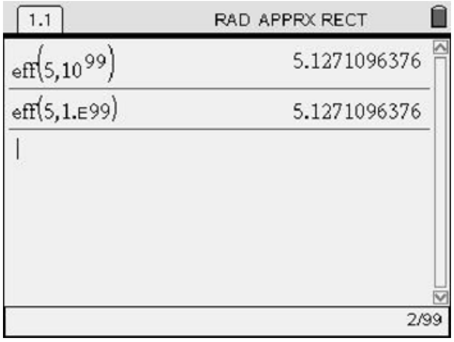
$$A = P\ \times e^{(0.01\,r\,n)}$$

The effective interest rate is $s\% = 100 \times \left(e^{(0.01r)} - 1\right)\%$

Using technology

Using the TI-Nspire:

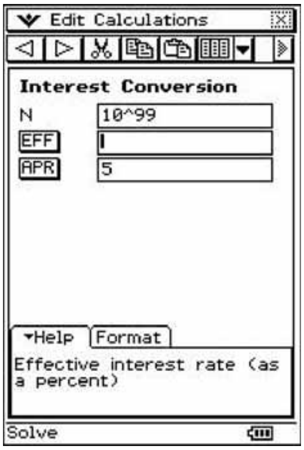
1 Using the green alphabetic keys type **eff(5, 10^99)** or **eff(5, 1E99)** and then press (enter).



```
1.1                    RAD APPRX RECT
eff(5,10^99)             5.1271096376
eff(5,1.E99)             5.1271096376
                                 2/99
```

Using the ClassPad:

1 Using the Interest Conversion program enter the variable values as shown.



Interest Conversion

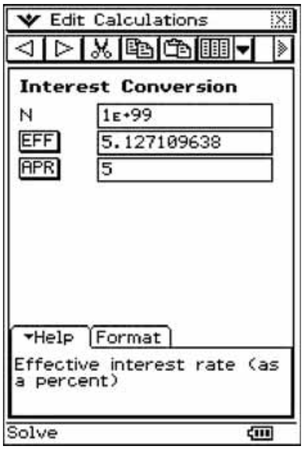
N 10^99

EFF

APR 5

Effective interest rate (as a percent)

2 Set the cursor to EFF then tap Solve in the bottom left-hand corner.



Interest Conversion

N 1E+99

EFF 5.127109638

APR 5

Effective interest rate (as a percent)

Example 32

An amount of $500 is invested at the nominal rate of 8% p.a., compounding continuously.

a Evaluate the effective interest rate.

b Calculate the investment return after 10 years.

Solution

a

$$\begin{aligned} r &= 8 \\ s\% &= 100 \times \left(e^{(0.01r)} - 1\right)\% \\ s\% &= 100 \times \left(e^{(0.01\times 8)} - 1\right)\% \\ &= 100 \times (1.083\,287\,068 - 1)\% \\ &= 8.33\% \text{ (to 2 decimal places)} \end{aligned}$$

A nominal rate of 8% p.a. equates to an effective continuous rate of 8.33%.

b $r = 8,\ n = 10,\ P = 500$

$$\begin{aligned} A &= P \times e^{(0.01r\,n)} \\ &= 500 \times e^{(0.01\times 8\times 10)} \\ &= 1112.77 \end{aligned}$$

The investment amount is $1113, to the nearest dollar.

Example 33

Rebecca inherited $30 800 from her grandfather. She knows the initial amount had been invested for 15 years, earning a compounding continuous rate of interest of 7.79% p.a.

a Evaluate the nominal interest rate.

b Calculate the initial value of the investment.

Solution

a

$$\begin{aligned} s &= 7.79 \\ s\% &= 100 \times \left(e^{(0.01r)} - 1\right)\% \\ 7.79 &= 100 \times \left(e^{(0.01\ \times\ r)} - 1\right) \\ 1.0779 &= e^{(0.01\ \times\ r)} \\ 0.01r &= \ln(1.0779) \\ \therefore\ r\% &= 7.5\% \end{aligned}$$

The nominal interest rate of the investment was 7.5% p.a.

b $s = 7.79,\ n = 15,\ A = 30\,800$

$$\begin{aligned} A &= P \times \left(1 + \frac{s}{100}\right)^n \\ 30\,800 &= P \times \left(1 + \frac{7.79}{100}\right)^{15} \\ 30\,800 &= P \times 3.080\,897 \\ P &= 9997.09 \end{aligned}$$

The initial amount invested by her grandfather was $9997, to the nearest dollar.

Conversion to base '*e*'

As noted previously, compound interest is an exponential function.

Given $A = P\left(1 + \frac{r}{(100\,m)}\right)^{m\,n}$, let $R = \left(1 + \frac{r}{(100\,m)}\right)^{m}$, where R is the annual growth factor.

$$\therefore A = PR^n \qquad \text{(i.e. } R \text{ is the base of the exponential function)}$$
$$A = P\left(e^{\ln R}\right)^n$$
$$\therefore A = Pe^{n\ln R} \qquad \text{(i.e. } e \text{ is the base of the exponential function)}$$

Thus, expressing the standard compound interest formula as an exponential function, base e.

Note:

- n, m and A are discrete variables.
- $A = P\,e^{n\ln R}$ represents a dilation of $A = Pe^n$ by factor $\frac{1}{\ln R}$, parallel to the n-axis.

Example 34

A compound interest investment, compounding monthly, is modelled by the exponential function $A = P\,e^{0.1194\,n}$, where n is the number of years, P is the initial investment and A is the final amount.

a Calculate the annual growth factor.
b Evaluate the effective interest rate.
c Determine the nominal rate of interest.

Solution

a $A = PR^n \quad \ldots\ldots(1)$

$A = P\,e^{0.1194\,n}$

$A = P\,\left(e^{0.1194}\right)^n \quad \ldots\ldots(2)$

Equating the bases of (1) and (2):

$$R = e^{0.1194}$$
$$= 1.1268$$

The annual growth factor is 1.1268.

b $s\% = 100 \times (1.1268 - 1)\,\%$

$= 12.68\%$

The effective interest rate is 12.68%.

c $R = 1.1268,\ m = 12$

$$R = \left(1 + \frac{r}{(100m)}\right)^m$$
$$\therefore 1.1268 = \left(1 + \frac{r}{(100 \times 12)}\right)^{12}$$
$$\sqrt[12]{1.1268} = 1 + \frac{r}{1200}$$
$$r \approx 12\%$$

The nominal interest rate is approximately 12%.

Exercise 8H

Example 32 **1** An amount of \$1000 is invested at the nominal rate of 9% p.a., compounding continuously.

- **a** Evaluate the effective interest rate.
- **b** Calculate the investment amount after 5 years. Validate your answer using the graphics calculator application FINANCE.

Example 32 **2** An amount of \$800 is invested at the nominal rate of 6.5% p.a., compounding continuously.

- **a** Evaluate the effective interest rate.
- **b** Calculate the investment amount after 12 years. Validate your answer using the graphics calculator application FINANCE.

3 A continuously compounding matured investment, which earned a nominal rate of 6% p.a. over 7 years, amounts to \$1576.

- **a** Evaluate the effective interest rate.
- **b** Calculate the initial amount invested.

Example 33 **4** Luke's 21st birthday present from his parents is a \$14 137 matured investment, which had commenced on the day of his birth. The investment earned a rate of 8.6% p.a., compounded continuously.

- **a** Evaluate the nominal interest rate.
- **b** Calculate the initial amount invested.

5 A compound interest investment, compounding weekly, is modelled by the exponential function $A = P\,e^{0.04\,n}$, where n is the number of years, P is the initial investment and A is the final amount.

Example 34
- **a** Calculate the annual growth factor.
- **b** Evaluate the effective interest rate.
- **c** Determine the nominal rate of interest.

MAPS **6** The exponential function $A = P\,e^{rn}$ models a daily compounding investment. If the investment increased in value by 50% in 4 years, evaluate the nominal interest rate.

MAPS **7** A sum is invested at a nominal rate of 9% p.a., compounding quarterly, for t years. On maturity the monies are re-invested immediately at a nominal rate of 6% p.a., compounding every 6 months, for $2t$ years. Given the initial amount has doubled over the total period, calculate the total period of the investment.

8.9 Depreciation and inflation

Depreciation means that the price of the item depreciates or reduces. Equipment, furniture, vehicles, computers, books and an inflating dollar are all examples of depreciating items.

Consider a car with an initial value of \$50 000 (\$$P$) depreciating at, say, 10% p.a. Therefore, with each passing year, its value will be 10% less than that of the previous year. The value (\$$A$) after n years can be illustrated by the table below.

n	\$$A$
0	50 000
1	45 000
2	40 500
n	$50\,000 \times 0.9^n$

Clearly, A is an exponential function of n.

$$A_n = 50\,000 \times \left(1 - \frac{10}{100}\right)^n$$

$$\therefore A_n = 50\,000 \times 0.9^n$$

The compound interest formula applies, but using a negative rate.

In general,

$$A = P\left(1 - \frac{r}{100}\right)^n$$

where

P = original investment

A = amount the investment depreciates to after n years

r = compound depreciating interest rate (r% p.a.)

n = number of years

Example 35

A computer was purchased in 2005 for \$3000. In 2009 it is valued at \$536. Calculate the annual rate of depreciation.

Solution

$P = \$3000,\ A = \$536,\ n = 4$

$$A = P\left(1 - \frac{r}{100}\right)^n$$

$$536 = 3000\left(1 - \frac{r}{100}\right)^4$$

$$\sqrt[4]{\frac{536}{3000}} = 1 - \frac{r}{100}$$

$$0.6501 \approx 1 - \frac{r}{100}$$

$$r \approx 34.99$$

The annual rate of depreciation is 34.99%.

Example 36

A factory's manufacturing machinery depreciates annually at 12%. After how many years will it be valued at 10% of its original purchase price?

Solution

$$A = 0.10P, \ r = 12$$

$$\begin{aligned} 0.10P &= P\left(1 - \frac{12}{100}\right)^n \\ 0.10 &= 0.88^n \\ n &= \frac{\ln 0.10}{\ln 0.88} \\ &= 18.01 \end{aligned}$$

After approximately 18 years, the machinery will be worth 10% of its original cost.

Inflation

Over the past few years, the annual rate of inflation in Australia has averaged around 3%. This means that goods will cost 3% more in 1 year's time, and in each subsequent year. The dollar's purchasing power erodes compared with that of the previous year.

The compound interest formula,

$$A = P\left(1 + \frac{r}{100}\right)^n$$

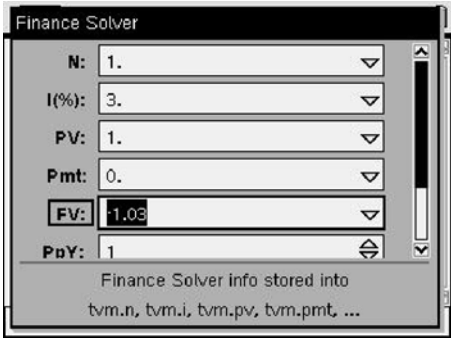
is usually expressed as

$$FV = PV\left(1 + \frac{r}{100}\right)^n$$

where PV = present value and FV = future value.

Using technology

Using the TI-Nspire:

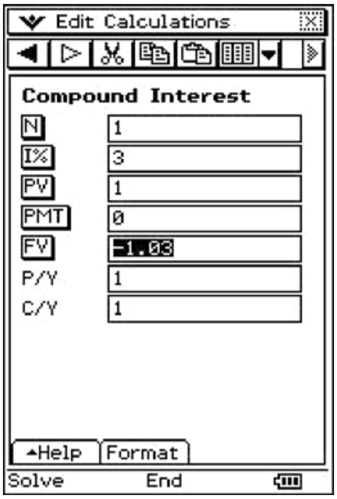


Finance Solver

N: 1.
I(%): 3.
PV: 1.
Pmt: 0.
FV: -1.03
PpY: 1

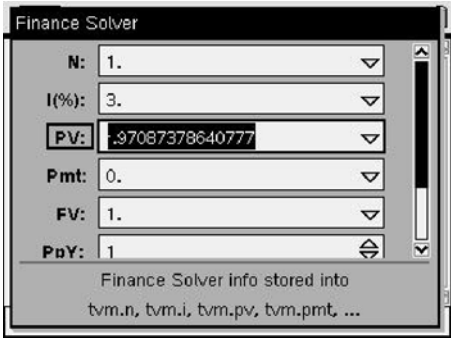
Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

Using the ClassPad:



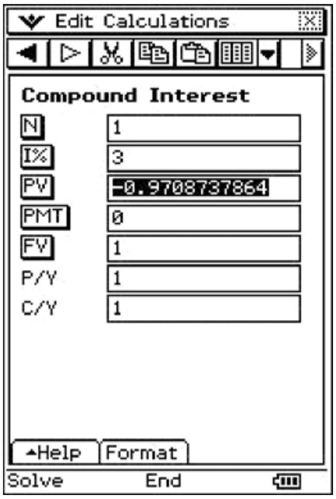
Compound Interest

N 1
I% 3
PV 1
PMT 0
FV -1.03
P/Y 1
C/Y 1



N: 1.
I(%): 3.
PV: -.97087378640777
Pmt: 0.
FV: 1.
PpY: 1

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...



Compound Interest

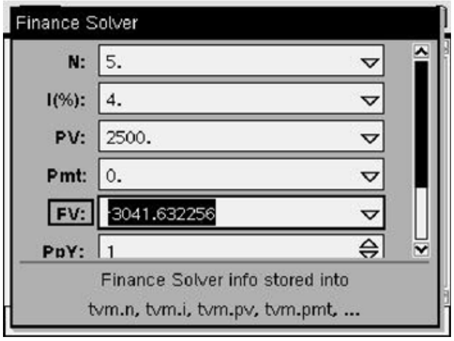
N 1
I% 3
PV -0.9708737864
PMT 0
FV 1
P/Y 1
C/Y 1

Example 37

If Australia’s average annual rate of inflation over the next 5 years is 4%, calculate what will be the equivalent of \$2500 today in 5 years’ time.

Solution

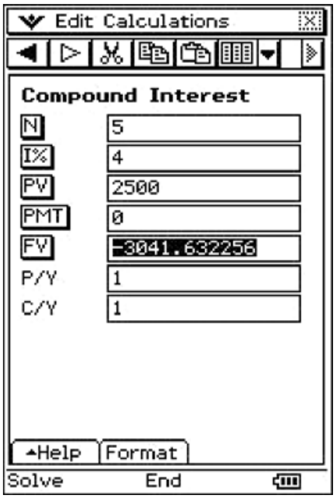
Using the TI-Nspire:



N: 5.
I(%): 4.
PV: 2500.
Pmt: 0.
FV: -3041.632256
PpY: 1

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

Using the ClassPad:



Compound Interest

N 5
I% 4
PV 2500
PMT 0
FV -3041.632256
P/Y 1
C/Y 1

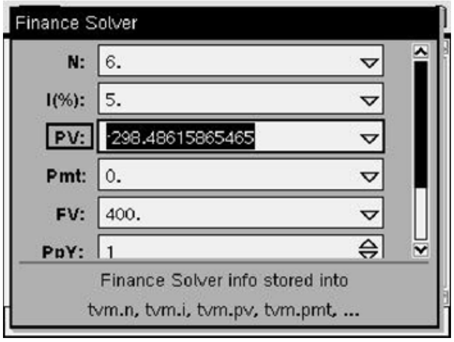
Approximately \$3041.63.

Example 38

Assuming an annual inflation rate of 5%, what is today's cost of an item worth \$400 in 6 years' time?

Solution

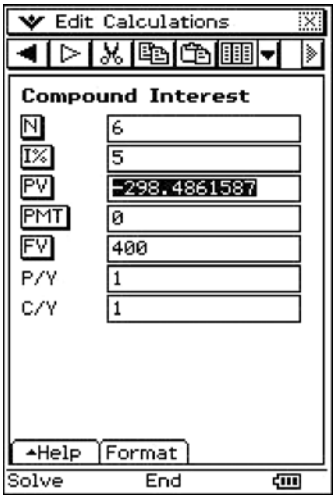
Using the TI-Nspire:



N: 6.
I(%): 5.
PV: -298.48615865465
Pmt: 0.
FV: 400.
PpY: 1

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

Using the ClassPad:



Compound Interest
N 6
I% 5
PV -298.4861587
PMT 0
FV 400
P/Y 1
C/Y 1

An item worth \$400 in 6 years' time costs \$298.49 today.

Example 39

On the 24th and 25th of October 1415, England's King Henry V's army fought those of France's King Charles VI at the battle of Agincourt. Given that £1 in 1415 approximates to £500 in 2008, evaluate the average annual nominal rate of inflation of the English pound over this period.

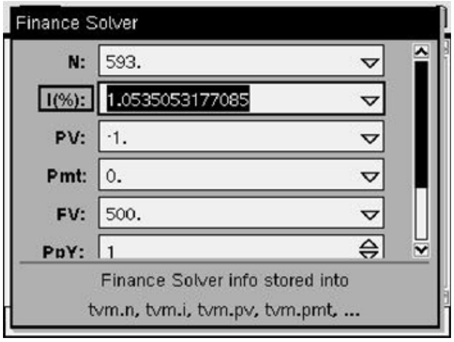
Solution

$PV = 1 \qquad FV = 500 \qquad n = 2008 - 1415 = 593$

$$\begin{aligned} FV &= PV\left(1+\frac{r}{100}\right)^n \\ 500 &= 1\left(1+\frac{r}{100}\right)^{593} \\ \sqrt[593]{500} &= 1+\frac{r}{100} \\ 1.010\,535 &= 1+\frac{r}{100} \\ r &= 1.0535 \end{aligned}$$

Approximately 1.05% average nominal inflation.

Using the TI-Nspire:



Finance Solver

N: 593.

I(%): 1.0535053177085

PV: -1.

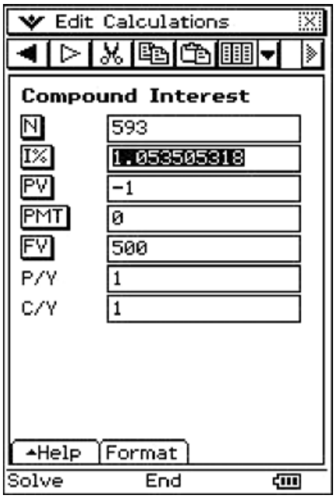
Pmt: 0.

FV: 500.

PpY: 1

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

Using the ClassPad:



Compound Interest

N 593

I% 1.053505318

PV -1

PMT 0

FV 500

P/Y 1

C/Y 1

The average nominal inflation is approximately 1.05%.

Exercise 8I

Example 35 **1** A car's present market value is $12 500. Given that the car is 7 years old and average depreciation for this model is 12.5% p.a., evaluate the initial purchase price.

2 An item purchased for $800 depreciates by 50% in successive years. What is its value after 3 years?

Example 36 **3** Office equipment is being depreciated at 17.5% for tax purposes. After how many years will it be valued at half its purchase price?

Example 37 **4** Assuming an average annual inflation rate of 3%, how much is a current dollar worth in 4 years' time?

Example 38 **5** Assuming 5% p.a. inflation over the past 3 years in New Zealand, what was the value of today's dollar 3 years ago?

Example 39 **6** A boat depreciates by 50% over 8 years. Calculate the average annual rate of depreciation.

7 Given a can of soft drink presently costs $2.20, calculate the cost in 10 years' time at 4% annual inflation.

8 Petrol cost 80 cents per litre in 2003. If the annual average inflation rate has been 3% p.a., what should be today's cost?

9 If inflation averaged 3% p.a. compounded quarterly for the past 8 years and 5% p.a. compounded quarterly for the 4 years previously, evaluate the cost of a $100 item 12 years ago.

10 A watch currently costs $220. Eight years ago, the same model watch cost $120. Given an inflation rate of 3.5% p.a., compounding monthly, is today's cost relatively cheaper or dearer than 8 years ago?

8.10 Annuities

Regular payments as part of a saving plan, loan contract, superannuation contributions, equipment leasing, pensions, term deposit credits, bursaries, government child endowment and lay-by deposits are all annuities.

An annuity is a regular equal monetary payment. Normally an annuity is for an extended period over a number of years.

Most annuities fall into two categories:

1. Payments received (e.g. pensions, retirement superannuation payments, term deposit credits, bursaries and trust fund disbursements)
2. Payments made (e.g. loan/credit card repayments, savings deposits, leasing, lay-by, life insurance and superannuation contributions)

As payments are made at regular intervals, it is common for funds to be deposited directly into the recipient's account (payments received) or for the payment to be debited directly from the payee's account (payments made).

Compound interest applies to both the accrual of funds when payments are made, and to the disbursement of existing monies when regular payments are received, which are called annuities in arrears.

Saving

Consider an amount, \$$Q$, is banked at the end of each month. Interest is paid at the end of the month on the average monthly account value at the nominal rate of r% per annum. A plan of this type is often called the '**future value of an ordinary annuity**'.

Let R be the growth factor per month,

then $R = \left(1 + \frac{r}{(100 \times 12)}\right)$.

The balance at the end of each month will be:

End of month	Amount (\$)	Amount as a series (\$)
1	Q	Q
2	$QR + Q$	$QR^1 + Q$
3	$(QR + Q)R + Q$	$QR^2 + QR^1 + Q$
4	$((QR + Q)R + Q)R + Q$	$QR^3 + QR^2 + QR^1 + Q$
$\vdots$ N	$\vdots$	$\vdots$ $QR^{N-1} \ldots + QR^3 + QR^2 + QR^1 + Q$

In general, after N months the amount A_N can be expressed as the series

$$\begin{aligned} A_N &= QR^{N-1} + \ldots QR^4 + QR^3 + QR^2 + QR^1 + Q \\ &= Q\left(1 + R + R^2 + R^3 + R^4 + \ldots R^{N-1}\right) \end{aligned}$$

$1 + R + R^2 + R^3 + R^4 + \ldots R^{N-1}$ is a geometric series with first term $a = 1$ and common ratio $r = R$.

$$\therefore\ 1 + R + R^2 + R^3 + R^4 + \ldots R^{N-1} = \frac{\left(R^{N-1}\right)}{(R-1)}$$

$$\therefore A_N = \frac{Q\left(R^N - 1\right)}{(R-1)}$$

Example 40

The sum of \$100 is banked at the end of each month as part of a savings plan. Interest is paid at the nominal rate of 5% p.a. Evaluate the total amount saved after 3 years.

Solution

$$Q = 100,\ r = 5,\ N = 36$$

$$\therefore \quad R = \left(1 + \frac{5}{(100 \times 12)}\right)$$

$$= 1.004\,167$$

$$A_N = \frac{Q\left(R^N - 1\right)}{(R-1)}$$

$$= \frac{100\left(1.004\,167^{36} - 1\right)}{(1.004\,167 - 1)}$$

$$= 3875.33$$

The sum of \$3875 has been saved over the 3 years, which includes the last 36th month \$100 payment for which no interest accrued.

Using technology

Example 41

James plans to save \$1 million for his retirement in 25 years' time. He estimates that the average nominal interest rate over this period to be 7% p.a. Given his bank compounds interest each month, calculate the monthly savings required if he:

a commences saving immediately

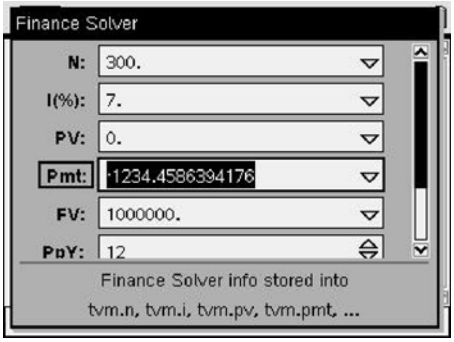
b delays for 10 years

Evaluate this difference in total contributions between parts **a** and **b**.

Solution

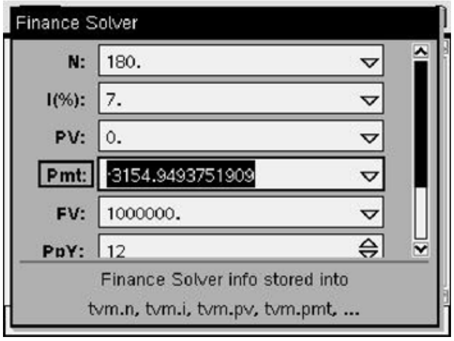
Using the TI-Nspire:

a Access Finance Solver, enter the values shown below then press enter at Pmt.



N: 300.
I(%): 7.
PV: 0.
Pmt: -1234.4586394176
FV: 1000000.
PpY: 12
Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

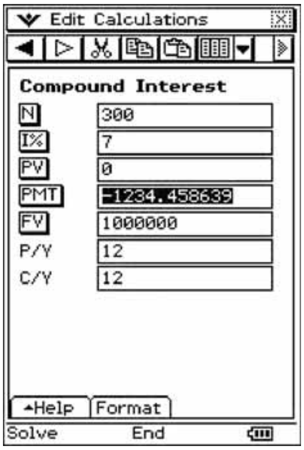
b Edit the screen and change N to 180. Press enter at Pmt.



N: 180.
I(%): 7.
PV: 0.
Pmt: -3154.9493751909
FV: 1000000.
PpY: 12
Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

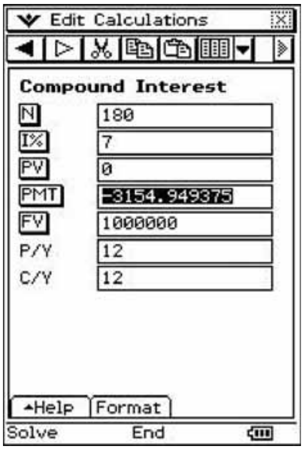
Using the ClassPad:

a Using the Compound Interest program, enter the variable values, as shown, and tap Solve when the cursor is set to PMT.



Compound Interest
N 300
I% 7
PV 0
PMT -1234.458639
FV 1000000
P/Y 12
C/Y 12

b Edit the screen and change N to 180. Tap Solve at PMT.



Compound Interest
N 180
I% 7
PV 0
PMT -3154.949375
FV 1000000
P/Y 12
C/Y 12

Note: Spreadsheets also provide a useful representation of an annuity.

Saving with an initial principal

Saving plans often commence with an initial payment to be followed by an annuity in the form of regular deposits. For example, Jane wishes to encourage her daughter Alexandra to save for her future financial security. The agreed plan is for Jane to deposit \$*P* in Alexandra's trust fund on the condition that Alexandra contributes \$*Q* per month. The fund attracts *r*% per annum nominal interest, compounding monthly.

Let R be the growth factor per month,

then $R = \left(1 + \frac{r}{(100 \times 12)}\right).$

Jane deposits \$$P$ at the beginning of the first month. Alexandra's saving of \$$Q$ is deposited at the end of the first month.

End of month	Amount (\$)	Amount as a series (\$)
0	P	
1	$PR+Q$	$PR+Q$
2	PR^2+QR+Q	PR^2+QR^1+Q
3	$PR^3+(QR+Q)R+Q$	$PR^3+QR^2+QR^1+Q$
4	$PR^4+((QR+Q)R+Q)R+Q$	$PR^4+QR^3+QR^2+QR^1+Q$
$\vdots$ N	$\vdots$	$\vdots$ $PR^N+QR^{N-1}\ldots+QR^3+QR^2+QR^1+Q$

After N months, the total amount A_N can be expressed as the sum of Jane's compounding single initial contribution and Alexandra's annuity.

$$A_N = PR^N + QR^{N-1}\ldots + QR^3 + QR^2 + QR^1 + Q$$
$$A_N = PR^N + Q\left(R^{N-1}\ldots + R^3 + R^2 + R^1 + 1\right)$$

$$R^{N-1} + \ldots + R^3 + R^2 + R^1 + 1 = \frac{\left(R^N - 1\right)}{(R-1)} \quad \text{(sum of a geometric series)}$$

$$\therefore A_N = PR^N + \frac{Q\left(R^N - 1\right)}{(R-1)}$$

Example 42

Let Jane contribute \$5000 and Alexandra agree to save \$400 per month. The trust fund pays 8% p.a. nominal interest, compounded monthly. Calculate:

a the expected fund balance in 10 years' time

b the total interest earned in 10 years' time

Solution

a $P = 5000,\ Q = 400,\ r = 8,\ N = 120$

$$\begin{aligned} R &= \left(1 + \frac{8}{(100 \times 12)}\right) \\ &= 1.00\dot{6} \\ A_N &= PR^N + \frac{Q\left(R^N - 1\right)}{(R-1)} \\ &= 5000 \times \left(1.00\dot{6}\right)^{120} + \frac{400\left(\left(1.00\dot{6}\right)^{120} - 1\right)}{\left(1.00\dot{6} - 1\right)} \\ &\approx 84\,277 \end{aligned}$$

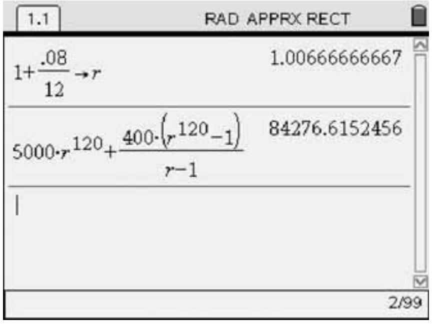
The balance is \$84 277, to the nearest dollar.

Using technology

To efficiently execute the calculation of A_N in Example 42, follow the screen entries as shown.

Using the TI-Nspire:

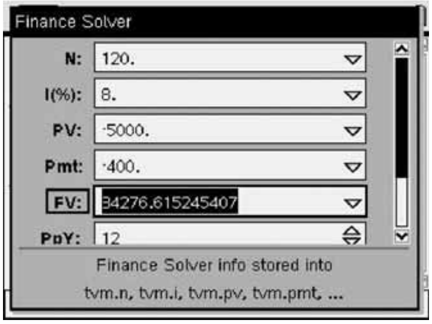
a



$1+\frac{.08}{12}\to r$ 1.00666666667

$5000\cdot r^{120}+\frac{400\cdot(r^{120}-1)}{r-1}$ 84276.6152456

Using Finance Solver to validate this, we have:



N: 120.
I(%): 8.
PV: ⁻5000.
Pmt: ⁻400.
FV: 84276.615245407
PpY: 12

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

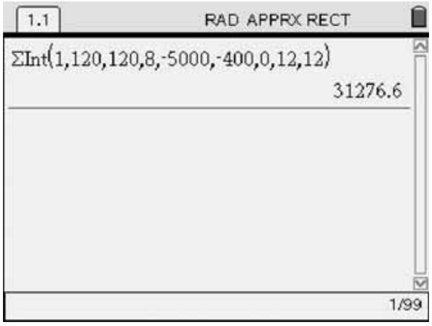
b

From the catalog, select *Finance → Amortization → Interest Paid*.

Now, type the following:

1, 120, 120, 8, –5000, –400, 0, 12, 12)

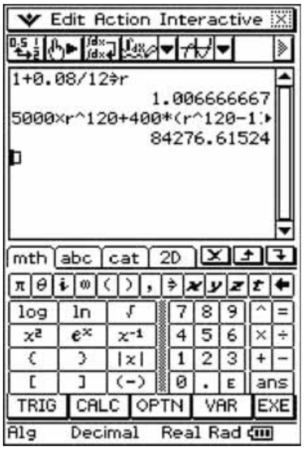
then press enter.



ΣInt(1,120,120,8,⁻5000,⁻400,0,12,12) 31276.6

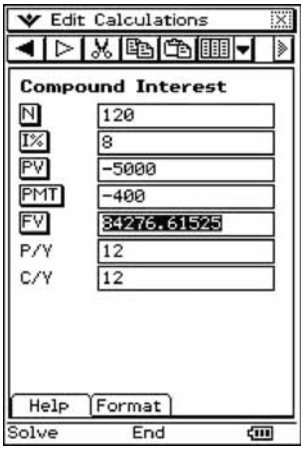
Using the ClassPad:

a



```
1+0.08/12⇒r
                 1.006666667
5000×r^120+400*(r^120-1:▸
                 84276.61524
```

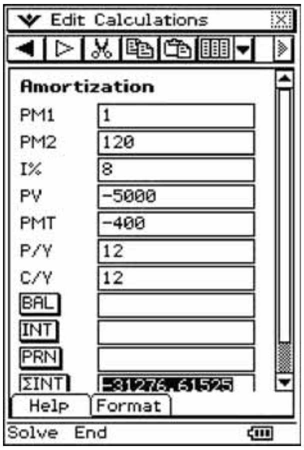
Using the Compound Interest program to validate this, we have:



Compound Interest
N 120
I% 8
PV -5000
PMT -400
FV 84276.61525
P/Y 12
C/Y 12

b

Using the Amortization program with the values stated below, we have:



Amortization
PM1 1
PM2 120
I% 8
PV -5000
PMT -400
P/Y 12
C/Y 12
BAL
INT
PRN
ΣINT -31276.61525

Exercise 8J

Validate, where possible, each of the answers to questions ***1–11***, *using each of the following calculator features*:

i *TVM Solver*

ii *graphing function*

Example 40

1 The sum of \$200 is banked each month as part of a savings plan. Interest is paid at the nominal rate of 8% per annum monthly. Evaluate the total amount saved after 5 years.

2 The sum of \$600 is banked at the end of each quarter as part of a savings plan. Interest is paid at the nominal rate of 8% per annum quarterly. Evaluate the total amount saved after 5 years.

3 Tom wishes to save \$4000 for a car deposit in 2 years' time. His bank is offering interest at a nominated rate of 7.5% per annum, compounding monthly. How much must Tom save each month?

4 Carmen is saving for a holdiay worth \$2500, which she wishes to take in 1 year's time. She is banking \$45 per week at a nominated rate of 7.5% per annum, compounding weekly. Will Carmen have saved enough for her holiday?

5 Than receives \$1000 on each of his 18th, 19th and 20th birthdays. He immediately invests the money at a nominal interest rate of 7% per annum, compounding yearly. Determine the value of the investment on his 21st birthday.

6 Which is the better investment?

- **i** A sum of \$1315 is deposited each quarter at a nominated 8% per annum, compounding quarterly.
- **ii** A sum of \$100 is deposited weekly, earning a nominated 8% per annum, compounding weekly.

7 David owes his father \$37 850 loaned to him for a home deposit. He agrees to repay the loan by depositing \$250 each fortnight into a bank account earning a nominal interest rate of 6% p.a., compounding fortnightly, until the total is reached. How long does David take to reach the total?

8 Claire plans for her retirement and begins investing \$1000 each week in an account earning interest at a nominal rate of 8.5% p.a., compounding weekly.

- **a** How long is it until she can retire with \$500 000?
- **b** If she increases her weekly deposits by 50% to \$1500, by what percentage does her time to retirement decrease?
- **c** Calculate the apparent extra cost of earlier retirement.

Examples 41, 42

9 Yen commences her investment plan with a deposit of \$1000. One month later she deposits \$100, at a nominal 6% p.a., compounding monthly, which she continues each month for 5 years. Calculate the:

a expected fund balance at the end of the 5 years
b total interest earned

10 Trevor has \$2500 as a start to saving \$10 000 over 3 years. Calculate his required weekly deposits in an account earning a nominal 6.95% p.a., compounding weekly.

11 An initial \$3000 is deposited into an account earning a nominal interest rate of 8% p.a., compounding quarterly. A further \$2400 is deposited at the end of each subsequent quarter. How long will it take for the account balance to exceed \$100 000?

12 Rob and Diana are saving for a deposit on an investment property, which they wish to purchase in 3 years' time. They can budget to save \$200 per week and will commence the plan with \$12 000 received from the sale of a parcel of shares. Evaluate the minimum nominal interest rate (to 2 decimal places) necessary to reach their target of \$50 000 if their bank offers weekly compounding interest.

Loans

Consider a loan of \$$P$ to be repaid by n equal instalments of \$$Q$ (i.e. an annuity). Funds still owing at the end of each period attract a nominal compound interest rate of r% per annum. The regular payments usually continue to be made until the total of \$$P$ and the accumulated interest has been repaid. Many loans (e.g. housing loans) are of this type. As the financial transaction commences in the present with the loan, and is repaid in the future by an annuity, it is termed the **present value of an ordinary annuity**.

Let R be the growth factor per month; and n be the number of repayments per year

then $R = \left(1 + \dfrac{r}{(100n)}\right)$

A_N = the amount still owing after N repayments

P = principal borrowed

End of month	Amount (\$)	Amount as a series (\$)
1	$PR - Q$	$PR^1 - Q$
2	$(PR^1 - Q)R - Q$	$PR^2 - QR^1 - Q$
3	$(PR^2 - QR^1 - Q)R - Q$	$PR^3 - QR^2 - QR^1 - Q$
4	$(PR^3 - QR^2 - QR^1 - Q)R - R$	$PR^4 - QR^3 - QR^2 - QR^1 - Q$
⋮	⋮	⋮
N		$PR^N - QR^{N-1} \ldots - QR^3 - QR^2 - QR^1 - Q$

Thus, after N repayments, the total amount still owing (A_N) can be expressed as:

$$A_N = PR^N - QR^{N-1} \ldots - QR^3 - QR^2 - QR^1 - Q$$
$$\therefore \quad A_N = PR^N - Q\left(R^{N-1} \ldots + R^3 + R^2 + R^1 + 1\right)$$

$$R^{N-1} \ldots + R^3 + R^2 + R^1 + 1 = \frac{\left(R^N - 1\right)}{(R-1)} \text{ (sum of a geometric series)}$$

$$\therefore \quad A_N = PR^N - \frac{Q\left(R^N - 1\right)}{(R-1)}$$

Amortisation

Amortisation is when the loan is fully repaid; that is, $A_N = 0$.

Therefore,
$$0 = PR^N - \frac{Q\left(R^N - 1\right)}{(R-1)}$$
$$PR^N = \frac{Q\left(R^N - 1\right)}{(R-1)}$$
$$\therefore \quad Q = \frac{PR^N(R-1)}{\left(R^N - 1\right)}$$

Therefore, $\$Q$ is the value of each of the N annuity payments necessary to fully repay a loan principal of $\$P$ plus nominal interest at r% per annum charged on the outstanding balance at the commencement of each period or rest. That is, the loan is **amortised**.

Example 43

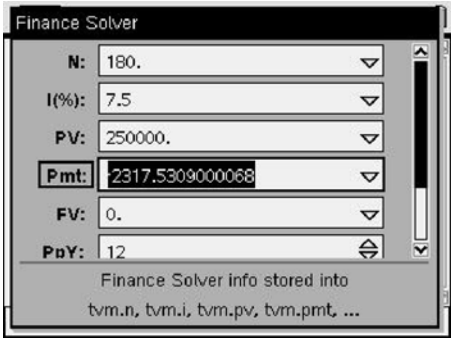
A $250 000 mortgage is amortised in 15, 20 or 30 years. Given monthly payments and a nominal rate of interest of 7.5% p.a., compounding monthly, evaluate the:

- **a** monthly repayment for each loan
- **b** total interest paid for each loan
- **c** amount still owing after 10 years for each loan
- **d** **i** principal repaid and the interest repaid on payment of the 12th instalment of the 30-year loan
  - **ii** principal repaid and the interest repaid on payment of the 350th instalment of the 30-year loan

Solution

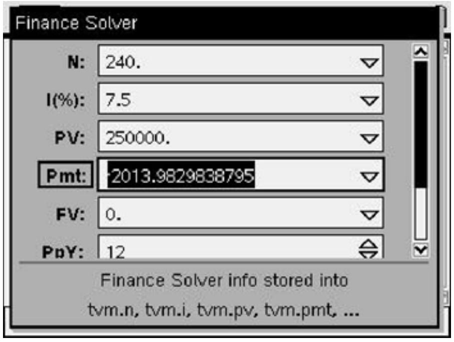
Using the TI-Nspire:

a



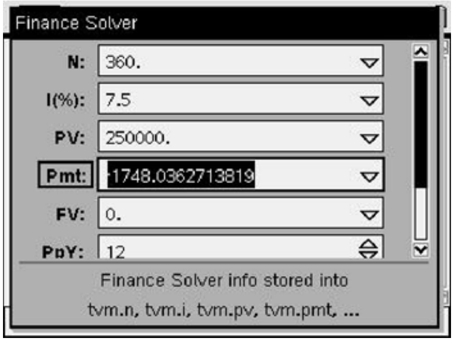
N: 180.
I(%): 7.5
PV: 250000.
Pmt: -2317.5309000068
FV: 0.
PpY: 12

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...



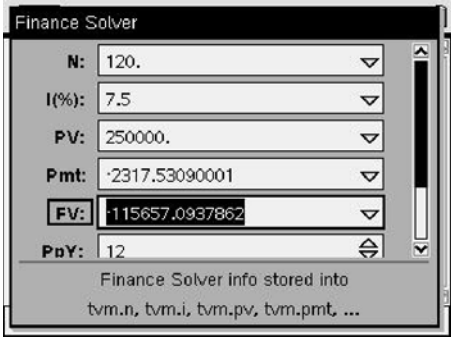
N: 240.
I(%): 7.5
PV: 250000.
Pmt: -2013.9829838795
FV: 0.
PpY: 12

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...



N: 360.
I(%): 7.5
PV: 250000.
Pmt: -1748.0362713819
FV: 0.
PpY: 12

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

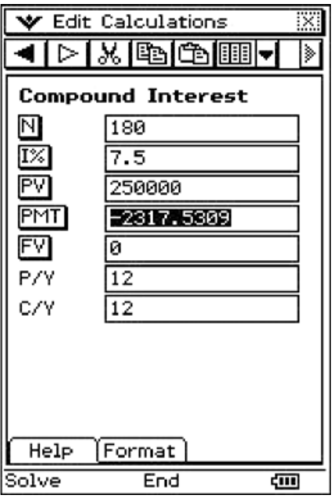


N: 120.
I(%): 7.5
PV: 250000.
Pmt: -2317.53090001
FV: -115657.0937862
PpY: 12

Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

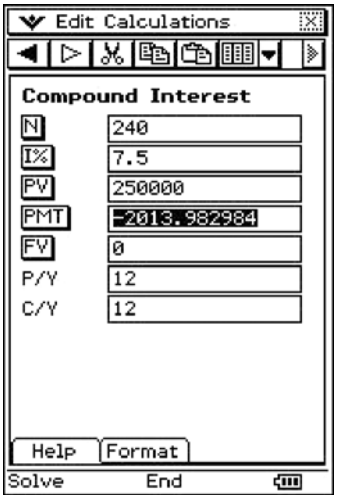
Using the ClassPad:

a



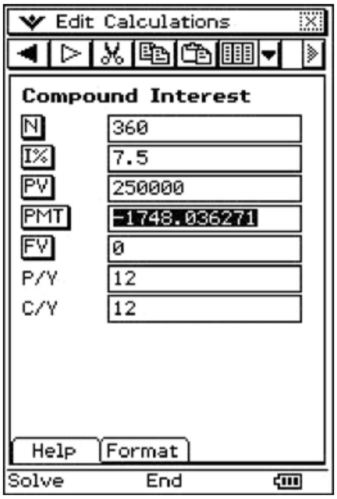
Compound Interest

N 180
I% 7.5
PV 250000
PMT -2317.5309
FV 0
P/Y 12
C/Y 12



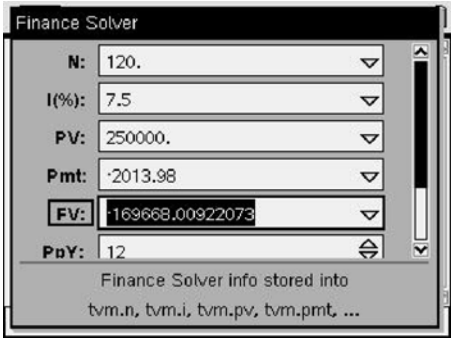
Compound Interest

N 240
I% 7.5
PV 250000
PMT -2013.982984
FV 0
P/Y 12
C/Y 12



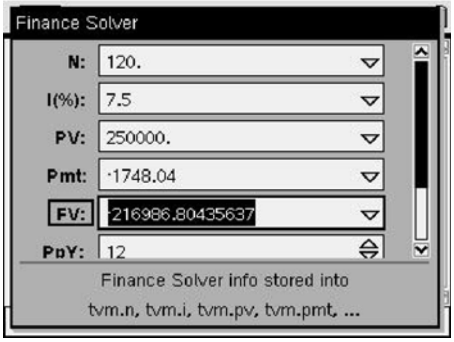
Compound Interest

N 360
I% 7.5
PV 250000
PMT -1748.036271
FV 0
P/Y 12
C/Y 12



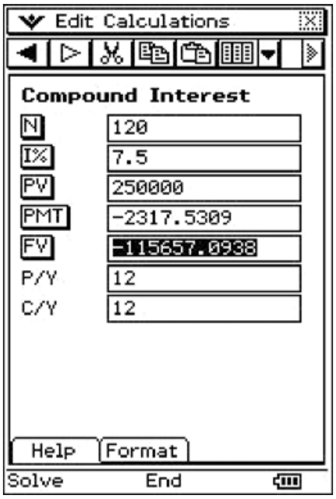
N: 120.
I(%): 7.5
PV: 250000.
Pmt: -2013.98
FV: -169668.00922073
PpY: 12

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

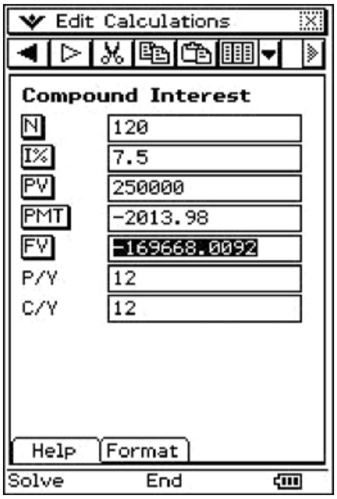


N: 120.
I(%): 7.5
PV: 250000.
Pmt: -1748.04
FV: -216986.80435637
PpY: 12

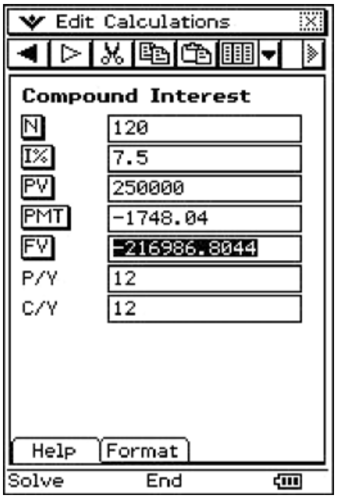
Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...



Compound Interest
N 120
I% 7.5
PV 250000
PMT -2317.5309
FV -115657.0938
P/Y 12
C/Y 12



Compound Interest
N 120
I% 7.5
PV 250000
PMT -2013.98
FV -169668.0092
P/Y 12
C/Y 12



Compound Interest
N 120
I% 7.5
PV 250000
PMT -1748.04
FV -216986.8044
P/Y 12
C/Y 12

Amounts owing after 10 years are \$115 657.25, \$169 668.01 and \$216 986.80, respectively.

d **i** At payment of the 12th instalment, the reduction of the amount owing is $A_{11} - A_{12}$.

$$\begin{aligned} A_{11} - A_{12} &= \$247\,894.11 - \$247\,695.41 \\ &= \$198.70 \\ \text{Interest paid} &= Q - \text{Principal reduction} \\ &= \$1748.04 - \$198.70 \\ &= \$1549.34 \end{aligned}$$

The 12th instalment of \$1748.04, which is composed of:

\$198.40 (i.e. reduction in amount owing) + \$1549.34 (i.e. interest)

ii At payment of the 350th instalment, the reduction of the amount owing is $A_{349} - A_{350}$.

$$\begin{aligned} A_{349} - A_{350} &= \$18\,526.44 - \$16\,894.20 \\ &= \$1632.24 \\ \text{Interest paid} &= Q - \text{Principal reduction} \\ &= \$1748.04 - \$1632.24 \\ &= \$115.80 \end{aligned}$$

The 350th instalment of \$1748.04, which is composed of:

\$1632.24 (i.e. reduction in amount owing) + \$115.80 (i.e. interest)

Exercise 8K

Validate, where possible, each of the answers to questions **1–6**, using each of the following calculator features:

i TVM Solver
ii graphing function

Example 43 **1** A \$100 000 mortgage is amortised in 10, 15 or 20 years. Given quarterly payments and a nominal rate of interest of 8% p.a., compounding quarterly, evaluate:

- **a** the quarterly payment for each loan
- **b** the total interest paid for each loan
- **c** the amount still owing after 5 years for each loan
- **d** **i** the principal repaid and the interest repaid on payment of the 12th instalment of the 20-year loan
  - **ii** the principal repaid and the interest repaid on payment of the 68th instalment of the 20-year loan

2 A \$5000 personal loan can be repaid in either 1, 2 or 5 years. Given weekly payments and a nominal rate of interest of 10.4% p.a., compounding weekly, evaluate:

- **a** the weekly payment for each loan
- **b** the total interest paid for each loan
- **c** the amount still owing after 6 months for each loan
- **d** **i** the principal repaid and the interest repaid on payment of the 6th instalment of the 1-year loan
  - **ii** the principal repaid and the interest repaid on payment of the 6th instalment of the 2-year loan
  - **iii** the principal repaid and the interest repaid on payment of the 6th instalment of the 5-year loan

Example 43

3 Anita and Raj are keen to purchase a property requiring them to sign a \$350 000 mortgage. They are able to repay \$2600 per month. Their bank is offering interest at a nominal rate of 7.75% p.a., compounding monthly for 30-year housing loans. Can Anita and Raj afford to purchase the property?

4 An amount of \$1000 is being repaid each fortnight over 25 years at 8.25% p.a. nominal interest, compounding fortnightly, for amortisation of a loan. Evaluate the size of the loan.

5 Tonya and Trevor signed a \$500 000 home mortgage over 25 years, repaying \$3222 per month. Given that the interest compounds monthly, evaluate the annual nominal interest rate compounded monthly, using your calculator's graphing facility.

6 A \$200 000 mortgage is offered at 10% p.a. interest, compounding yearly, with annual payments of \$30 000. Use algebra to calculate the time it will take to reduce the balance owed to less than \$100 000.

8.11 Modelling and problem solving

Exercise 8L

1 $f(x) = e^x$ and $g(x) = -x$. Solve for x: $f(g(x)) + g(f(x)) = 0$.

2 The population of a colony of small, interesting insects is modelled by the following hybrid function:

$$N(t) = \begin{cases} 20e^{0.2t}, & 0 \leq t \leq 50 \\ 20e^{10}, & 50 \leq t \leq 70 \\ 10e^{10}\left(e^{70-t}+1\right), & t > 70 \end{cases}$$

where t is the number of days.

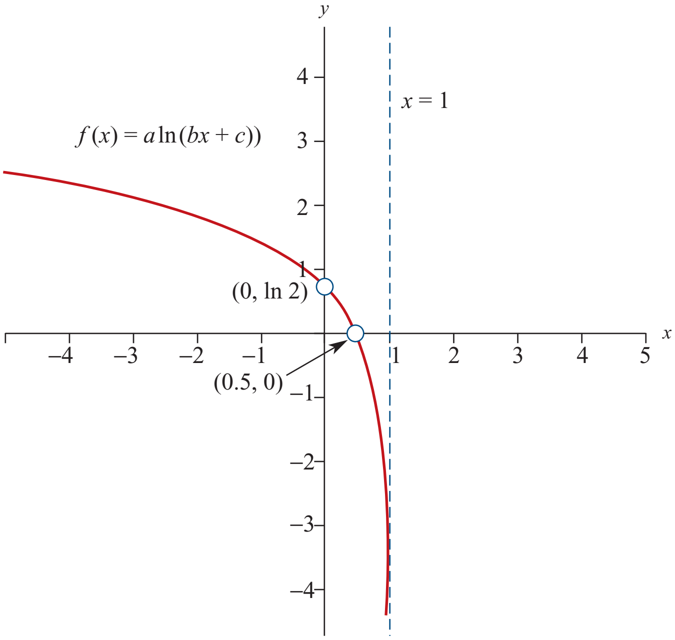
- **a** Sketch the graph of $N(t)$ against t.
- **b** Find:

  i $N(10)$ **ii** $N(40)$ **iii** $N(60)$ **iv** $N(80)$

c Find the number of days for the population to reach:
  i 2968 **ii** 21 932

3 Find a and k, such that the graph of $y = a\,e^{(x+k)}$ passes through the points (0, 1) and $(1, e)$.

4 Evaluate a, b and c for the plot $f(x) = a\ln(bx + c)$, given below.



5 A type of bacteria is modelled by the formula $n = A\left(1 - e^{-Bt}\right)$, where n is the size of the population at time t hours. A and B are positive constants.

- **a** If $t = 2$ when $n = 10\,000$ and $t = 4$ when $n = 15\,000$:
  - **i** Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.
  - **ii** Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.
  - **iii** Solve this equation for a.
  - **iv** Find the exact value of B.
  - **v** Find the exact value of A.
- **b** Sketch the graph of n against t.
- **c** After how many hours is the population of bacteria 18 000?

6 A radioactive substance is decaying such that the amount A grams at time t years is given by the formula $A = A_0e^k$. If $t = 1$ when $A = 60.7$ and $t = 6$ when $A = 5$, find the values of the constants A_0 and k.

7 In a chemical reaction, the amount (x grams) of a substance that has reacted is given by $x = 8\left(1 - e^{-0.2t}\right)$, where t is the time from the beginning of the reaction, in minutes.

a Sketch the graph of x against t.

b Find the amount of substance that has reacted after:

  i 0 minutes **ii** 2 minutes **iii** 10 minutes

c Find the time at which exactly 7 grams of the substance has reacted.

8 Newton's law of cooling for a body placed in a medium of constant temperature states:

$$T - T_s = (T_0 - T_s)\, e^{-kt}$$

where

T is the temperature (in $^\circ$ C) of the body at time t (in minutes)

T_s is the temperature of the surrounding medium; and

T_0 is the initial temperature of the body.

An egg at 96° C is placed in a sink of water at 15° C to cool. After 5 minutes the egg's temperature is found to be 40° C. (Assume that the temperature of the water does not change.)

a Find the value of k.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30° C?

9 Each time Joan rinses her hair after washing it, the result is to remove a quantity of shampoo from the hair. With each rinsing, the quantity of shampoo removed is one-tenth of the previous rinse.

a If Joan rinses out 90 mg of shampoo with the first rinse, how much will she have washed out altogether after six rinses?

b How much shampoo do you think was present in her hair at the beginning?

10 A prisoner is trapped in an underground cell that is inundated with a sudden rush of water, which comes up to a depth of 1 m; that is, one-third of the height of the ceiling (which measures 3 m). After an hour a second inundation occurs, but this time the water level rises by only $\frac{1}{3}$ m. After a second hour another inundation of water raises the level by $\frac{1}{9}$ m. If this process continues for 6 hours, write down:

a the amount the water level will rise at the end of the sixth hour

b the total height of the water level at this time

If this process continues, do you think the prisoner, who cannot swim, will drown? Why?

11 After an undetected leak in a storage tank, the staff at an experimental research station were subjected to 500 curie hours of radiation the first day, 400 curie hours the second day, 320 the third day and so on.

Find the number of curie hours they were subjected to:

a on the 14th day

b during the first five days of the leak

12 A rubber ball is dropped from a height of 81 metres. Each time it strikes the ground, it rebounds two-thirds of the distance through which it has fallen.

- a Find the height the ball reaches after the sixth bounce.
- b Assuming the ball continues to bounce indefinitely, find the total distance travelled by the ball.

13 In payment for loyal service to the king, a wise peasant asked to be given one grain of rice for the first square of a chessboard, two grains for the second square, four for the third square and so on, for all 64 squares of the board. The king thought this seemed fair and readily agreed, but was horrified when the court mathematician informed him of how many grains of rice he would have to pay the peasant.

How many grains of rice did the king have to pay? (Leave your answer in index form.)

14 Mathematician Kathryn is an astute investor. She boasts that her compound interest investment with yearly rests is easy to remember, as $\sqrt{72}$ is both the number of years and the nominal interest rate required for her investment to double in value. Is she correct?

15 An investment compounds annually. In the second year the nominal rate is twice that for the first year. Given the investment has grown by a total of 20% over the 2 years, evaluate the nominal rate for each year, using algebra.

16 The exponential function $A = Pe^{rn}$ models a daily compounding investment. If the investment increases in value by 25% in 2 years, evaluate the nominal interest rate.

17 Convert a continuously compounding interest rate of 5% p.a. as a quarterly interest rate.

18 Jake finds €100 in his suitcase while packing to return to Ireland after 2 years spent away. Inflation in the Republic of Ireland during this time was 4% p.a. Evaluate the average annual depreciation rate in the value of Jake's €100.

19 The sum of $1000 is banked at the end of each month for 10 years, earning interest on monthly balances. The investment returns the amount of $204 840 after the last payment is made. Calculate the nominal interest rate, using a graphical method. Validate your answer using the TVM Solver.

20 A mortgage loan of $350 000 is offered at 8% p.a. interest, compounding monthly, with monthly payments of $3000. Calculate the time taken to reduce the balance owed to less than $100 000.

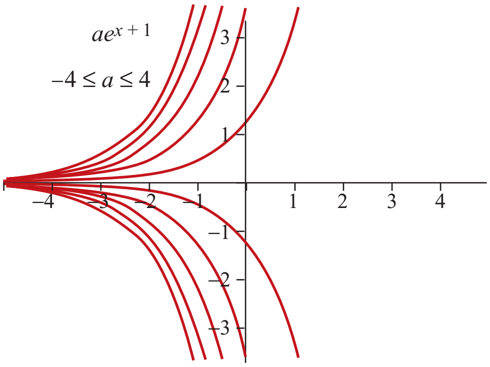
21 A mortgage loan of $300 000 is to be repaid over 20 years, requiring repayments of $2500 per month. Given interest compounds monthly, evaluate the annual nominal interest rate.

Chapter summary

- **Definition** of 'e' $\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
- **Sketch transformations** and families of

  $y = ae^{(b(x-c))} + d$

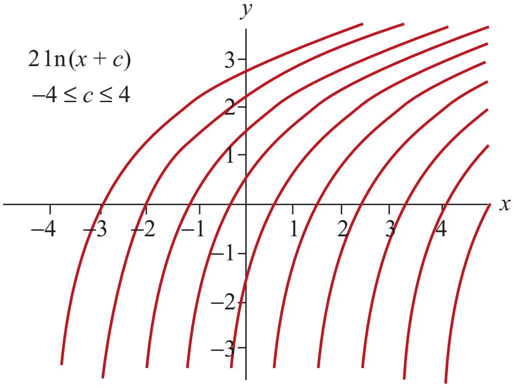
  For example:



- **Sketch transformations** and families of

  $y = a\ln(b(x-c)) + d$

  For example:



- Law of exponential change.

  Let A be the quantity at time t. Then,

  $A = A_0e^{kt}$, where A_0 is a constant.

  Growth: $k > 0$ $\qquad$ Decay: $k < 0$

  The number k is the **rate constant** of the equation.
- For a **geometric sequence**: $t_n = r\,t_{n-1}$, where r is a constant.

  $t_n = ar^{n-1}$, where a is the first term.
- The **geometric mean** of two numbers a and c (of the same sign) is $\pm\sqrt{ac}$.
- The sum of a **geometric series** is $\quad S_n = \frac{a(r^n-1)}{r-1}, \quad -1 < r < 1.$
- The **sum of an infinite geometric series** is $\quad S_\infty = \frac{a}{1-r}, \quad -1 < r < 1.$
- Yearly compounding interest

  If P = original investment

  A = amount the investment grows to after n years

  r = compound interest rate r% per annum

  n = number of years invested

$$A = P\left(1+\frac{r}{100}\right)^n$$

- **'Rule' of 72**: $n \times r \approx 72$ is to approximate the number of years (n) for an investment to double at r% p.a. compound interest.
- Compound interest

$$A = P\left(1 + \frac{r}{(100\,m)}\right)^{m\,n}$$

where
r% is the nominal rate of interest
m is the number of rests per year
n is the number of years
P is the initial principal
A is the final amount

- The **effective rate** of an investment $s\% = 100\left[\left(1 + \frac{r}{(100\,m)}\right)^{m} - 1\right]\%$
- A **continuously compounding** investment is given by:

$$A = P \times e^{rn} \text{ and } A = Pe^{n \ln R}$$

where
r% is the nominal rate of interest
n is the number of years
P is the initial principal
A is the final amount
$R = \left(1 + \frac{r}{100}\right)$ is the growth factor

- **Depreciation** can be modelled by:

$$A = P\left(1 - \frac{r}{100}\right)^{n}$$

where
P = original investment
A = amount the investment depreciates to after n years
r = compound depreciating interest rate (r% p.a.)
n = number of years

- **Inflation** is given by:

$$FV = PV\left(1 + \frac{r}{100}\right)^{n}$$

where PV = present value and FV = future value

- Future value (A_N) of an ordinary annuity

$$A_N = \frac{Q\left(R^N - 1\right)}{(R - 1)}$$

where $Q =$ payments/rest, $R =$ growth factor, $N =$ number of rests.

- Future value (A_N) of an annuity with an initial payment (P)

$$A_N = PR^N + \frac{Q\left(R^N - 1\right)}{(R - 1)}$$

where $Q =$ payments/rest, $R =$ growth factor, $N =$ number of rests.

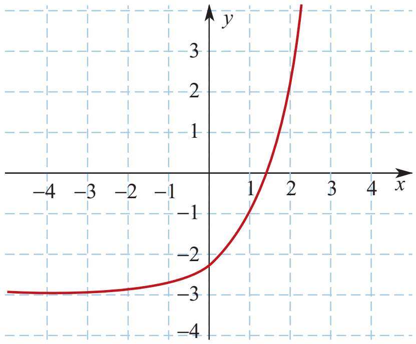
- **Amortisation** occurs when the loan is fully repaid.

$$\therefore\ A_N = 0 \text{ and } Q = \frac{PR^N (R - 1)}{\left(R^N - 1\right)}$$

where $Q =$ payments/rest, $R =$ growth factor, $N =$ number of rests.

Multiple-choice questions

1 The exponential function best describing the graph below is:



A $y = e^{x-2} - 3$ **B** $y = 2e^{x+1} - 3$ **C** $y = 2e^{x-1} - 3$

D $y = 2.5\ e^{x-1.5} - 3$ **E** $y = e^{x+2} - 3$

2 Given $f(x) = \ 2\ln(3x - 1) + 1$, $f^{-1}(x)$ is:

A $\frac{1}{3}e^{2x-2} + 1$ **B** $\frac{1}{3}e^{2x+2} + 1$ **C** $\frac{1}{3}e^{2x+2} - 1$

D $e^{2x-2} + 3$ **E** $3e^{2x+2} - 1$

3 Given the graph of the function $y = ae^{bx}$ passes through the points (0, –1) and $\left(-2, -\frac{1}{e}\right)$, the values of a and b are:

A $a = -1, \quad b = 2$ **B** $a = -\frac{1}{2}, \quad b = 1$ **C** $a = 1, \quad b = -\frac{1}{2}$

D $a = -1, \quad b = \frac{1}{2}$ **E** $a = \frac{1}{2}, \quad b = -1$

4 If the third term of a geometric sequence is 3 and the fifth term is 48, then the first term is:

A $\frac{3}{16}$ **B** $-\frac{3}{16}$ **C** 4 **D** −4 **E** $\pm\frac{3}{16}$

5 For the geometric sequence $t_4 = 2$ and $t_7 = -2$, S_3 equals:

A 2 **B** 0 **C** -1 **D** 1 **E** -2

6 If a geometric sequence is defined by $a = 3$ and $r = 2$, then S_∞ is:

A -3 **B** ∞ **C** 3 **D** 0 **E** 1

7 The effective interest rate of an investment at 10% p.a. nominal rate with monthly rests is:

A 10% **B** 10.47% **C** 120% **D** 9.5% **E** 12%

8 A property was rented for $200 per week in 2005. If the average annual inflation rate is 3.5% during the next 4 years, the expected rental cost in 2009 will be:

A $214 **B** $224 **C** $230 **D** $250 **E** $228

9 A nominal interest rate of 8% p.a., compounding continuously, converts to an effective interest rate of:

A 8.33% **B** 10% **C** 7.67% **D** 8.5% **E** 7.5%

10 If the value of a new car depreciates by half every 4 years, it will be worth less than 10% of its initial cost during the:

A 13th year **B** 12th year **C** 16th year **D** 14th year **E** 10th year

Short-response questions

1 For each of the following, find y in terms of x:

a $\log_e y = (\log_e x) + 2$ **b** $\log_{10} y = \log_{10} x + 1$

c $\log_2 y = 3 \log_2 x + 4$ **d** $\log_{10} y = -1 + 5 \log_{10} x$

e $\log_e y = 3 - \log_e x$ **f** $\log_e y = 2x - 3$

2 Solve each of the following equations for x (to 3 decimal places):

a $3^x = 11$ **b** $2^x = 0.8$ **c** $2^x = 3^{x+1}$

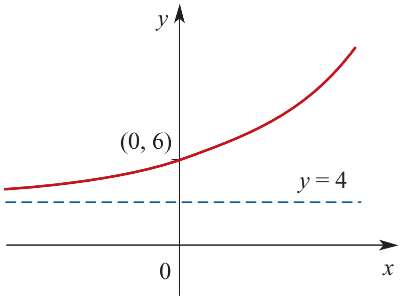
3 Find the value of x for which $4e^{3x} = 287$, giving the answer to 3 significant figures.

4 The graph of the function with equation $f(x) = e^{2x} - 3ke^x + 5$ intersects the axes at (0, 0) and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

5 The graph shown has rule

$$y = ae^x + b$$

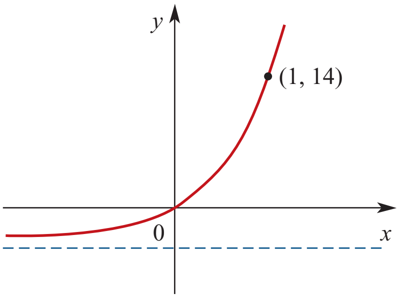
Find the values of a and b.



6 The rule for the function of the graph shown is of the form

$$y = ae^x + b$$

Find the values of a and b.



7 The points (3, 10) and (5, 12) lie on the graph of the function with rule $y = a\ \log_e(x - b) + c$. The graph has a vertical asymptote with equation $x = 1$. Find the values of a, b and c.

8 The graph of the function with rule $f(x) = a \log_e(-x) + b$ passes though the points $(-2, 6)$ and $(-4, 8)$. Find the values of a and b.

9 The sixth term of a geometric sequence is 9 and the tenth is 729. Find the fourth term.

10 An amount of \$1000 is invested at 3.5% p.a., compounded annually.

- **a** Find the value of the investment after 10 years.
- **b** Find how long it will take for the original investment to double in value.

11 The first term of a geometric sequence is 9 and the third term is 4. Find the possible values for the second and fourth terms.

12 The sum of three consecutive terms of a geometric sequence is 24 and the sum of the next three terms is also 24. Find the sum of the first 12 terms.

13 A pendulum is set swinging so that its first oscillation is through 40°. If each succeeding oscillation is $\frac{4}{5}$ of the preceding one, what is the total angle through which it has swung after 25 oscillations?

14 The sum to infinity of the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$ is:

a 2 **b** 1 **c** $\frac{1}{2}$ **d** $\frac{1}{3}$ **e** $\frac{2}{3}$

15 A geometric series is defined by

$$\frac{x+1}{x^2} - \frac{1}{x} + \frac{1}{x+1} - \ldots$$

- **a** Let r be the common ratio. Find r in terms of x.
- **b** **i** Find the infinite sum if $x = 1$.
  - **ii** Find the infinite sum if $x = -\frac{1}{4}$.
  - **iii** Find the infinite sum if $x = 2$.
- **c** Find the possible values of x for which the infinite sum is defined.

16 Evaluate the amount (A) an investment grows to when \$2000 is invested for 21 years at 9% p.a. compound interest, compounded annually.

17 Calculate the principal amount (P) invested at 10% p.a. compound interest, compounding annually, if the amount grows to $1610.51 in 10 years.

18 Evaluate the nominal rate of interest (r%), given that a principal amount of $7000 invested for 8 years, compounding annually, returns a balance of $12 391.74.

19 Calculate the number of years required for $1000 invested at 10.5% p.a. compound interest, compounded annually, to mature to $3313.96.

20 The sum of $6000 is invested for 5 years at an interest rate of 6.99% p.a., compounding monthly. Evaluate the final amount.

21 An investment returned $4676.55 after 4 years at 7.25% p.a., compounding weekly. How much was the principal amount invested?

22 An amount of $4000 was invested at a nominal rate of 12% p.a., compounding daily, returning a balance of $5732.98. Calculate the term (i.e. number of years) of the investment.

23 An amount of $25 000 was invested at a nominal rate of 12% p.a., returning $82 774 after 10 years. Calculate the period of each rest. *Hint*: Use your graphic calculator Plot Intersect feature.

24 Which is the better effective rate?

a a nominal rate of 8.25% p.a., compounding quarterly

b a nominal rate of 8.20% p.a., compounding monthly

25 A sum of $500 is invested at the nominal rate of 8% p.a., compounding continuously.

a Evaluate the effective interest rate.

b Calculate the investment return after 10 years.

26 The matured value of an investment is $30 800. The investment was for 15 years and the compounding continuous rate of interest was 7.79% p.a.

a Evaluate the nominal interest rate.

b Calculate the initial value of the investment.

27 An item purchased for $100 depreciates by 50% in successive years. What's its value after 4 years?

28 A jet ski depreciates by one-eighth of the previous year's value for each successive year. After how many full years will it be first valued at less than one-eighth of its purchase price?

29 Assuming an average annual inflation rate of 2.5%, how much is one current dollar worth in 5 years' time?

30 A developing country has suffered from the effects of a crippling inflation rate of 22.5% p.a. for the past 3 years. What was the value of €1000 3 years ago?

31 A £100 note is hidden for safekeeping for 4 years, during which inflation averaged 3.5% p.a. Evaluate the average annual rate of depreciation of the note's value.

32 A sum of $100 is banked at the end of each month as part of a savings plan. Interest is paid at the nominal rate of 5% p.a., compounding monthly. Evaluate the total amount saved after 3 years.

33 Which is the better investment?

- **a** \$1000 deposited each quarter at a nominated rate of 12% p.a., compounding quarterly
- **b** \$330 per month at a nominated rate of 12% p.a., compounding monthly
- **c** \$11 per day at a nominated rate of 8.99% p.a., compounding daily

34 An investment plan commences with a deposit of \$25 000 and payments of \$500 per month thereafter. The account earns a nominal interest rate of 6.8% p.a., compounding monthly. Calculate:

- **a** the expected fund balance in 10 years
- **b** the total interest earned in 10 years

35 A mortgage loan of \$150 000 is amortised in 5, 10 or 20 years. Given monthly payments and a nominal rate of interest of 7.5% p.a., compounding monthly, evaluate:

- **a** the monthly payment for each loan
- **b** the total interest paid for each loan
- **c** the amount still owing after 2 years for each loan
- **d**
  - **i** the principal repaid and the interest repaid on payment of the 60th instalment of the 5-year loan
  - **ii** the principal repaid and the interest repaid on payment of the 60th instalment of the 10-year loan
  - **iii** the principal repaid and the interest repaid on payment of the 60th instalment of the 20-year loan

CHAPTER

9

Rates of change

Objectives

- To express changes in two related variables as a **rate**.
- To distinguish between **constant** and **variable** rate of change.
- To evaluate an **average** rate of change as the **gradient** of the **secant** in both practical and purely mathematical situations.
- To recognise **instantaneous** rate of change as the **gradient** of the **tangent**.
- To understand simple concepts of **limit**.
- To define the **derivative** of a function at a point.
- To interpret the **derivative** of a polynomial at a point as the **instantaneous** rate of change and the **gradient** of the **tangent**.
- To develop and use the following **rules** for **differentiation** in both practical and purely mathematical situations.

$$\frac{d}{dx}(x^n) = n\,x^{n-1} \text{ for rational } n$$

$$\frac{d}{dx}(k\,f(x)) = k\,f'(x),\ k = \text{constant}$$

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

In Chapters 1 to 8, when the relationship between two variables was considered, the idea that one variable, y, for example, is a function of another variable, such as x, has been used and developed. The variable x is usually considered to be the **independent** variable and y the **dependent** variable. The value of y **depends** on the value of x and is found by applying the rule connecting the two variables to the given value of x. So the rule that connects x and y enables the value of y to be determined for a given value of x. For example, for the rule $y = x^2 - 3x + 5$, if $x = 3$, then $y = 5$.

Furthermore, such relationships have been represented graphically and key features, such as axis intercepts, turning points and asymptotes, have been considered.

Analysis of the graphical and tabular representation also can be used to see how that relationship is changing. A graphics calculator's multi-modal presentation is important technology in facilitating this understanding.

How the relationship is changing is of critical importance in establishing how accurately a given rule models the relationship between the variables in question. As changes in values of x are considered, the question of how the corresponding values of y are changing arises. If, as x, for example, increases, does y also increase, or does it decrease or, in fact, remain unaltered? And if it does change, does it do so consistently, quickly, slowly, indefinitely, and so on?

A connection can be made between the manner and rate at which a function is changing and the slope or gradient of the graph representing the function.

9.1 Rate

A **rate** is a **ratio** between two related quantities.

For example, height/time (metre/year), persons/area (number/hectare), distance/time (kilometre/hour).

Example 1

A car uses 20 litres of fuel to travel 250 km. Evaluate:

a the number of kilometres per litre

b the amount of fuel used for each kilometre travelled

c the amount of fuel used per 100 km

Solution

a
$$\begin{aligned}\text{rate (km/L)} &= \frac{250}{20}\\ &= 12.5\end{aligned}$$

b
$$\begin{aligned}\text{rate (L/km)} &= \frac{20}{250}\\ &= 0.08\end{aligned}$$

c
$$\begin{aligned}\text{rate (L/100 km)} &= \frac{20}{2.5}\\ &= 8\end{aligned}$$

Example 2

A student saves \$2000 in the 40 weeks of Year 11. Calculate:

a the rate of saving (\$/week)

b the rate of time for each dollar saved (week/\$)

c the rate of saving (\$/month)

Solution

a $$\text{Rate } (\$/\text{week}) = \frac{2000}{40} = 50$$

b $$\text{Rate } (\text{week}/\$) = \frac{40}{2000} = 0.02$$

c $$\text{Rate } (\$/\text{month}) = \frac{2000}{(40 \div 4)} = 200 \quad \text{(using the approximation of 4 weeks per month)}$$

Example 3

Convert the rate of distance/time (speed) of 100 km/hour to:

a metres per second

b miles/hour, given 1 mile = 1.61 km

Solution

a $$\frac{100\text{ km}}{\text{hour}} = \frac{10\,000\text{ m}}{3600\text{ s}} = 27.\dot{7}\text{ m/s}$$

b $$\frac{100\text{ km}}{\text{hour}} = \frac{100/1.61\text{ miles}}{\text{hour}} = 62.1\text{ miles/h}$$

Example 4

Evaluate the following changes:

a the change in height from 1.60 m to 1.83 m

b temperature change from 23° C to 14° C

Solution

a $$\begin{aligned}\text{Change} &= \text{Final} - \text{Initial}\\ &= 1.83 - 1.60\\ &= 0.23\text{ m}\end{aligned}$$

b $$\begin{aligned}\text{Change} &= \text{Final} - \text{Initial}\\ &= 14 - 23\\ &= 9^\circ\text{ C}\end{aligned}$$

Change = Final state − Initial state

Exercise 9A

Examples 1,2 **1** A tree grows 2 m in 5 years. Evaluate the tree's rate of growth.

Example 3 **2** Convert a rate of 2 millilitres per second to an equivalent rate of litres per minute.

Examples 1,2 **3** A chef prepares 48 meals in 4 hours. Evaluate:

- **a** his work rate (meals/hour)
- **b** the time required for each meal (minute/meal)
- **c** the time required to prepare meals for a table of 10, given he works at the same rate

Example 3 **4** Given AU\$1 $\equiv$ UK £0.40, evaluate:

- **a** the exchange rate UK£/AU\$
- **b** the exchange rate AU\$/UK£
- **c** AU\$500 in UK£ (to 2 decimal places)
- **d** UK£500 in AU\$ (to 2 decimal places)

5 A six-wheel light truck wears out eight tyres over 72 000 km. Calculate:

- **a** the number of tyres per 100 000 km
- **b** the rate of tyre wear (km/tyre)

6 A farmer uses 5 tonnes of fertiliser per 40 hectares. Evaluate:

- **a** the rate of fertiliser use (tonne/hectare)
- **b** the rate of fertiliser 'effective coverage' (hectare/tonne)
- **c** the quantity of fertiliser required for a further 150 hectares if the farmer decides this soil requires a fertilisation rate of only 50%

Example 4 **7** The initial rate of influenza infections in a school population of 1200 is 16.5 infections per 100 persons. After 2 weeks the epidemic has spread to an infection rate of 25.0 infections per 100 persons. Determine:

- **a** the change in the rate of infection
- **b** the change in the number of persons infected

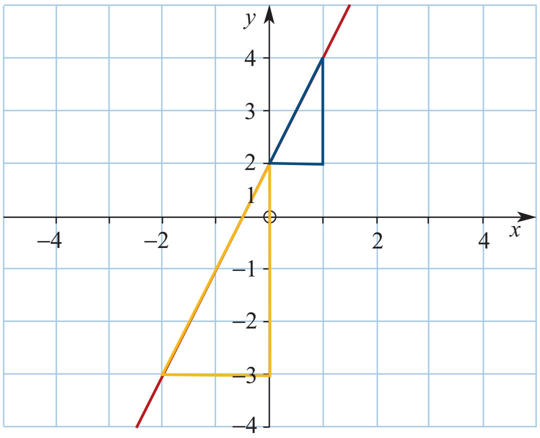
9.2 Constant rate of change

A **rate of change** between two related quantities is the **ratio** of the **change** in the **dependent** quantity (vertical or y-axis) to its matching **change** in the **independent** quantity (horizontal or x-axis).

For example:

$$\begin{aligned}\text{Rate of change of height (growth)} &= \text{Change in height/Change in time}\\ &= (\text{Final height} - \text{Initial height}) / (\text{Final time} - \text{Initial time})\end{aligned}$$

Any function that is **linear** will have a **constant** rate of change. The **ratio** is **constant** irrespective of the size or location of the change.



That rate of change is simply the **gradient** of the graph. The gradient of a linear function is the rate of change of the dependent variable with respect to the independent variable.

If the function is stated as $y = mx + c$, then the gradient is m and, thus, the rate of change is m.

Example 5

A car travels from Copahunga to Charlegum, a distance of 150 km, in 2 hours (i.e. 120 minutes). If the car travels at a constant speed, draw a distance–time graph and calculate the rate of change of distance over time (speed).

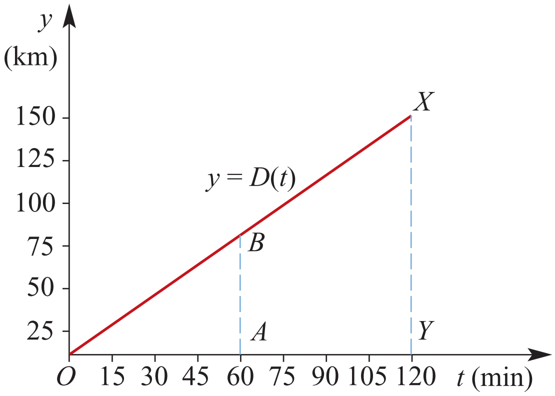
Solution

We denote the function by D. The graph of the function is shown below.

The rule of the function may be written as:

$$D(t) = \frac{150}{120}t = \frac{5}{4}t$$

Note that $\frac{XY}{YO} = \frac{BA}{AO} = \frac{5}{4}$



The gradient of the graph gives the speed in kilometres per minute.

Therefore, the speed of this car is $\frac{5}{4}$ kilometres per minute (i.e. $\frac{5}{4}$ km/min).

This speed may be expressed in kilometres per hour (km/h).

$$\text{Speed (km/h)} = \frac{5}{4} \times 60 = 75$$

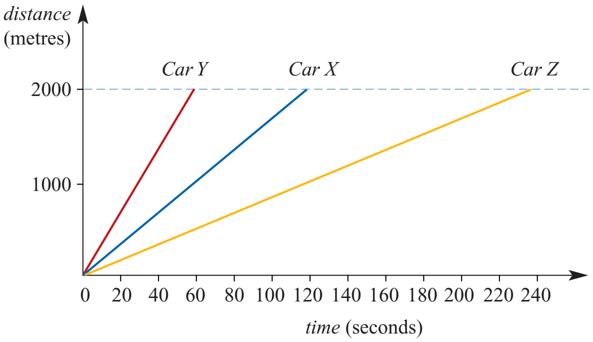
Example 6

Three cars are driven over a 2-kilometre straight track. They all start at point A and finish at point B. Each car travels with constant speed. It is not a race as:

- the speed of car Y is twice that of car X, and
- the speed of car Z is half that of car X.

Assume that car X travels at 1 km/min. Illustrate this situation with a distance–time graph.

Solution



Note:

Car X

The gradient of the graph for car X is $\frac{2000}{120} = 16\frac{2}{3}$

$\therefore$ Rate of change of distance over time (speed) $= 16\frac{2}{3}$ (m/s)

The linear function is $d(t) = 16\frac{2}{3}t$

Car Y

The gradient of the graph for car Y is $\frac{2000}{60} = 33\frac{1}{3}$

$\therefore$ Rate of change of distance over time (speed) $= 33\frac{1}{3}$ (m/s)

The linear function is $d(t) = 33\frac{1}{3}t$

Car Z

The gradient of the graph for car Z is $\frac{2000}{240} = 8\frac{1}{3}$

$\therefore$ Rate of change of distance over time (speed) $= 8\frac{1}{3}$ (m/s)

The linear function is $d(t) = 8\frac{1}{3}t$

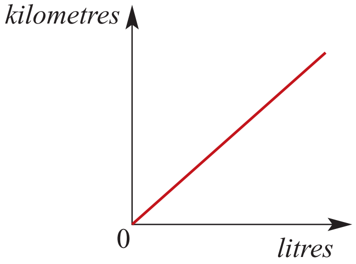
An object whose motion may be described by a linear distance–time graph is travelling at a constant speed, which is represented by the gradient of the linear graph.

There are many other examples, as well as constant speed motion, where a real-life situation is usefully modelled by a straight-line graph in such a way that the gradient of the graph is meaningful.

In all these situations, the **gradient** of the straight-line graph represents a **rate**. For example:

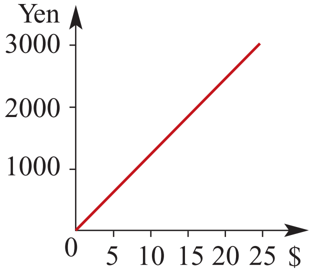
i *Petrol consumption of a car*

A straight-line graph has been used as a model, and its gradient represents the rate of consumption of petrol, which is measured in km per litre. Such a model makes fairly large assumptions. What are these?



ii *The exchange rate for currencies*

The gradient of the graph gives the exchange rate for Australian dollars to Japanese yen (i.e. yen/\$).



Example 7

The height of water stored in a leaking cylindrical tank of radius 5 metres is dropping at the constant rate of 2 cm/day. Given that the initial water height of the full tank was 4 metres:

a Plot the height h (cm) as a function of time t (days).

b Determine the function $h(t)$.

c Calculate the rate of change of stored water volume (m^3/day).

d Convert your answer to part **c** to a rate in litres per hour (i.e. L/h).

Solution

a

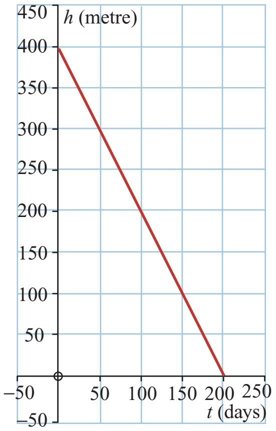
$$\text{Rate} = \frac{\text{Change in height}}{\text{Change in time}}$$

$$-2 = (0 - 400)\,/\,(t - 0)$$

$$t = 200$$

i.e. The tank will be empty in 200 days.

Plot and join intercept points (0, 400), (200, 0).



b

$$\frac{h - h_1}{t - t_1} = m$$

$$\frac{h - 400}{t - 0} = -2$$

$$h\,(t) = -2t + 400$$

c Volume change/day

$$= -\pi r^2 h$$
$$= -\pi \times 5^2 \times 0.02$$
$$\approx -1.57$$
$$\therefore \quad \text{Rate} \approx -1.57\,(\text{m}^3/\text{day})$$

d Rate $= -1.57\,(\text{m}^3/\text{day})$

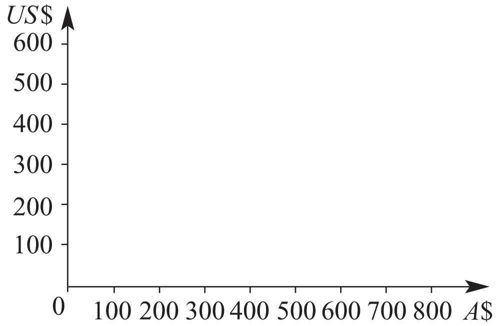
$$= -1.57 \times 1000/24\,(\text{L/h})$$
$$\approx 65.4\,(\text{L/h})$$

Exercise 9B

Example 5

1 A car travels from Mumbai to Poona a distance of 200 km at a constant speed. The journey takes 150 minutes. Draw a distance–time graph and calculate the speed.

2 The exchange rate for the Australian dollar in terms of the American dollar is AU\$1 = US\$0.80. Draw a straight-line graph that illustrates this relationship. The axes should be as shown.



Example 6

3 Assuming constant speed, find the speed for each of the following:

a Distance travelled 120 km, time taken 2 hours

b Distance travelled 60 m, time taken 20 seconds

c Distance travelled 8000 m, time taken 20 minutes

d Distance travelled 200 km, time taken 5 hours 40 minutes

e Distance travelled 6542 m, time taken 5 minutes 20 seconds

Example 7

4 Find the rate of flow of the following taps in litres/minute:

a A tap that fills a 40 litre drum in 5 minutes

b A tap that fills a 600 litre tank in 12 minutes

c A tap that takes 17 minutes to fill a 200 litre tank

d A tap that takes 17 minutes 20 seconds to fill a 180 litre tank

5 Water flows out of a tap at the rate of 15 litres/minute.

a Copy and complete this table, showing the total volume that has flowed from the tap at each time t:

Time (in minutes) t	0	0.5	1	1.5	2	3	4	5
Volume (in litres) V	0							

b Draw a graph from the table.

6 A worker is paid \$200 for 13 hours work. What is her rate of pay per hour?

7 An aircraft travelling at a constant speed took 24 seconds to travel 5000 metres. What is the speed of the plane in metres per second?

8 A spherical balloon is inflated such that its volume is increasing by a constant 8 cm^3 every second. Draw a sketch graph to show how the volume of the sphere changes with time.

9 A bank account records a credit of $5800 on 1st January, 2007. On 1st January, 2009 the account is overdrawn by $200. Evaluate the rate of change of the account's credit per year over this period.

10 A house painter's records show that she used 26 litres (L) of exterior paint to cover an area of 300 m^2 of cladding. Another similar job required 41 L for 480 m^2.

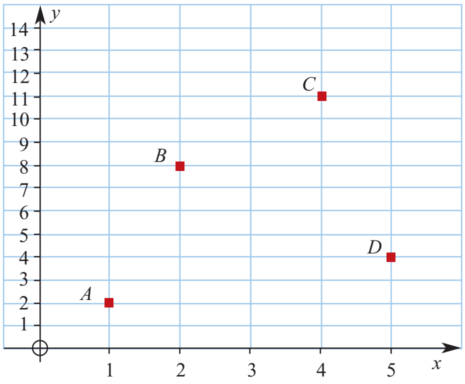
a Calculate the rate of coverage (m^2/L).

b Plot a graph of area A(m^2) vs capacity V(L).

c Interpret the plot's intersection with the V-axis.

d Establish the function $A(V)$.

11 Consider the plot shown below.



Determine the rate of change between points:

a A and B **b** B and C **c** C and D **d** A and D **e** D and A

12 In a 2-D Cartesian plane what is the rate of change of a line:

a parallel to the x-axis?

b parallel to the y-axis?

13 An electricity account reads as shown.

Tariff	Meter No.	Reading 30/05	Reading 31/08	kWh	Days	GST	Cost (excl. GST)
11	092371	66979	67805	826	93	11.61	$127.66

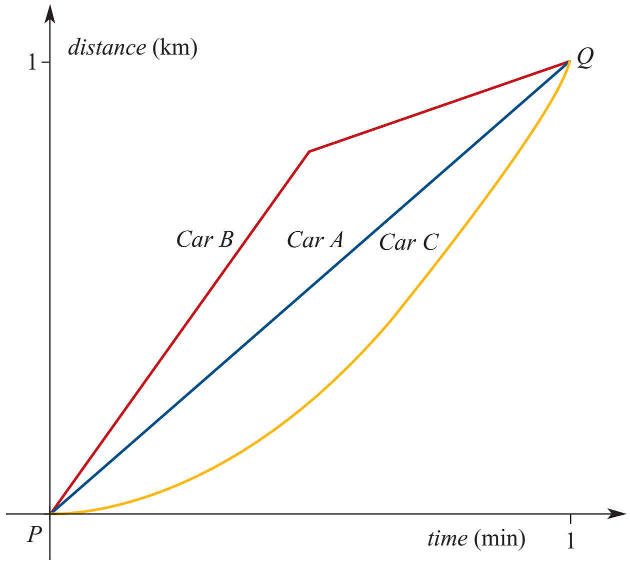
a Evaluate the cost rate ($/kWh).

b Evaluate the rate of change of electricity used over the period shown.

9.3 Non-constant rate of change and average rate of change

Constant speed is not the only form of motion for an object. Many moving objects do not travel with constant speed. For example, the speedometer of a car travelling in city traffic rarely stays still (constant) for long. The gradient of the car's distance versus time graph and, therefore, the rate of change is not constant. The function is not linear.

Consider cars A, B and C travelling from location P to location Q (1 km). All three cars start at P at the same time and arrive at Q at the same time, 1 minute later.



Car A travels at a constant speed of 1 km/min. The speed of cars B and C vary, as shown by the graph. Nevertheless, all three cars have travelled distance PQ (1 km) in the same time of 1 minute.

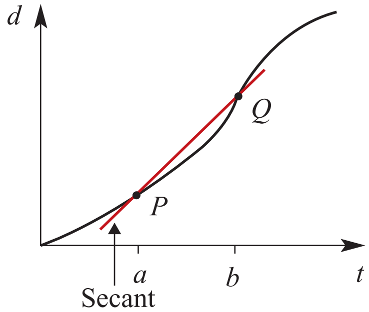
Cars B and C have non-constant rates of change or speed over the journey, being sometimes greater and sometimes less than car A's constant speed of 1 km/min.

Cars B and C *average* speed is 1 km/min.

$$\text{The average speed is given by } \frac{\text{Distance } PQ}{\text{Time taken}} = \frac{1\text{ km}}{1\text{ minute}}$$
$$= 1\text{ km/min}$$

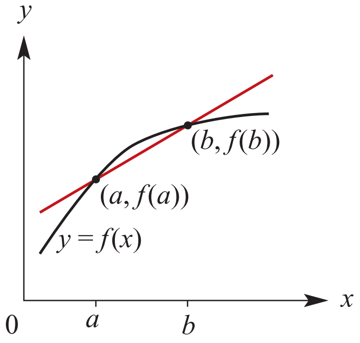
The average speed of cars B and C is the gradient of the **line** PQ representing car A.

In general, average speed $= \dfrac{\text{Total distance travelled}}{\text{Total time taken}}$, so the average speed of an object for $a \leq t \leq b$ is given by the gradient of the chord PQ or the secant PQ, which is the *line* passing through points P and Q.



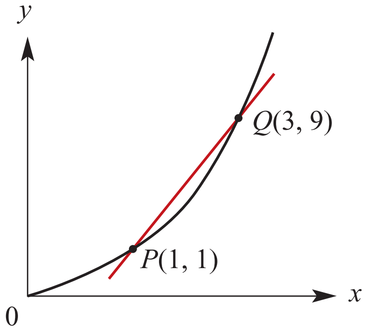
In general, for any function with rule $y = f(x)$, the **average rate of change** of y with respect to x over the interval $a \leq x \leq b$ is the gradient of the chord joining $(a, f(a))$ to $(b, f(b))$.

i.e. $$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}$$



For example, the average rate of change of y with respect to x over the interval $1 \leq x \leq 3$ for the function graphed is given by the gradient of

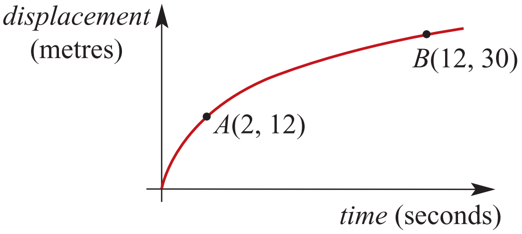
$$PQ = \frac{9 - 1}{3 - 1} = 4$$



Example 8

The displacement (metres) against time (seconds) graph for the motion of an object is shown below.

Find the average speed of the object in m/s over the interval from $t = 2$ to $t = 12$.



Solution

$$\begin{aligned}\text{Average speed} &= \frac{\text{Total distance travelled}}{\text{Total time taken}}\\ &= \frac{30 - 12}{12 - 2}\\ &= \frac{18}{10}\\ &= 1.8 \text{ m/s}\end{aligned}$$

Example 9

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{Change in } y}{\text{Change in } x}$$

$$f\ (1) = (1)^2 - 2\,(1) + 5 = 4$$

$$f\ (5) = (5)^2 - 2\,(5) + 5 = 20$$

$$\begin{aligned}\text{Average rate of change} &= \frac{20-4}{5-1}\\ &= 4\end{aligned}$$

Example 10

The air temperature T (° C) at a weather station on a particular day is modelled by the equation $T = \dfrac{600}{t^2 + 2t + 30}$, where t is the time after 6.00 p.m.

a Find the temperature at 6.00 p.m.

b Find the temperature at midnight.

c Find the average rate of change of the air temperature from 6.00 p.m. till midnight.

Solution

a At 6.00 p.m., $t = 0$.

Hence, $T = \dfrac{600}{(0)^2 + 2(0) + 30} = 20^\circ$ C

b At midnight, $t = 6$.

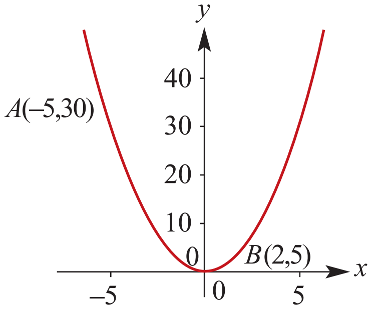
Hence, $T = \dfrac{600}{(6)^2 + 2(6) + 30} \approx 7.69^\circ$ C

c Average rate of change of temperature $\approx \dfrac{7.69 - 20}{6 - 0} = -2.05^\circ$ C/h

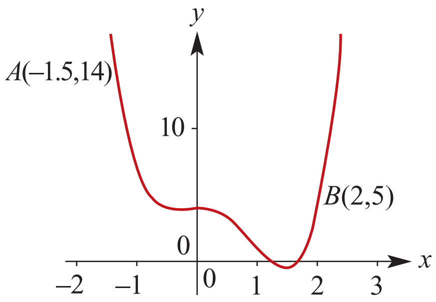
Exercise 9C

1 Find the average rate of change of y with respect to x from point A to point B for each of these graphs.

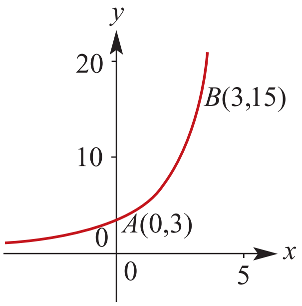
a



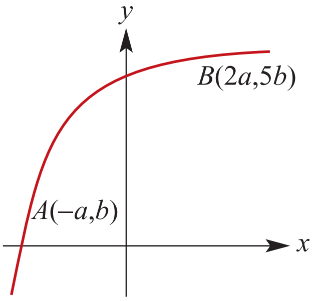
b



c



d



2 For each of these functions, find the average rate of change over the stated intervals.

Example 8

a $f(x) = 2x + 5,\ 0 \le x \le 3$ **b** $f(x) = 3x^2 + 4x - 2,\ 0 \le x \le 2$

c $f(x) = \dfrac{2}{(x-3)} + 4,\ 4 \le x \le 7$ **d** $f(x) = \sqrt{5 - x},\ 0 \le x \le 4$

Example 9

3 The displacement, S m, of an object, t seconds after it starts to move is given by the function $S(t) = t^3 + t^2 - 2t,\ t > 0$.

Find the average rate of change of displacement (speed) of the object:

a in the first 2 seconds **b** in the next 2 seconds

Example 10

4 A person invests \$2000 dollars, which will increase in value by 7% per year for the next 3 years.

a Calculate the value of the investment after 3 years.

b Calculate the average rate of change in the value of the investment over that time.

Example 9 **5** The depth, d cm, of water in a bath tub, t minutes after the tap is turned on is modelled by the function

$$d(t) = \frac{-600}{(2t + 12)} + 50, \ t > 0$$

Find the average rate of change of the depth of the water in the tub over the first 10 minutes after the tap is turned on.

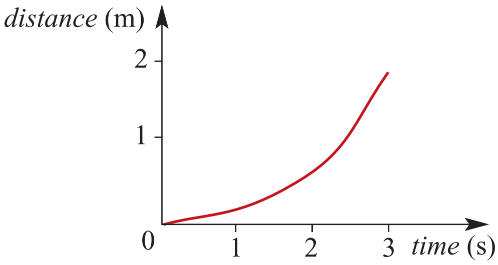
Example 8 **6** In the graph, the average speed from $t = 0$ to $t = 3$ is:

a 2 m/s

b 1 m/s

c $\frac{2}{3}$ m/s

d $1\frac{1}{2}$ m/s



7 Car 1 and car 2 both start together and travel with constant speed over a 1 kilometre straight track.

Car 1 has speed 60 km/h; car 2 travels at $\frac{3}{4}$ of this speed.

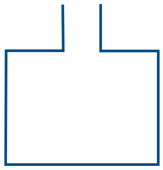
Illustrate this situation with distance–time graphs for both cars on the one set of axes.

9.4 Recognising relationships

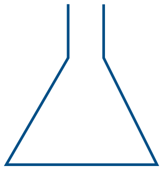
In previous chapters polynomial, exponential, logarithmic and periodic functions have been considered and it has been shown how many real-life situations may be modelled by these functions. We now consider several real situations concerning two particular variables and, in particular, recognising the form of the relationships between the variables through graphs. The algebraic relationship is not established.

Example 11

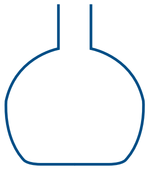
a



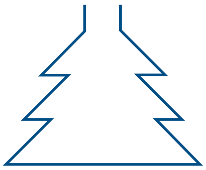
b



c



d

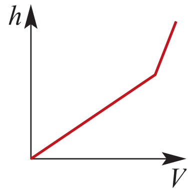


Water is being poured steadily into each of these vessels.

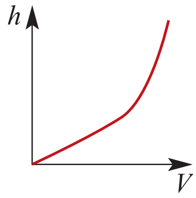
Draw a graph that shows the relationship between the height of the water and the volume that has been poured in.

Solution

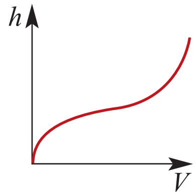
a



b



c



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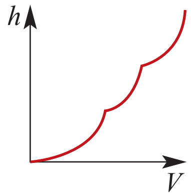
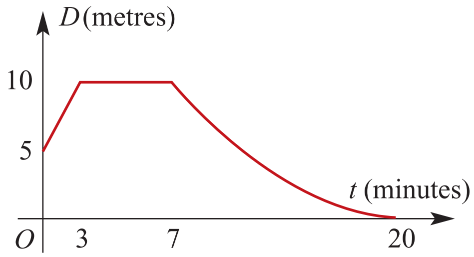


Example 12

The graph opposite shows the displacement, D metre of a particle from a fixed point O, over a period of 20 minutes.
Describe the motion of the particle.

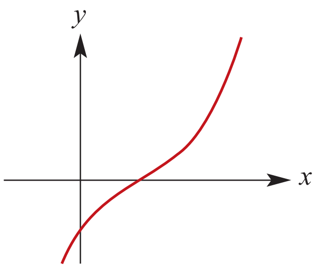


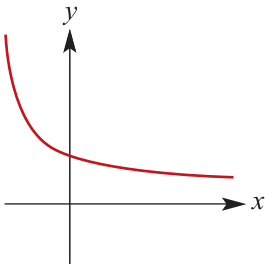
Solution

The object is initially 5 m from point O. It travels away from O for 3 minutes at a constant speed of $\frac{5}{3}$ m/min. It then remains stationary at a distance of 10 m from O for 4 minutes before returning to O at a speed that is gradually decreasing, until it comes to rest at O at time $t = 20$ minutes.

By examining the graph representing a function, it can be determined whether the rate of change is positive, negative or in fact neither.

If the graph of y against x for a function with rule $y = f(x)$, for a given domain shows an increase in y as x increases, it can be said that the **rate of change** of y with respect to x is **positive** for that domain. It is also said that the function is increasing for that domain. The *slope* of the curve is positive.





If the graph of y against x for a function with rule $y = f(x)$, for a given domain shows a decrease in y as x increases, it can be said that the **rate of change** of y with respect to x is **negative** for that domain. It is also said that the function is decreasing for that domain. The *slope* of the curve is negative.

If y remains the same value as x changes, the corresponding graph is a horizontal line and the rate of change of y with respect to x is said to be **zero** (refer to Ex. 9B, Q12a).

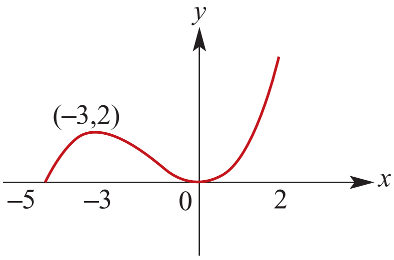
All of this is consistent with linear functions discussed in Chapter 1.

Example 13

Given $-5 \leq x \leq 2$, describe the domain for which:

a the value of y is increasing as x is increasing

b the rate of change of y with respect to x positive



Solution

a The value of y is increasing for $-5 \leq x < -3, 0 < x \leq 2$.

b The rate of change of y with respect to x is positive for $-5 \leq x < -3, 0 < x \leq 2$.

Exercise 9D

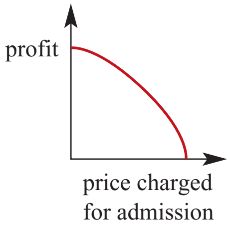
*For questions **1–7**, there may not be single correct answers. Your written explanations are an important part of the exercise.*

1 The manager of a theatre wishes to know what effect changing the price of admission will have on the profit he makes.

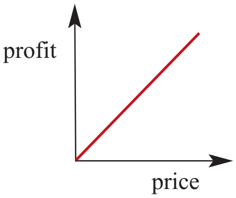
a Which one of the following graphs would show the effect of change?

b Explain your choice, including comments on scales, axes and what the point of intersection of the axes represents.

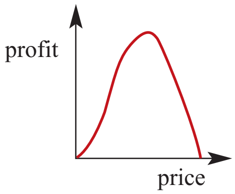
a



b



c



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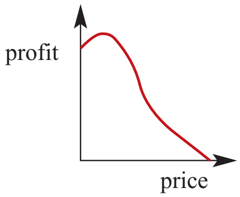
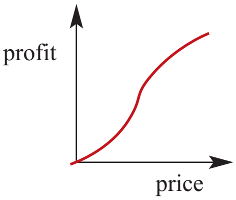
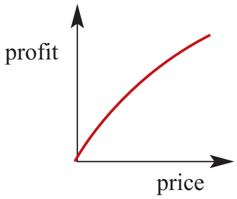


e



f

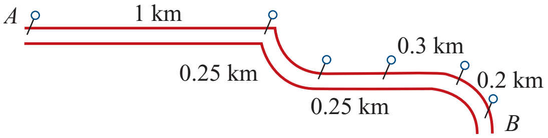


2 Sketch a graph to show how the height of a person might vary with age.

Example 12

3 A motorist starts a journey at the point marked *A* on a country road, drives 2 km along the route shown, and stops at a point marked *B*. He is able to drive at 100 km/h but must slow down at corners.

- **a** Explain briefly how the car's speed varies along the route.
- **b** Sketch a graph showing how the car's speed varies along the route.

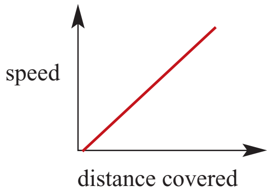


(Label the vertical axis 'Car's speed', the horizontal axis 'Distance travelled from *A*'.)

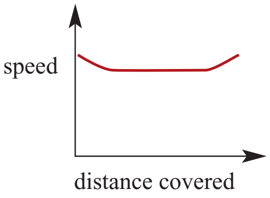
4 An athlete is a competitor in a 10 000 m race. Below are a selection of graphs that could show the relationship between the speed of the runner and the distance covered.

- **a** Explain the meaning of each graph in words.
- **b** Which graph is the most realistic for a winning athlete?

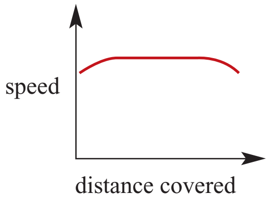
a



b



c



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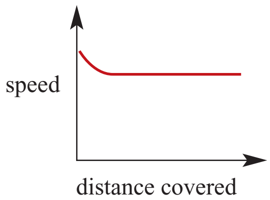

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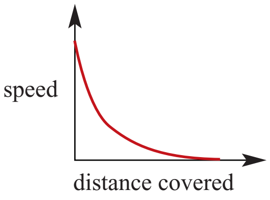

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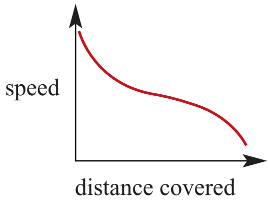


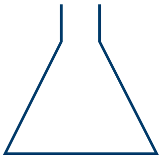
If you do not think any of these graphs are realistic draw your own version and explain it fully.

5 A sprinter covers 100 metres at a constant speed of 10 m/s. Sketch:

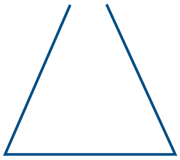
a the distance–time graph **b** the speed–time graph

Example 11 **6** Water is being poured steadily into these vessels.

a



b



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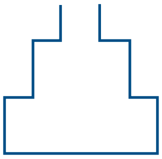
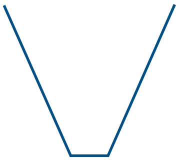
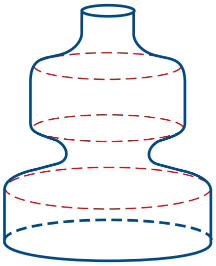


d

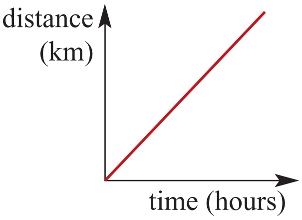


Draw a graph that shows the relationship between the height of the water and the volume that has been poured in.

7 For the vessel shown, sketch a reasonable curve for the volume, V, of water in the vessel as a function of the height, h, of the water in the vessel.



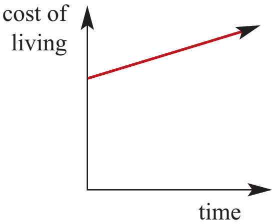
8 The graph relating the distance a car travels to the time taken is a straight line, as shown.
The graph shows that the car is:



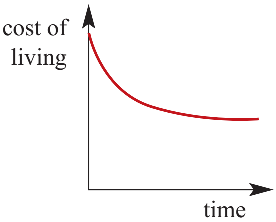
- **a** speeding up
- **b** slowing down
- **c** travelling uphill
- **d** travelling at a constant speed
- **e** stationary

9 Which one of these graphs best represents the rate of cost of living slowing down?

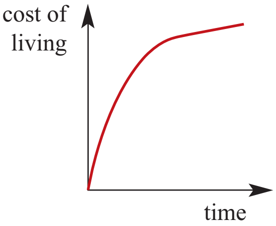
a



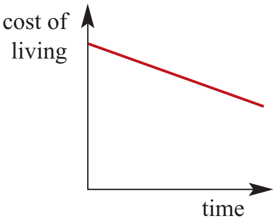
b



c



d

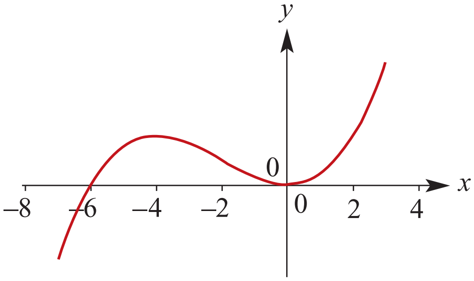


10 In the graph shown below for $-7 \leq x \leq 3$, determine the domain for which:

a the value of y is increasing as x is increasing

Example 13

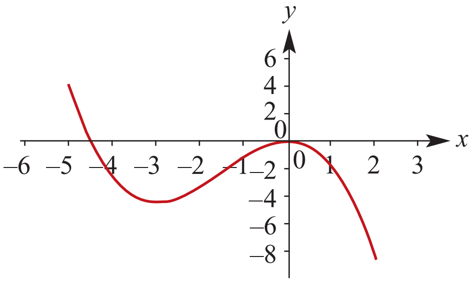
b the rate of change of y with respect to x is positive



11 In the graph shown below for $-5 \leq x \leq 2$, determine the domain for which:

a the value of y is decreasing as x is increasing

b the rate of change of y with respect to x is negative



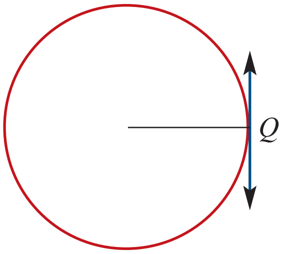
9.5 Instantaneous rate of change

In Section 9.3 the average rate of change of a function over a stated domain has been considered. It has also been established that, in general (except for linear functions), the rate of change of a function is different at each different point on the graph of the function. If we are to fully analyse how a function is behaving, we need to establish the rate of change of the function at every given point.

Defining tangent

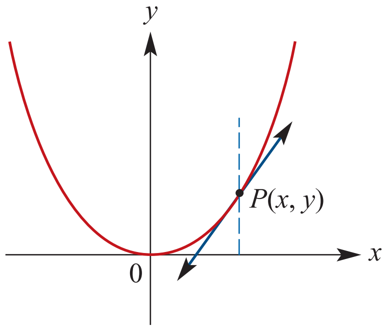
For a circle, a **tangent** is a line that intersects the circle at one and *only* one point.

There are other curves for which tangency can be defined in this way, for example, an ellipse. However, for some very simple curves, the tangent cannot be described by the definition that we use to describe tangents to circles.



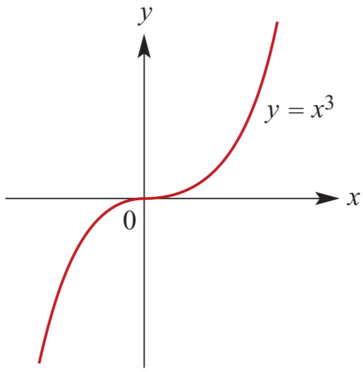
Consider, for example, a parabola.

The tangent at P intersects the graph only at (x, y). But the line parallel to the y-axis through P also has this property. We may try to get around this problem by indicating that the tangent line must only touch the curve, but does not cross it.



For many curves, however, this is not true.

The tangent to this curve at the point (0, 0) is the x-axis, and this certainly crosses the curve.

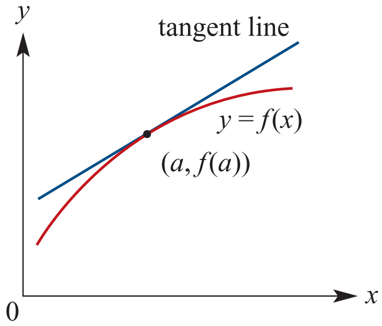


To define tangent adequately for all curves demands mathematics that we will meet later. For the present we state that the tangent at a point P of a curve is a line through P that has a gradient that is the same as the gradient of the curve at P.

The rate of change at a given point can be determined by finding the gradient of the tangent at that point.

We refer to this rate of change at a specific point as the **instantaneous rate of change**.

If $y = f(x)$, then the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Initially, the instantaneous rate of change at a given point will be estimated by drawing the tangent at the specified point and finding the gradient of the tangent drawn. Any such attempt to draw a tangent by hand is unlikely to be very accurate, so this technique will provide an approximation only.

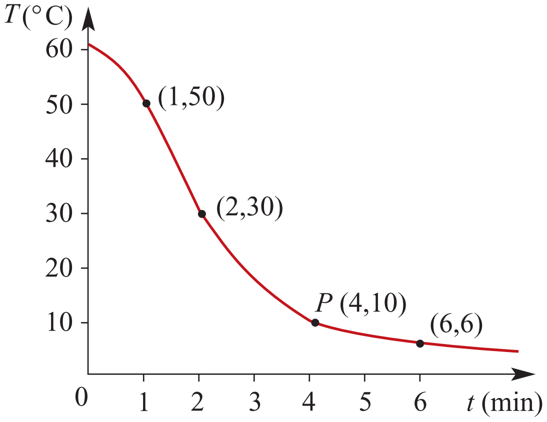
Example 14

This graph represents the temperature T (° C) of a kettle at time t (min) after taking it from the heat and putting it outside.

a What does the gradient represent?

b Find the average rate of change of temperature between $t = 2$ and $t = 6$.

c Draw the tangent at P and, hence, find an approximate value for the rate of hange at $t = 4$.

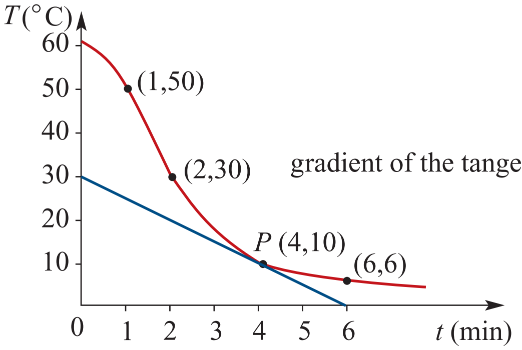


Solution

a The rate of increase of temperature with respect to time.

b
$$\text{Average rate of change} = \frac{6 - 30}{6 - 2} = \frac{-24}{4} = -6° \text{ C per minute}$$

c

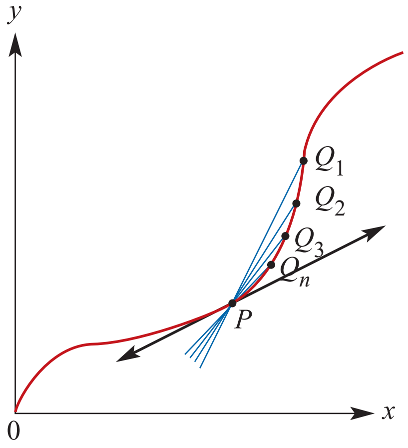


$$\text{gradient of the tangent at } P = \frac{-30}{6} = -5$$

Another technique that may be used to find an approximation for the instantaneous rate of change is to use the gradient of a chord from the point in question, P, to another point on the curve, Q, that is very close to P.

So the gradient of the chord PQ gives us an approximation to the gradient of the tangent at P, which is, of course, equal to the gradient of the curve at P.

We can consider chords $PQ_1,\ PQ_2,\ \ldots,\ PQ_n, \ldots$ As the point Q_n is chosen 'closer' to P, it can be seen that the gradient of the chord PQ_n is 'closer' to the gradient of the tangent at P. The gradient of the curve at P is the gradient of the tangent at P.

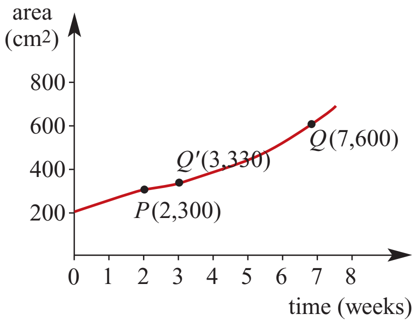


Example 15

The graph represents the area covered by a spreading plant. Area is measured in cm^2 and time in weeks.

a Find the gradient of the chord PQ.

b Q' has coordinates (3, 330). Find the gradient of PQ' and, hence, give an approximate value of the gradient of the graph at P.



Solution

a Gardient of chord $PQ = \dfrac{600 - 300}{7 - 2}$

$$= \frac{300}{5}$$

$$= 60$$

i.e. the average rate of increase of area for the plant from $t = 2$ to $t = 7$ is 60 cm^2 per week.

b Gardient of $PQ' = \dfrac{330 - 300}{1}$

$$= 30$$

$\therefore$ Gradient at P is approximately 30.

i.e. the rate of change of area with respect to time is approximately 30 cm^2 per week when $t = 2$.

Example 16

Estimate the gradient of the curve $y = x^3 + 1$ at the point (2, 9).

Solution

Consider the points $P(2, 9)$ and $Q(2.01, (2.01)^3 + 1)$.

$$\begin{aligned}\text{The gradient of } PQ &= \frac{(2.01)^3 + 1 - 9}{2.01 - 2} \\ &= 12.0601\end{aligned}$$

An even better approximation may be made by choosing $P(2, 9)$ and $Q(2.001, (2.001)^3 + 1)$. Why?

Example 17

By considering the chord joining the points where $x = 3$ and $x = 3.1$, estimate (without accurate drawing) the gradient of the curve $y = 2^x$ at $x = 3$.

Repeat for $x = 3$ and $x = 3.001$.

Solution

When $x = 3$, $y = 8$.

When $x = 3.1$, $y = 8.5742$ (correct to 4 decimal places).

The gradient of the line joining (3, 8) to (3.1, 8.5742) = 5.7419.

An estimate of the gradient of $y = 2^x$ at $x = 3$ is 5.742.

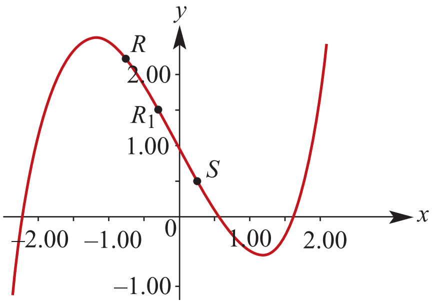
When $x = 3.001$, $y = 8.005\,547$ and the gradient = 5.547.

(*Note*: The true gradient (to 4 decimal places) is 5.5452.)

The method above of approximating the rate of change at a single point may be modified slightly.

The gradient of $y = 0.5x^3 - 2x + 1$ at the point (0, 1) will be investigated.

The graph of $y = 0.5x^3 - 2x + 1$ is shown.



First, find the gradient of RS where $R = (-0.75, 2.2891)$, $S = (0.25, 0.5078)$.

Gradient of $RS = -1.7813$ (values of R and S were found to 4 decimal places).

Consider a new chord R_1S where $R_1 = (-0.25,\ 1.4922)$ and $S = (0.25,\ 0.5078)$.

Gradient of $R_1S = -1.9688$.

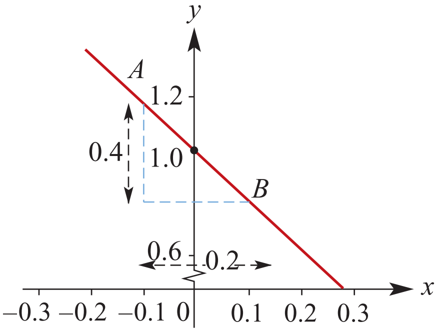
Zoom in on the region of the graph in the vicinity of $x = 0$, and it can be seen that the section of the graph appears increasingly linear. By assuming this section is in fact linear and finding the gradient of the section of the curve, an approximation to the gradient of the curve at $x = 0$ can be found.

Consider a chord AB where $A = (-0.1,\ 1.1995)$ and $B = (0.1,\ 0.8005)$.

A particular section of the graph around (0, 1) can be plotted.

From this section of graph it can be seen that the gradient of the section of the curve being considered is –2, and the final approximation can be made that the gradient of the curve $y = 0.5x^3 - 2x + 1$ is –2 at the point with coordinates (0, 1).

This method may be used even without sketching the graph.



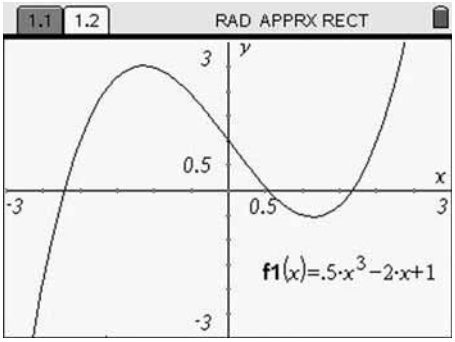
Using technology

A graphics calculator may be used in this process, as shown below.

Method A

Using the TI-Nspire:

1 Enter the equation into $f\,1(x)$, press enter and choose a suitable Window.

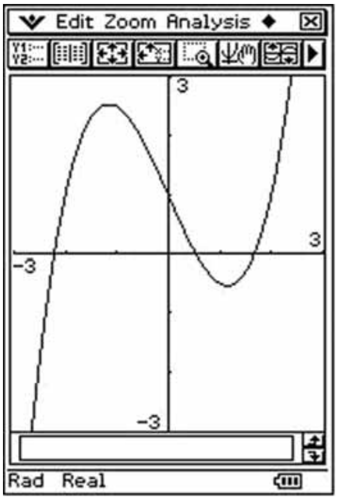


f1(x)=.5·x³−2·x+1

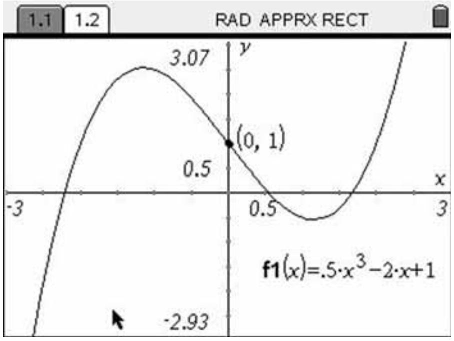
Method A

Using the ClassPad:

1 Enter the equation into $y1$, press EXE, tap and choose a suitable Window.



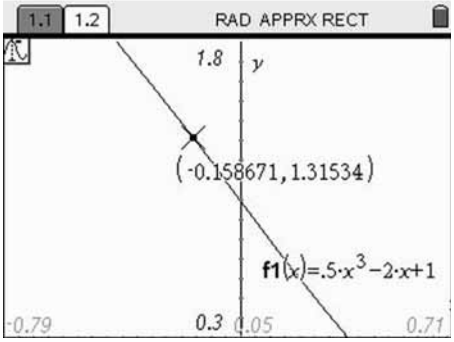
2 Use *Graph Trace* to take the cursor to the point (0, 1).

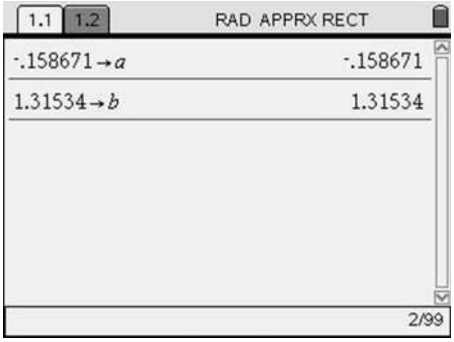


3 Choose **3:Zoom In** from the Window menu and press enter.

4 Take the cursor to the point (0, 1) and again choose **3:Zoom In** from the Window menu.

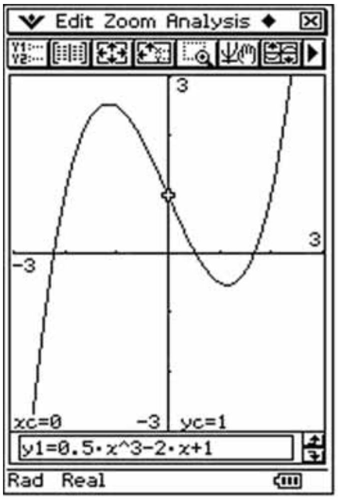
5 Choose two points on this section of the curve through (0, 1) and save the values as (A, B) and (C, D).





```
-.158671→a                     -.158671
1.31534→b                       1.31534
```

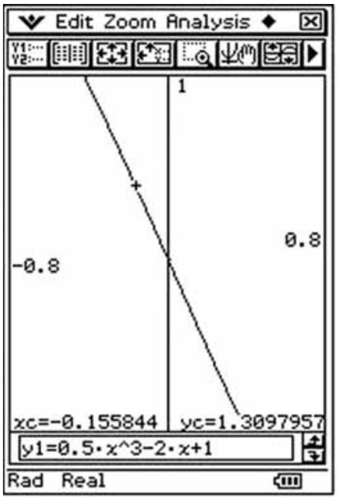
2 Use Trace to take the cursor to the point (0, 1).

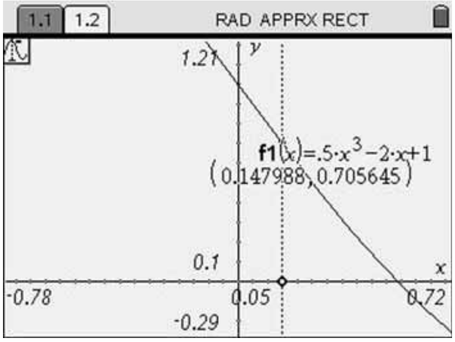


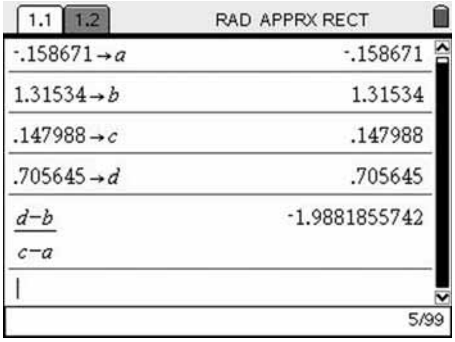
3 Tap **Zoom In** from the Zoom menu.

4 Take the cursor to the point (0, 1) and again tap **Zoom In** from the Zoom menu.

5 Choose two points on this section of the curve through (0, 1) and save the values as (A, B) and (C, D).

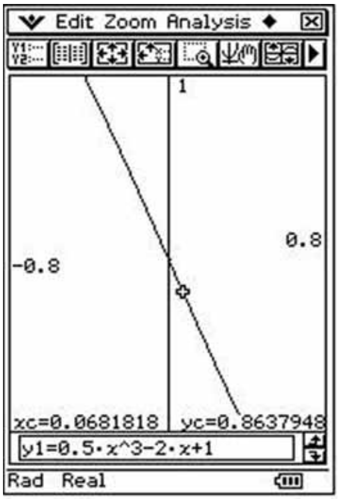


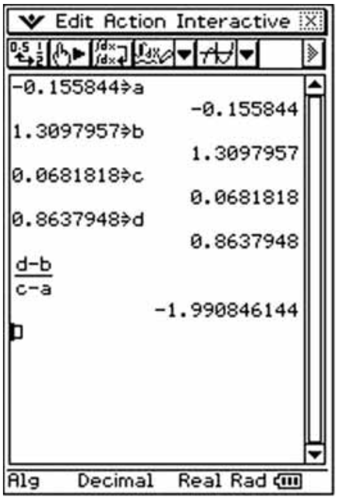




```
-.158671→a                   -.158671
1.31534→b                     1.31534
.147988→c                     .147988
.705645→d                     .705645
d−b                     -1.9881855742
c−a
```

```
-0.155844⇒a
                     -0.155844
1.3097957⇒b
                     1.3097957
```





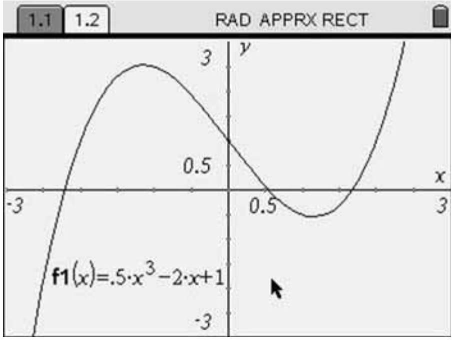
```
-0.155844⇒a
                     -0.155844
1.3097957⇒b
                     1.3097957
0.0681818⇒c
                     0.0681818
0.8637948⇒d
                     0.8637948
d−b
c−a
                  -1.990846144
```

Hence, the gradient is close to –2.

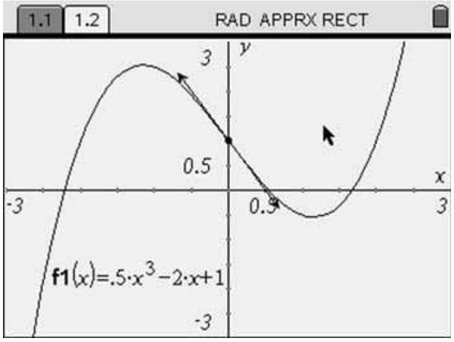
Method B

Using the TI-Nspire:

1 Display the plot $y = 0.5x^3 - 2x + 1$.

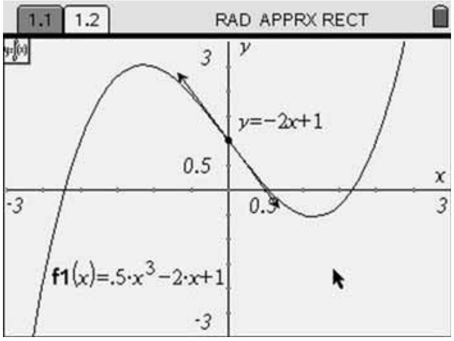


2 To draw a tangent line, select *Tangent* from the Points & Lines submenu. Move the cursor to the point (0, 1) and press enter to see the tangent.



3 To view the equation of the tangent line, select **6: Coordinates and Equations** from the Actions submenu.

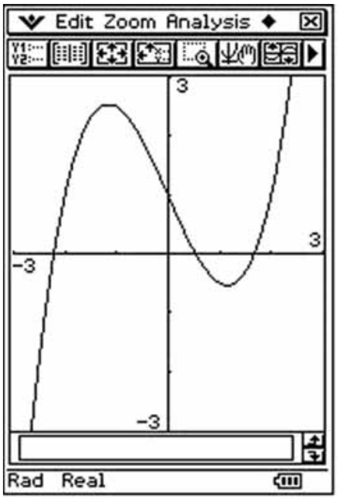
4 Move the cursor to the tangent line and press enter to display its equation.



Method B

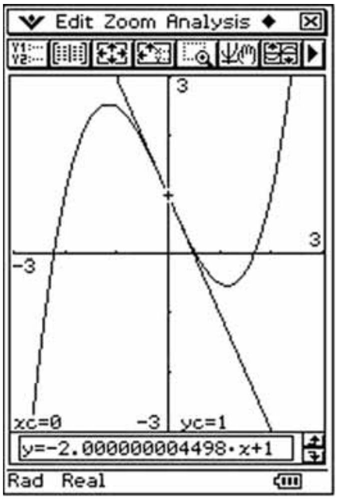
Using the ClassPad:

1 Display the plot $y = 0.5x^3 - 2x + 1$.



2 To draw a tangent line, tap Analysis, then select *Tangent* from the Sketch submenu.

3 Move the cursor to the point (0, 1) and then press EXE.



xc=0 yc=1

y=-2.000000004498·x+1

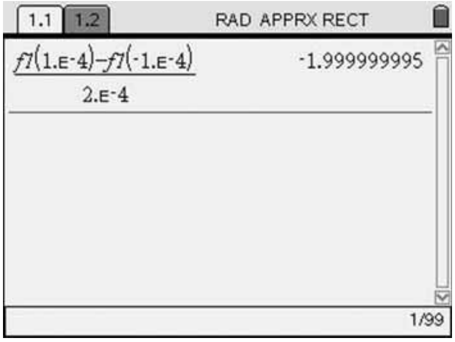
4 The equation of the tangent is displayed on the screen.

The gradient of the tangent is –2.

Alternatively, the rate of change may be approximated directly using the Calculator/Main application.

Method C

With the equation stored in $f1(x)$, type the following:



```
f1(1.E-4)-f1(-1.E-4)
--------------------   -1.999999995
       2.E-4
```

Method C

With the equation stored in $y1$, type the following:

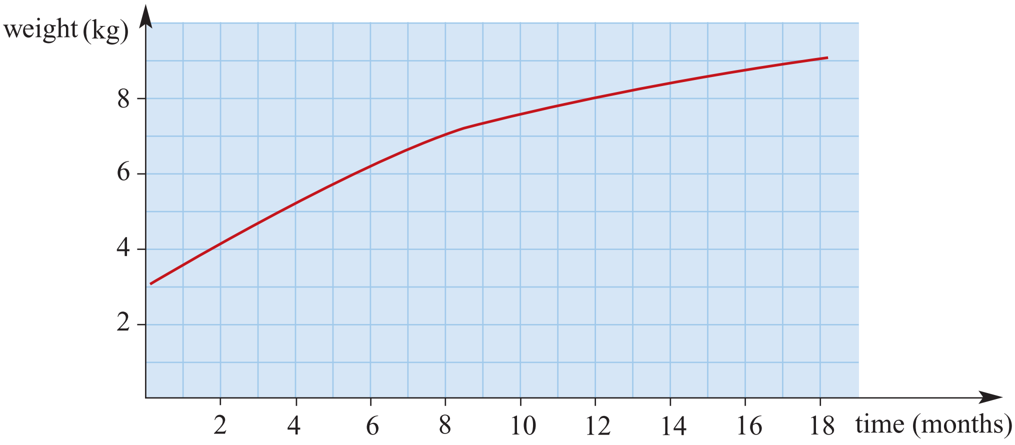
```
y1(0.0001)-y1(-0.0001)
       0.0002
              -1.999999993
```

Gradient $= -1.999999\ldots \approx -2$

Exercise 9E

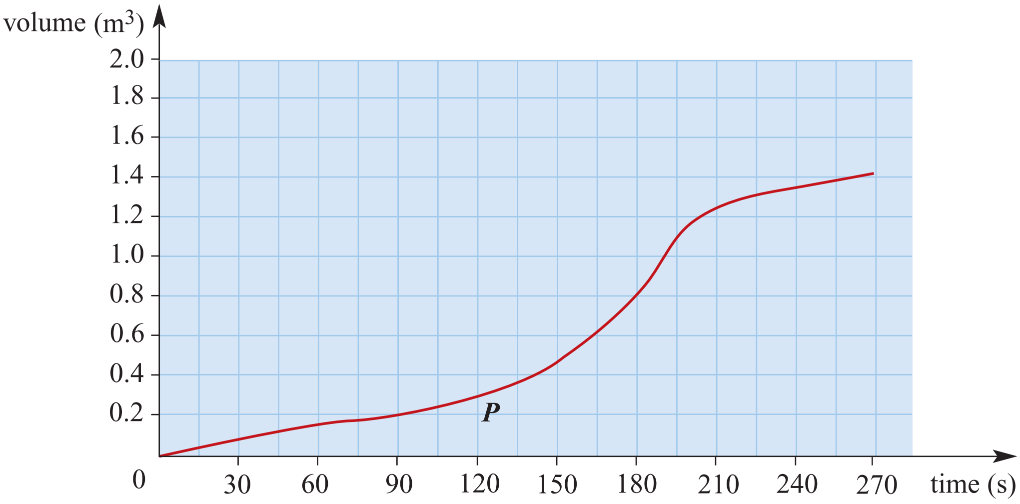
Example 14 **1** The graph shown below is the weight–time graph for the growth of a baby boy measured at 2-monthly intervals for the first 18 months of his life. By drawing a line at the appropriate points, estimate:

- **a** the average rate of growth of the baby over these 18 months
- **b** the rate of growth of the baby at 6 months
- **c** the rate of growth of the baby at 15 months



Example 14

2 A building is heated by gas. The volume–time graph for a 270-second period is shown in the graph:



a Place a rule along the tangent at P and, hence, find the rate of use of gas at time $t = 120$.

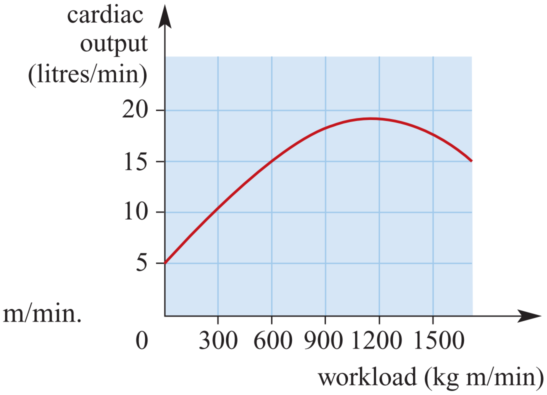
b Find the rate of use at $t = 160$.

c Find the rate of use at $t = 220$.

3 Cardiac output is an important factor in athletic endurance. The graph shows a stress–test graph of cardiac output (measured in litres/min of blood) versus workload (measured in kg m/min).

a Estimate the average rate of change of cardiac output with respect to workload as the workload increases from 0 to 1200 kg m/min.

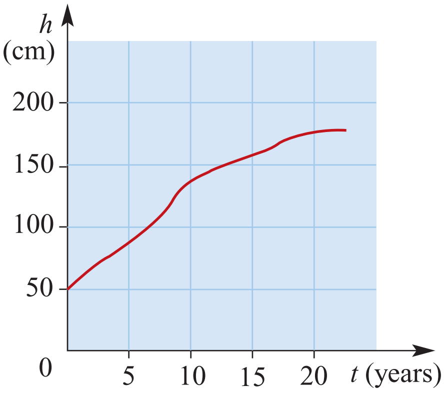
b Estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 450 kg m/min.



4 The graph of an individual's height h (cm) versus t (years) from some time after birth to age 20 is shown.

a When is the growth rate greatest?

b Estimate the growth rate after 5 years.

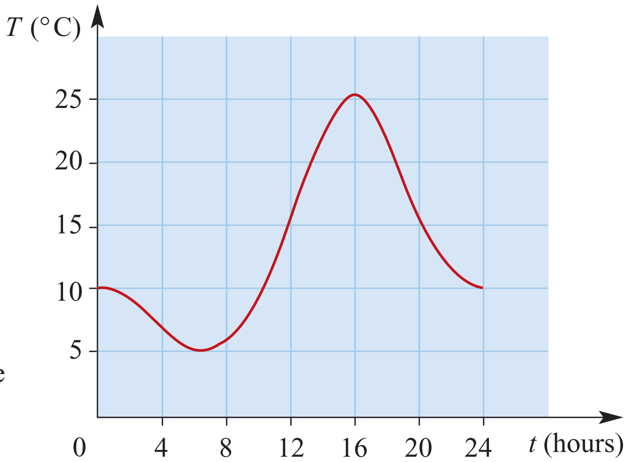


5 Temperature (T ° C) varies with time (t hours) over a 24-hour period, as illustrated in the graph.

a Estimate the maximum temperature and the time at which this occurs.

b The temperature rise between 10.00 and 14.00 is approximately linear. Estimate the rate at which the temperature is increasing in this period.

c Estimate the instantaneous rate of change of temperature at $t = 20$.



Example 15 **6** By considering the chord joining the points where $x = 1.2$ and $x = 1.4$, estimate (without accurate drawing) the gradient of the curve $y = \frac{1}{x}$, where $x = 1.3$.

Example 16 **7** Draw the graph of $y = \sqrt{16 - x^2}$, $\quad -4 \leq x \leq 4$. Use an appropriate technique to find an estimate for the gradient of the curve at the points:

a $x = 0$ **b** $x = 2$ **c** $x = 3$

8 It is known that the straight line $y = 4x - 4$ touches the curve $y = x^2$ at the point (2, 4). Sketch the graphs of both of these functions on the one set of axes.
Find the gradient of $y = x^2$ at (2, 4).

9 Water is being collected in a water tank, The volume, $V(\text{m}^3)$, of water in the tank after time t minutes is given by $V = 3t^2 + 4t + 2$.

a Find the average rate of change of volume between times $t = 1$ and $t = 3$.

b Find an estimate for the rate of change of volume at $t = 1$.

10 A population of bacteria is growing. The population, P million, after time t minutes is given by $P = 3 \times 2^t$.

a Find the average rate of change of population between times $t = 2$ and $t = 4$.

b Find an estimate for the rate of change of population at $t = 2$.

11 Water is flowing out of a water tank. The volume, $V(\text{m}^3)$, of water in the tank after t minutes is given by $V = 5 \times 10^5 - 10^2 \times 2^t$, $0 \leq t \leq 12$.

a Find the average rate of change of volume between times $t = 0$ and $t = 5$.

b Find an estimate for the rate of change of volume when $t = 6$.

c Find an estimate for the rate of change of volume when $t = 12$.

Example 17 **12** By considering the chord joining the points where $x = 1.2$ and $x = 1.3$, estimate the gradient of the curve $y = x^3 + x^2$ at $x = 1.3$.

13 Use the technique of Examples 16 and 17 to estimate the gradients of each of the following at the stated point:

a $y = x^3 + 2x^2,\ (1,\ 3)$ **b** $y = 2x^3 + 3x,\ (1,\ 5)$
c $y = -x^3 + 3x^2 + 2x,\ (2,\ 8)$ **d** $y = 2x^3 - 3x^2 - x + 2,\ (3,\ 26)$

14 The volume, V, of a cube with edge length x is given by $V = x^3$.

a Find the average rate at which the volume of the cube changes with respect to x as x increases from $x = 2$ to $x = 4$.
b Find the instantaneous rate at which V changes with respect to x when $x = 2$.

15 Let $y = 2x^2 - 1$.

a Find the average rate at which y changes with respect to x over the domain $1 \leq x \leq 4$.
b Find the instantaneous rate at which y changes with respect to x when $x = 1$.

16 Let $y = \sin x$.

a Find the average rate at which y changes with respect to x over each of the following domains:
i $0 \leq x \leq \frac{\pi}{2}$ **ii** $0 \leq x \leq \frac{\pi}{4}$ **iii** $0 \leq x \leq 0.5$ **iv** $0 \leq x \leq 0.1$
b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

17 Let $y = 10^x$.

a Find the average rate at which y changes with respect to x over each of the following domains:
i $0 \leq x \leq 1$ **ii** $0 \leq x \leq 0.5$ **iii** $0 \leq x \leq 0.1$ **iv** $0 \leq x \leq 0.01$
b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

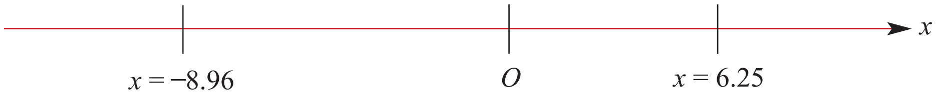
9.6 Displacement, velocity and acceleration

One of the key applications of rates of change is in the study of the motion of a particle.

In this section only motion in a straight line is considered.

The **displacement** of a particle is a specification of its position.

Consider motion on a straight line with reference point O.



We say that displacement to the right of O is positive and to the left negative.

A particle is moving along the straight line, let x metres denote its displacement relative to O.

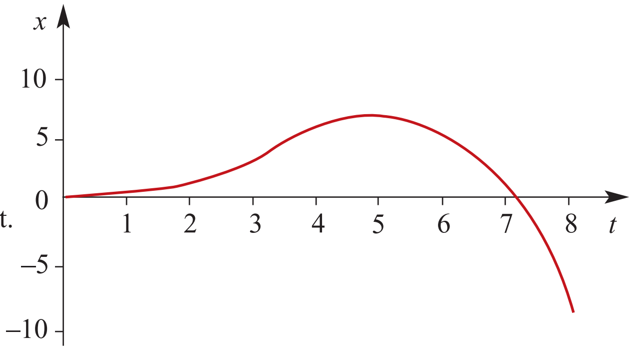
At time $t = 0,\ x = 0$ (time is measured in seconds)

At time $t = 5,\ x = 6.25$

At time $t = 8,\ x = -8.96$

At $t = 0$, the particle starts from rest and moves to the right. At $t = 5$, the particle stops and moves back in the opposite direction. Its displacement–time graph is shown below.

Note that from $t = 0$ until $t = 7.1$ the displacement is positive; that is, the position of the particle is to the right of O. For t greater than 7.1 the particle is to the left of O; it has negative displacement.

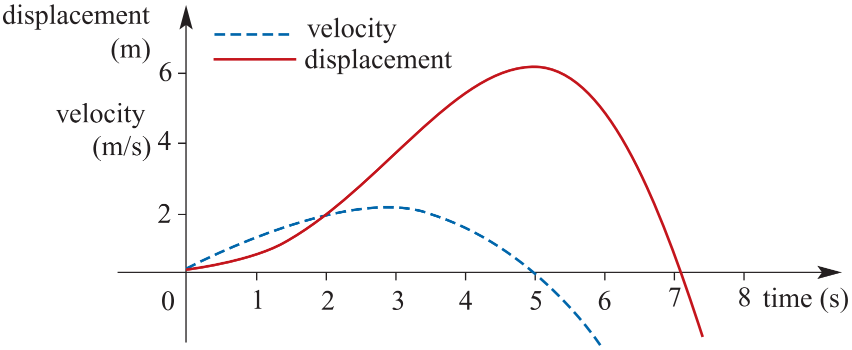


Velocity is the rate of change of displacement with respect to time.

$$\text{The average velocity} = \frac{\text{Change in displacement}}{\text{Time elapsed}}$$

For example, for the time interval $0 \leq t \leq 5$,

$$\text{average velocity} = \frac{6.25}{5} = 1.25 \text{ metres per second (m/s)}$$



For the time interval $5 \leq t \leq 8$,

$$\begin{aligned}\text{the average velocity} &= \frac{-8.96 - 6.25}{3}\\ &= -5.07 \text{ metres per second (m/s)}\end{aligned}$$

For $0 < t < 5$ the velocity is positive.

For $t > 5$ the velocity is negative.

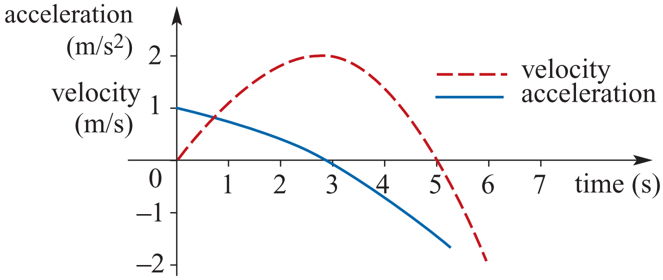
The negative velocity indicates that the particle is travelling from right to left. The relationship between displacement and velocity is illustrated in the preceding graph. The vertical axis is in both metres per second for velocity and metres for displacement.

Note that when $t = 5$ the velocity is 0.

Acceleration is the rate of change of velocity with respect to time. We show both the velocity–time and acceleration–time graphs on the one set of axes. The units of acceleration are metres per second per second and we abbreviate this to m/s^2.

Note that the acceleration is 0 for $t \approx 3$ and this occurs when the velocity changes from increasing to decreasing.

The instantaneous velocity of the particle at time t is given by the gradient of the displacement–time graph for the point corresponding to time t.



Example 18

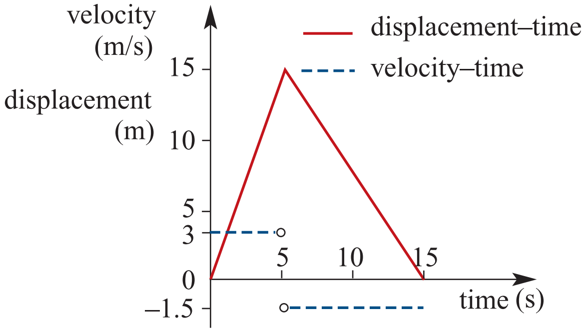
A particle is moving in a straight line. It was initially at rest at a point O. It moves to the right of O with a constant velocity and reaches a point, A, 15 metres from O after 5 seconds. It then returns to O. The return trip takes 10 seconds. It stops at O.

On the one set of axes draw the displacement–time graph and the velocity–time graph for the motion.

Solution

Note: The gradient of the displacement–time graph for $0 < t < 5$ is 3. The gradient for $5 < t < 15$ is –1.5.

The gradient of the displacement–time graph determines the velocity–time graph.

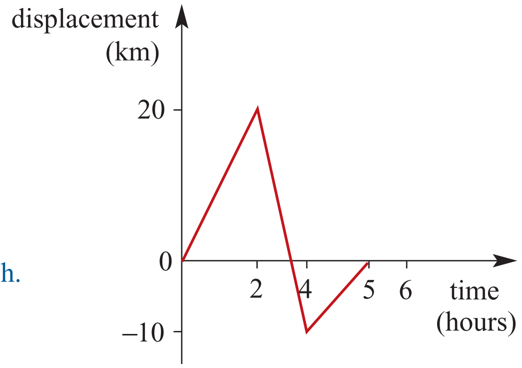


Example 19

The graph shown is the displacement–time graph that corresponds to the cycle trip of a boy who lives at the edge of a long straight road. The road runs north–south and north is chosen to be the positive direction.

a Describe his trip.

b Draw the corresponding velocity–time graph.



Solution

a The boy heads north for 2 hours.

His velocity for this period is $\frac{20-0}{2} = 10$ km/h.

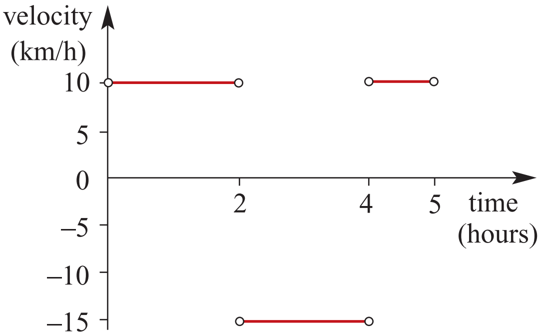
He turns and rides south for 2 hours.

His velocity for this period is $\frac{-10-20}{2} = -15$ km/h.

He turns and rides north until he reaches the house.

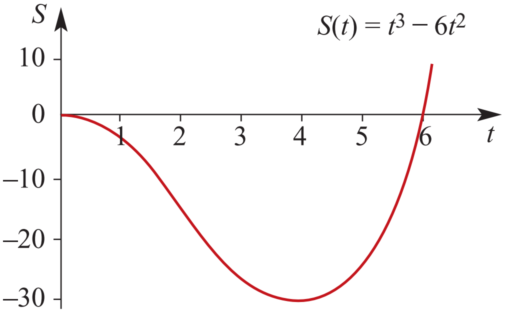
His velocity for this period is $\frac{0-(-10)}{1} = 10$ km/h.

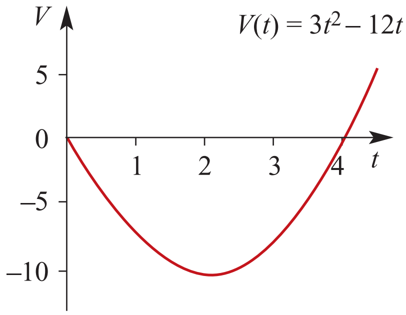
b



Example 20

The displacement of a particle moving in a straight line is given by the function $S(t) = t^3 - 6t^2,\ t \geq 0$. The graph of S against t is shown. The corresponding velocity–time graph is also shown. The function describing the velocity is $V(t) = 3t^2 - 12t$.





a Find the average velocity of the particle for the domains:
i $3.5 \leq x < 4.5$ **ii** $3.9 \leq x < 4.1$ **iii** $3.99 \leq x < 4.01$

b From part **a** what is the instantaneous velocity when $t = 4$?

c **i** For what values of t is the velocity positive?
ii For what values of t is the velocity negative?

Solution

a **i** Average velocity $= \dfrac{S(4.5) - S(3.5)}{1} = \dfrac{-30.375 + 30.625}{1} = 0.25$

ii Average velocity $= \dfrac{S(4.1) - S(3.9)}{0.2} = \dfrac{-30.939 + 31.941}{0.2} = 0.01$

iii Average velocity $= S(4.01) - S(3.99) = \dfrac{-31.999\,399 + 31.999\,401}{0.02}$

$= 0.0001$

b The results of part **a** give the instantaneous velocity as zero when $t = 4$, and this is consistent with the result suggested by both graphs.

c **i** From distance–time graph it can be seen that the velocity is positive for $t > 4$.

ii From the distance–time graph it can be seen that the velocity is negative for $0 \leq t < 4$.

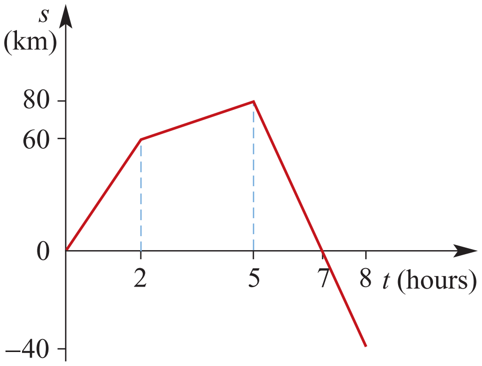
Exercise 9F

1 Let $s(t) = 6t - 2t^3$ be the displacement function of a particle moving on a straight line, where t is in seconds and s is in metres.

- **a** Find the average velocity for the time domain $0 \leq x < 1$.
- **b** Find the average velocity for the time domain $0.8 \leq x < 1$.
- **c** Find an estimate for the instantaneous velocity for $t = 1$.
- **d** State the values of t for which the displacement, s, is positive.
- **e** State the values of t for which the velocity is positive.

Examples 18, 19

2 Alongside is the displacement–time graph for a train travelling on a straight track. Displacement is measured from the door of the station master's office at Jimbara station.



- **a** What was the train's velocity over each of the following time intervals?
  - **i** $0 \leq x < 2$
  - **ii** $2 \leq x < 5$
  - **iii** $5 \leq x < 8$
- **b** Describe the train journey.
- **c** Draw a velocity–time graph for the train's motion for the domain $0 \leq x < 8$.

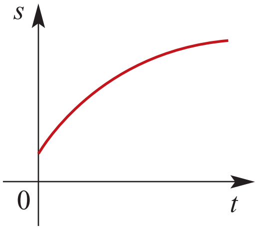
3 The motion of a particle moving in a straight line is described by the following information:

- For the time period (0, 2), velocity $= -3$.
- For the time period (2, 5), velocity $= 3$.
- For the time period (5, 7), velocity $= 4$.

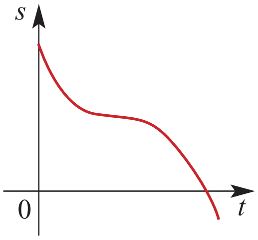
Draw the displacement–time graph for the domain $0 \leq x < 7$.

4 Match the displacement–time graphs shown with their velocity–time graph.

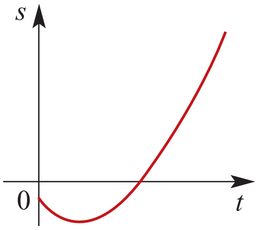
a



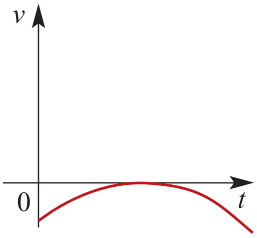
b



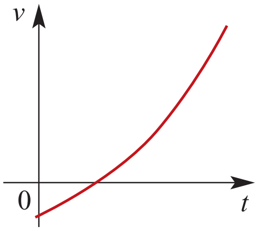
c



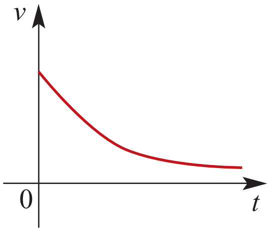
A



B

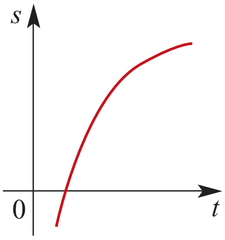


C

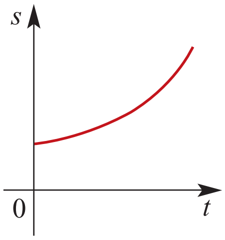


5 The graphs of four displacement–time graphs are shown. In each case determine the sign of the velocity and whether the particle is speeding up or slowing down.

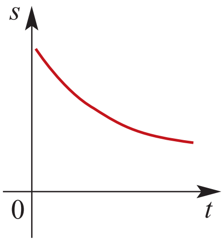
a



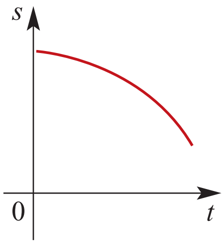
b



c



d

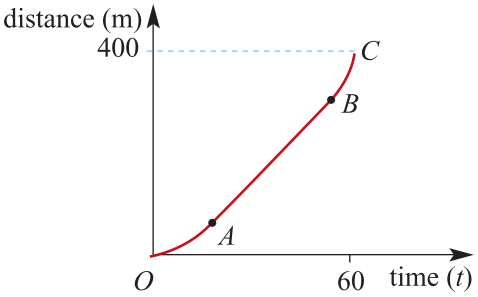


6 The distance–time graph for a 400-metre runner is shown. Explain what is happening between:

a 0 and A

b A and B

c B and C



Example 20

7 The table shows the distance, d metres, of a ball from its starting position, t seconds after being thrown into the air.

t	0	1	2	3	4	5	6
d	0	25	40	45	40	25	0

Use scales 2 cm = 1 second and 1 cm = 5 m and draw the graph of d against t. From your graph find:

a when the ball returns to the starting point

b the average velocity of the ball from $t = 1$ to $t = 2$

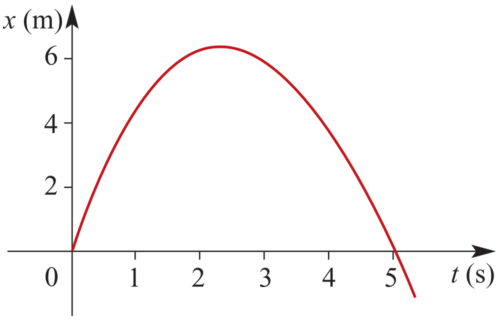
c the average velocity of the ball from $t = 1$ to $t = 1.5$

d an estimate of the velocity of the ball when $t = 1$

e an estimate of the velocity of the ball when $t = 4$

f an estimate of the velocity of the ball when $t = 5$

8 A particle moves along a horizontal straight line. It starts from rest at a point O. The graph is the displacement–time graph for this motion.



a At which time is the velocity zero?

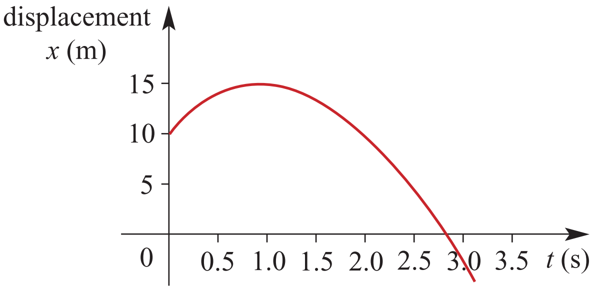
b For which values of t is the velocity positive?

c If to the right of O is taken as positive displacement, how far does the particle go to the right?

d How long does it take to return to O?

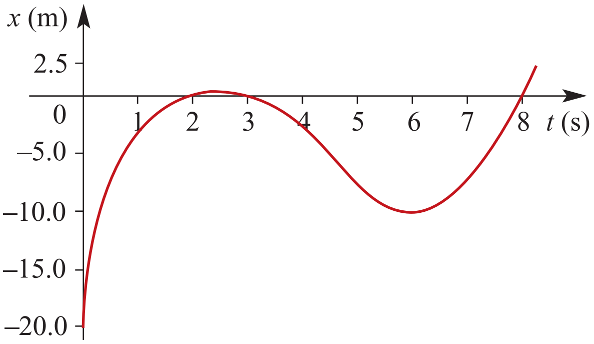
e Estimate the velocity of the particle at $t = 1$.

9 A stone is thrown vertically upwards from the edge of a platform that is 10 m above the ground. The displacement–time graph for the motion of the stone is shown. The motion of the stone is in a straight line and the reference point for displacement is taken as a point at ground level, directly below where the stone was thrown.



- **a** From the graph estimate the velocity with which the stone is thrown.
- **b** What is the maximum height reached by the stone?
- **c** At what time does the stone reach its maximum height?
- **d** At what time does the stone hit the ground?
- **e** From the graph estimate the speed at which the stone hits the ground.

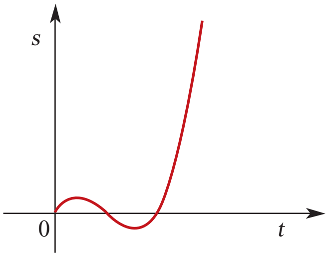
10 A particle is moving in a horizontal straight line. Displacement is measured from a point O. The particle starts at a point 20 m to the left of O. The displacement–time graph for the motion of the particle is as shown.



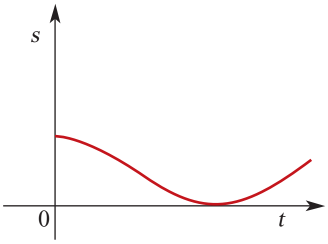
- **a** At which times is the particle at O?
- **b** For which values of t is the particle moving to the right?
- **c** For which values of t is the particle stationary?

11 Match each displacement–time graph (row 1) shown with its velocity–time graph (row 2) and its acceleration–time graph (row 3).

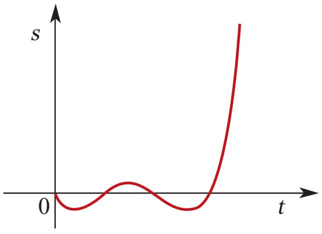
a



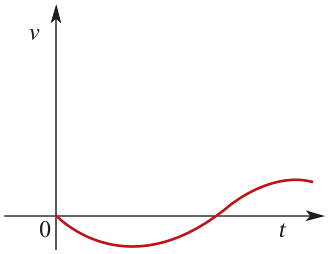
b



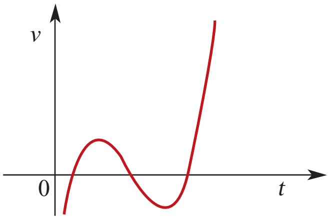
c



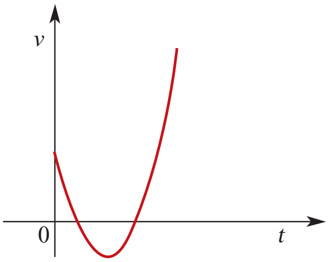
A



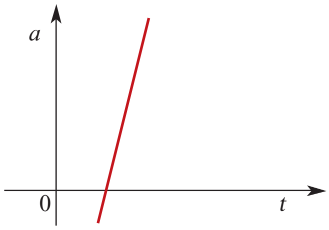
B



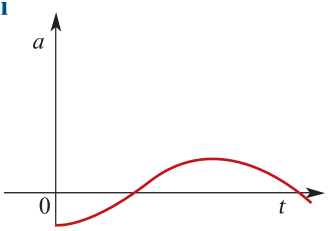
C



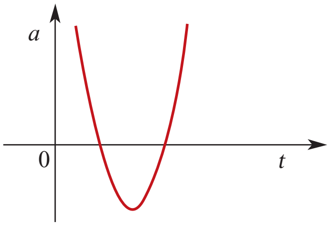
i



ii



iii

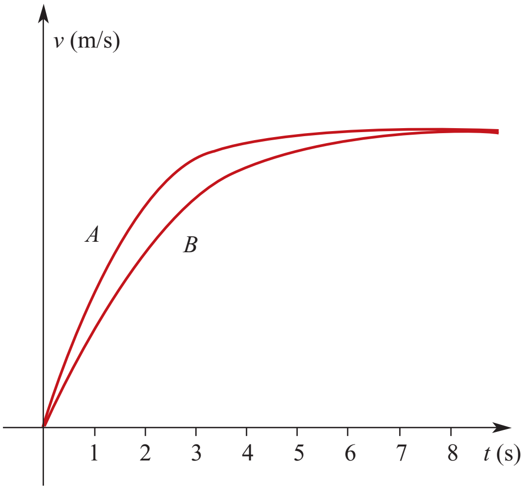


12 A body falling from rest on the surface of Earth will accelerate at approximately 10 m/s^2. At time t (seconds), the velocity, v (m/s), and displacement, s (metres), are related by the following:

$$v = -10\,t$$
$$v^{2} = -20\,s$$

Plot a suitable s versus t graph.

13 The plots of cars A and B are shown below. Approximate the time per second when their acceleration is equal. Write a brief report comparing the cars' velocity and relative position over this journey.

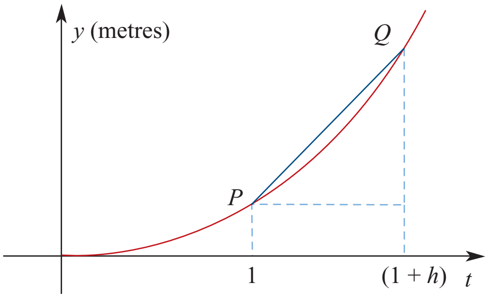


9.7 The gradient of a curve at a point and the gradient function

At the Earth's surface, an object falls a distance of y metres in t seconds, where $y \approx 4.9t^2$.

Can a general expression for the instantaneous speed of such an object after t seconds be approximated?

The gradient of chord PQ approximates the speed of the object at P. The 'shorter' this chord is made the better the approximation.



Let P be the point on the curve where $t = 1$. Let Q be the point on the curve corresponding to a small time interval h seconds after $t = 1$.

i.e. Q is the point on the curve where $t = 1 + h$.

$$\text{The gradient of chord } PQ = \frac{4.9\left((1+h)^2 - 1^2\right)}{(1+h) - 1}$$

$$= \frac{4.9\left(1 + 2h + h^2 - 1\right)}{h}$$

$$= 4.9\,(2 + h)$$

The table gives the gradient of chord PQ for different values of h. Use your calculator to check these.

h	Gradient of PQ
0.1	10.29
0.01	9.849
0.001	9.8049
0.0001	9.80049
0.00001	9.800049
⋮	⋮
⋮	⋮

If values of h of smaller and smaller magnitude are taken, it is found that the gradient of chord PQ gets closer and closer to 9.8. The gradient at the point where $t = 1$ is 9.8. Thus, the speed of the object at the moment $t = 1$ is 9.8 metres per second.

The speed of the object at the moment $t = 1$ is the limiting value of the gradients of PQ as Q approaches P.

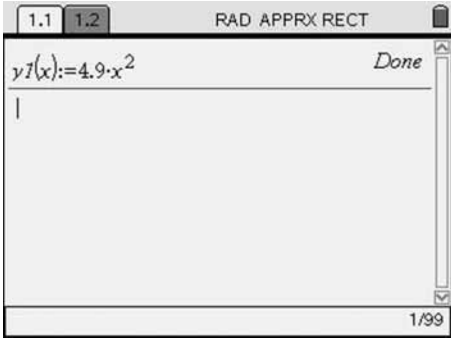
Using technology

The graphics calculator may be used to validate the values shown in the previous table.

Using the TI-Nspire:

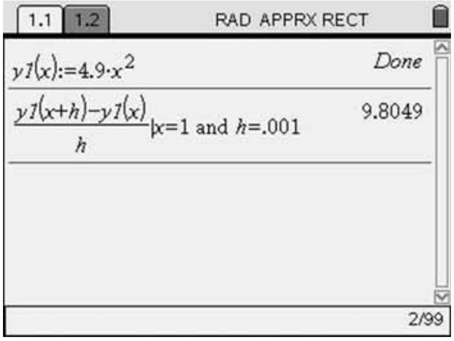
In the Calculator application do the following:

1 Define the function $y = 4.9x^2$ by typing **y1(x): = 4.9x^2** and then press enter.



```
y1(x):=4.9·x^2                    Done
```

2 Now type **(y1(x + h) – y1(x))/ h|x = 1 and h = 0.001** and press enter.



```
y1(x):=4.9·x^2                    Done
(y1(x+h)-y1(x))/h |x=1 and h=.001  9.8049
```

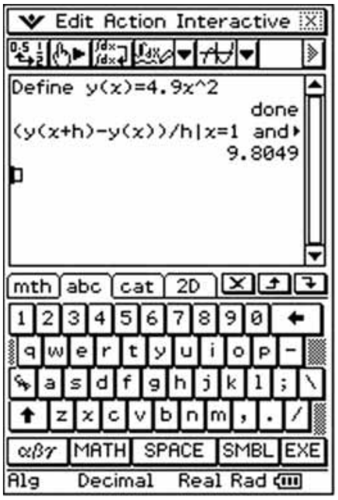
Using the ClassPad:

In the Main application do the following:

1 Define the function $y = 4.9x^2$ by typing **Define y(x) = 4.9x^2** and then press EXE.

```
Define y(x)=4.9x^2
                    done
```

2 Now type **(y(x + h) – y(x))/ h|x = 1 and h = 0.001** and press EXE.



```
Define y(x)=4.9x^2
                    done
(y(x+h)-y(x))/h|x=1 and▸
                  9.8049
```

A formula for the speed of the object at any time t is required. Let P be the point with coordinates $(t, 4.9t^2)$ on the curve and Q be the point with coordinates $(t+h, 4.9(t+h)^2)$.

$$\text{The gradient of chord } PQ = \frac{4.9\ \left((t+h)^2 - t^2\right)}{(t+h)-t}$$
$$= 4.9(2t+h)$$

From this an expression for the speed can be found. Consider the limit as h approaches 0; that is, the value of $4.9(2t+h)$ as h becomes arbitrarily small.

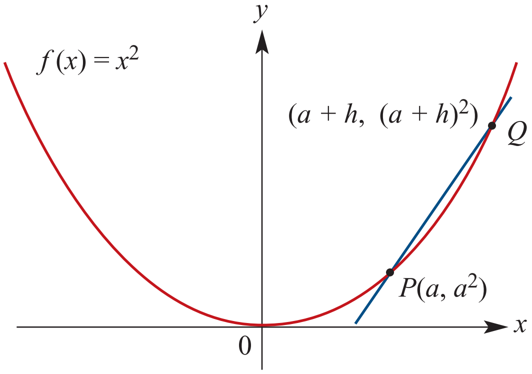
$$\text{i.e.}\qquad \text{Speed} = \lim_{h\to 0}(4.9\,(2t+h))$$
$$= 4.9 \times 2t$$
$$= 9.8t$$

The speed at time t is $9.8t$ metres per second. (The gradient of the curve at the point corresponding to time t is $9.8t$.) This technique can be used to investigate the gradient of similar functions.

Now consider the function $f\colon R \to R,\ f(x) = x^2$.

The gradient of the chord PQ shown in the figure is:

$$= \frac{(a+h)^2 - a^2}{a+h-a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{a+h-a}$$
$$= 2a + h$$



The limit as h approaches 0 of $(2a+h)$ is $2a$.

i.e. $\lim\limits_{h\to 0}\ (2a+h) = 2a$

The gradient at P is $2a$.

It can be seen that there is nothing special about a. So if x is a real number a similar formula holds.

So the gradient of the function $y = x^2$ at any point x is equal to $2x$.

It is said that $2x$ is **the derivative of x^2 with respect to x** or, more briefly, **the derivative of x^2 is $2x$**.

We refer to the function $2x$ as the **gradient** (or **derived**) function of x^2.

The straight line that passes through $P(a, a^2)$ and which has gradient $2a$ is called the **tangent** to the curve at P.

From the discussion at the beginning of this section we found that the derivative of $4.9t^2$ is $9.8t$, which is the equation to the tangent to the curve at point $P(t, 4.9t^2)$.

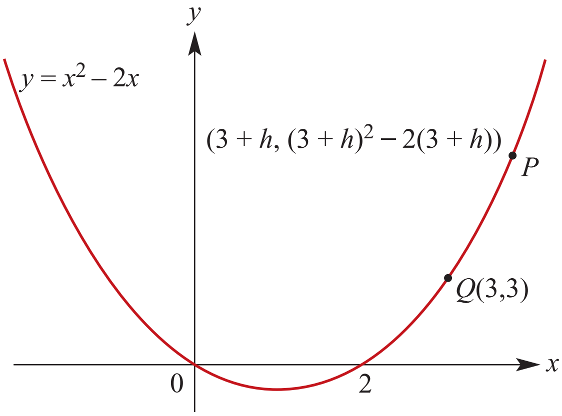
Example 21

By first considering the gradient of the chord PQ, find the gradient of $y = x^2 - 2x$ at the point Q with coordinates $(3, 3)$.

Solution

Consider chord PQ:

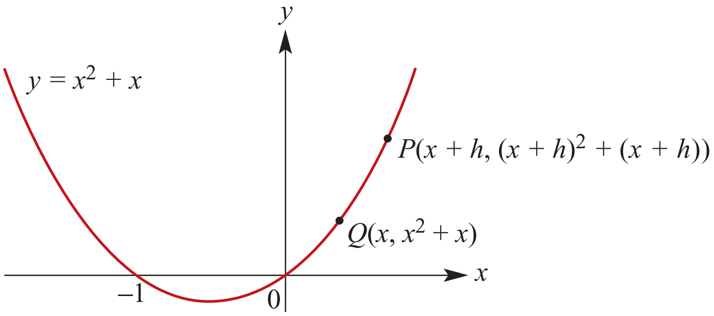
$$\begin{aligned}\text{Gradient of } PQ &= \frac{(3+h)^2 - 2(3+h) - 3}{3+h-3} \\ &= \frac{9 + 6h + h^2 - 6 - 2h - 3}{3+h-3} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h\end{aligned}$$



Consider h approaching zero. The gradient at the point $(3, 3)$ is 4.

Example 22

Find the gradient of chord PQ and, hence, the derivative of $x^2 + x$.



Solution

$$\begin{aligned}\text{The gradient of chord } PQ &= \frac{(x+h)^2+(x+h)-(x^2-x)}{x+h-x}\\ &= \frac{x^2+2xh+h^2+x+h-(x^2+x)}{h}\\ &= \frac{2xh+h^2+h}{h}\\ &= 2x+h+1\end{aligned}$$

From this it is seen that the derivative of x^2+x is $2x+1$.

The notation for limit as h approaches zero of $2x+h+1$ is $\lim\limits_{h\to 0}(2x+h+1)$.

The derivative of a function with rule $f(x)$ may be found by first finding an expression for the gradient of the chord from $Q(x, f(x))$ to $P(x+h,\ f(x+h))$ and then finding the limit of this expression as h approaches zero.

i.e.

$$\text{The derivative of } f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example 23

By first considering the gradient of the chord from $Q(x, f(x))$ to $P(x+h,\ f(x+h))$ for the curve $f(x)=x^3$, find the derivative of x^3.

Solution

$$\begin{aligned}f(x) &= x^3\\ f(x+h) &= (x+h)^3\\ \text{The gradient of chord } PQ &= \frac{f(x+h)-f(x)}{(x+h)-x}\\ &= \frac{(x+h)^3-x^3}{(x+h)-x}\\ \text{The derivative of } f(x) &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{(x+h)-x}\\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3h^2x+h^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3h^2x+h^3}{h}\\ &= \lim_{h\to 0}3x^2+3hx+h^2\\ &= 3x^2\end{aligned}$$

We have found that the derivative of x^3 is $3x^2$. The following example provides practise in determining limits.

Example 24

Find:

a $\lim_{h \to 0} 22x^2 + 20xh + h$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h \to 0} 3x$

d $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} 22x^2 + 20xh + h = 22x^2$

b $$\lim_{h \to 0} \frac{3x^2h + 2h^2}{h} = \lim_{h \to 0} 3x^2 + 2h$$
$$= 3x^2$$

c $\lim_{h \to 0} 3x = 3x$

d $\lim_{h \to 0} 4 = 4$

Exercise 9G

Examples 21, 22

1 For a curve with equation $y = 3x^2 - x$:

a Find the gradient of chord PQ, where P is the point $(1, 2)$ and Q is the point $((1 + h), 3(1 + h)^2 - (1 + h))$.

b Find the gradient of PQ when $h = 0.1$.

c Find the gradient of the curve at P.

Example 23

2 Find:

a $\lim_{h \to 0} \dfrac{(x + h)^2 + 2(x + h) - (x^2 + 2x)}{h}$ i.e. the derivative of $y = x^2 + 2x$

b $\lim_{h \to 0} \dfrac{(5 + h)^2 + 3(5 + h) - 40}{h}$ i.e. the gradient of $y = x^2 + 3x$, where $x = 5$

c $\lim_{h \to 0} \dfrac{(x + h)^3 + 2(x + h)^2 - (x^3 + 2x^2)}{h}$ i.e. the derivative of $y = x^3 + 2x^2$

Example 24

3 Find:

a $\lim_{h \to 0} \dfrac{2x^2h^3 + xh^2 + h}{h}$

b $\lim_{h \to 0} \dfrac{3x^2h - 2xh^2 + h}{h}$

c $\lim_{h \to 0} 20 - 10h$

d $\lim_{h \to 0} \dfrac{30hx^2 + 2h^2 + h}{h}$

e $\lim_{h \to 0} 5$

4 A space vehicle moves so that the distance travelled over its first minute of motion is given by $y = 4t^4$, where y is the distance travelled in metres and t the time in seconds. By finding the gradient of the chord between the points where $t = 4$ and $t = 5$, estimate the speed of the space vehicle when $t = 5$.

5 A population of insects grows so that the population, P, at time t (days) is given by $P = 1000 + t^2 + t$, where $t > 0$. By finding the gradient of the chord between the points where $t = 3$ and $t = 3 + h$, find an estimate for the rate of growth of the insect population at time $t = 3$.

6 For a curve with equation $y = \frac{2}{x}$:

  a Find the gradient of chord AB, where A is the point (2, 1) and B is the point $\left((2+h), \frac{2}{2+h}\right)$.

  b Find the gradient of AB when $h = 0.1$.

  c Find the gradient of the curve at A.

7 For a curve with equation $y = x^2 + 2x - 3$:

  a Find the gradient of chord PQ, where P is the point (2, 5) and Q is the point $((2+h), (2+h)^2 + 2(2+h) - 3)$.

  b Find the gradient of PQ when $h = 0.1$.

  c Find the gradient of the curve at P.

9.8 The derived function

In Section 9.7 we saw how the gradient of a function with rule $f(x)$ could be derived by considering the gradient of a chord PQ on the curve of $y = f(x)$.

Consider the graph $y = f(x)$ of the function f: $R \to R$.

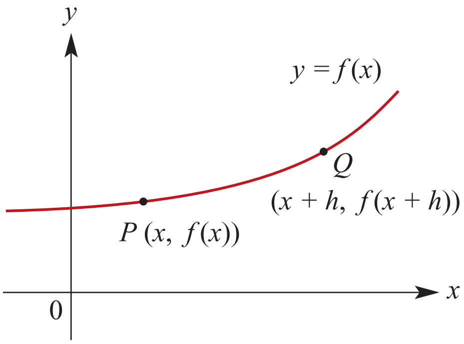
$$\text{The gradient of the chord } PQ = \frac{f(x+h) - f(x)}{x+h-x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

Therefore, the gradient of the graph at P is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient or derived function is denoted by $f(x)$, where

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$



In this chapter only polynomial functions are considered. For polynomial functions the derived function always exists and is defined for every number in the domain of f.

Determining the derivative of an expression or the derived function through evaluating the limit is called **differentiation by first principles**.

Example 25

For $f(x) = x^2 + 2x$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\
&= \lim_{h \to 0} \; 2x + h + 2 \\
&= 2x + 2 \\
\therefore f^x &= 2x + 2
\end{aligned}$$

Example 26

For $f(x) = 2 - x^3$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{2 - (x+h)^3 - \left(2 - x^3\right)}{h} \\
&= \lim_{h \to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - (2 - x^3)}{h} \\
&= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h \to 0} \; -3x^2 - 3xh - h^2 \\
&= -3x^2
\end{aligned}$$

Similarly, the following results are obtained:

For $f(x) = x$, $f'(x) = 1$.

For $f(x) = x^2$, $f'(x) = 2x^2$

For $f(x) = x^3$, $f'(x) = 3x^2$.

For $f(x) = x^4$, $f'(x) = 4x^3$.

For $f(x) = 1$, $f'(x) = 0$.

This suggests the following general result:

$$f(x) = x^n,\ f'(x) = nx^{n-1},\ n = 1, 2, 3, \ldots$$

and

$$f(x) = c,\ f'(x) = 0,\ c = \text{constant}$$

From the previous section we have seen that for k, a constant, if

$f(x) = kx^n$, the derivative function f' has rule $f'(x) = knx^{n-1}$.

It is worth making a special note of the results.

If $g(x) = kf(x)$, where k is a constant, then $g'(x) = kf'(x)$.

That is, **the derivative of a number multiple is the multiple of the derivative**.

For example, for $g(x) = 5x^2$, the derived function $g'(x) = 5(2x) = 10x$.

Another important rule for differentiation is:

If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

That is, **the derivative of the sum is the sum of the derivatives**.

For example, for $f(x) = x^2 + 2x$ the derived function is $f'(x) = 2x + 2$.

The process of finding the derivative function is called **differentiation**.

Example 27

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

$$\begin{aligned} & f(x) = x^5 - 2x^3 + 2 \\ \text{then} \quad & f'(x) = 5x^4 - 2\left(3x^2\right) + 2(0) \\ & = 5x^4 - 6x^2 \end{aligned}$$

Example 28

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and $f'(1)$.

Solution

$$\begin{aligned} & f(x) = 3x^3 - 6x^2 + 1 \\ \text{then} \quad & f'(x) = 3\left(x^2\right) - 6(2x) + 1(0) \\ & = 9x^2 - 12x \\ & f'(1) = 9 - 12 \\ & = -3 \end{aligned}$$

Example 29

Find the gradient of the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$ at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and $f'(1) = 9 - 12 = -3$.

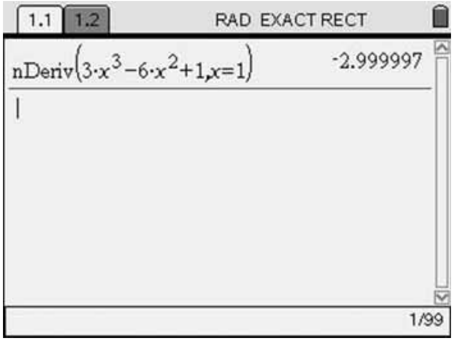
The gradient of the curve is –3 at the point (1, –2).

Using technology

The graphics calculator function **nDeriv** evaluates the numerical derivative at an **X** value.

Using the TI-Nspire:

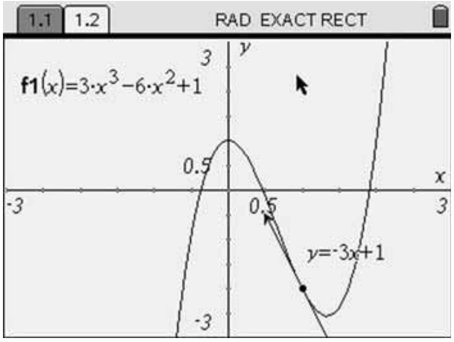
1 In the Calculator application, type **nDeriv** $(\mathbf{3x^3 - 6x^2 + 1, x = 1})$ and press enter.



nDeriv$(3\cdot x^3-6\cdot x^2+1,x=1)$ ‾2.999997

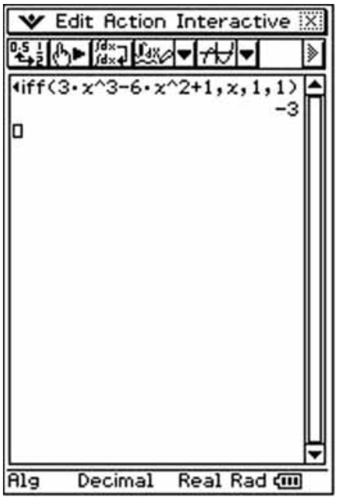
1/99

As described earlier, draw a tangent line and display its equation.



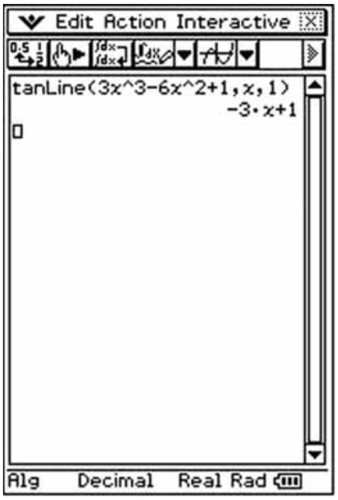
Using the ClassPad:

1 In the Main application, type **diff** $(\mathbf{3x^3 - 6x^2 + 1, x, 1, 1})$ and press EXE.



◂iff(3·x^3-6·x^2+1,x,1,1)
-3

In the Main application, type **tanLine** $(\mathbf{3x^3 - 6x^2 + 1, x, 1})$ and then press EXE.



tanLine(3x^3-6x^2+1,x,1)
-3·x+1

The tangent gradient is –3.

An alternative notation for the derivative is the following:

If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, so that $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, the derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

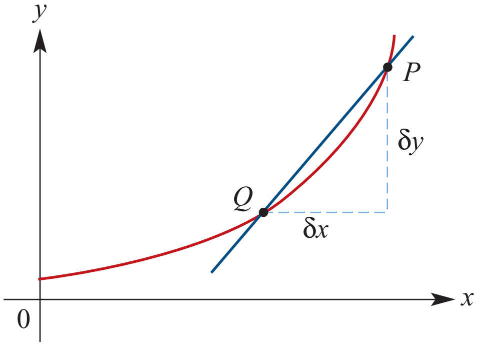
i.e. If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.

Similarly, if s is a function of t, the derivative of s with respect to t is $\frac{ds}{dt}$.

In this notation d is not a factor and cannot be cancelled. This came about because in the eighteenth century the standard diagram for finding the limiting gradient was labelled as shown in the figure. ('δ' is the lower case Greek letter for 'd', and is pronounced *delta*.)

δx means a difference in x.

δy means a difference in y.



Example 30

a If $y = t^2$, find $\frac{dy}{dt}$. **b** If $x = t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a If $y = t^2$, then $\frac{dy}{dt} = 2t$ **b** If $x = t^3 + t$, then $\frac{dx}{dt} = 3t^2 + 1$

c If $z = \frac{1}{3}x^3 + x^2$, then $\frac{dz}{dx} = x^2 + 2x$

Example 31

a For $y = (x + 3)^2$, find $\frac{dy}{dx}$. **b** For $z = (2t - 1)^2(t + 2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$. **d** Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First, it is necessary to write $y = (x + 3)^2$ in expanded form.

$$\therefore y = x^2 + 6x + 9$$

$$\text{and } \frac{dy}{dx} = 2x + 6$$

b First, expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$

$$\therefore \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First, dividing by x, $x \neq 0$:

$$y = x + 3$$

$$\therefore \frac{dy}{dx} = 1$$

d $y = 2x^3 - 1$

$$\therefore \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as $\frac{d}{dx}(2x^2 - 4x)$.

In general, $\boxed{\frac{d}{dx}(f(x)) = f'(x)}$

Example 32

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$

$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$

$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$

$= 18z^2 - 8z$

Example 33

Find the coordinates of the points on curves determined by each of the following equations at which the gradient has the given values:

a $y = x^3$; gradient $= 8$ **b** $y = x^2 - 4x + 2$ gradient $= 0$

c $y = 4 - x^3$; gradient $= -6$

Solution

a

$$y = x^3$$
$$\therefore \frac{dy}{dx} = 3x^2$$
$$\therefore 8 = 3x^2$$
$$\therefore \quad x = \frac{\pm 2\sqrt{6}}{3}$$
$$\therefore \left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right), \left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$$

b

$$y = x^2 - 4x + 2$$
$$\therefore \frac{dy}{dx} = 2x - 4$$
$$\therefore \quad 0 = 2x - 4$$
$$x = 2$$
$$\therefore \quad (2, -2)$$

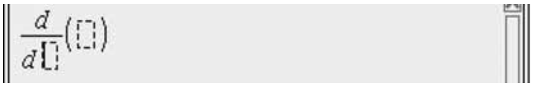
c

$$y = 4 - x^3$$
$$\therefore \frac{dy}{dx} = -3x^2$$
$$\therefore \quad -6 = -3x^2$$
$$\therefore \quad x = \pm\sqrt{2}$$
$$\therefore \quad (\sqrt{2}, 4 - 2\sqrt{2}), \left(-\sqrt{2}, 4 + 2\sqrt{2}\right).$$

Using technology

Using the TI-Nspire:

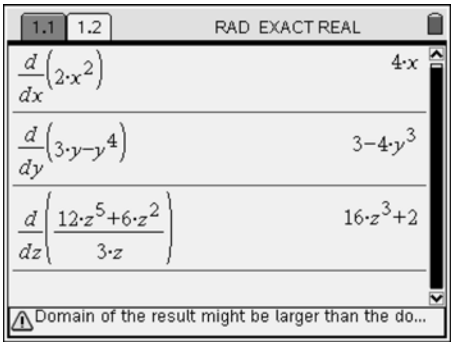
1 In the calculator application, press menu and select *Derivative* from the Calculus submenu.



$\frac{d}{d\square}(\square)$

2 In the denominator enter the variable with which you wish to differentiate.

3 In the brackets enter the expression you wish to differentiate, then press enter.



$\frac{d}{dx}(2\cdot x^2)$ $4\cdot x$

$\frac{d}{dy}(3\cdot y-y^4)$ $3-4\cdot y^3$

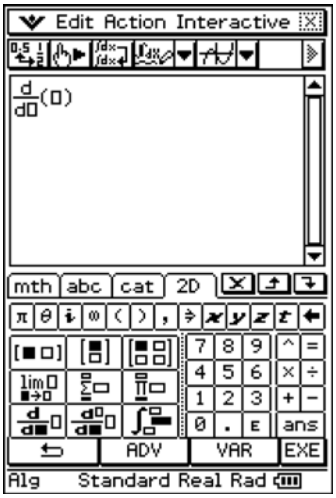
$\frac{d}{dz}\left(\frac{12\cdot z^5+6\cdot z^2}{3\cdot z}\right)$ $16\cdot z^3+2$

Domain of the result might be larger than the do...

Using the ClassPad:

1 In the main application, press Keyboard and then tap the 2D tab.

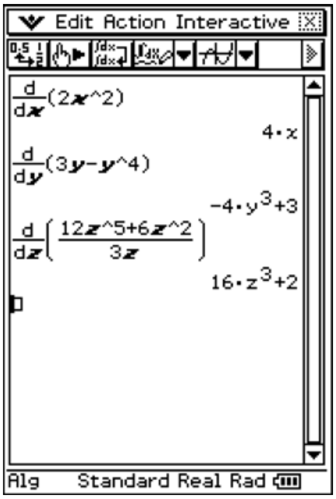
2 Now tap the CALC tab to access the derivative operator shown below.



$\frac{d}{d\square}(\square)$

3 In the denominator enter the variable with which you wish to differentiate.

4 In the brackets enter the expression you wish to differentiate and then press EXE.



$$\frac{d}{dx}(2x\text{^}2)$$

$$4\cdot x$$

$$\frac{d}{dy}(3y-y\text{^}4)$$

$$-4\cdot y^3+3$$

$$\frac{d}{dz}\left(\frac{12z\text{^}5+6z\text{^}2}{3z}\right)$$

$$16\cdot z^3+2$$

Exercise 9H

Examples 25, 26 **1** In each of the following, find $f'(x)$ by finding $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$:

a $f(x) = 3x^2$ **b** $f(x) = 4x$ **c** $f(x) = 3$
d $f(x) = 3x^2 + 4x + 3$ **e** $f(x) = 2x^3 - 4$ **f** $f(x) = 4x^2 - 5x$
g $f(x) = 3 - 2x + x^2$

Example 27 **2** Find the derivative of each of the following with respect to x:

a $x^2 + 4x$ **b** $2x + 1$ **c** $x^3 - x$
d $\frac{1}{2}x^2 - 3x + 4$ **e** $5x^3 + 3x^2$ **f** $-x^3 + 2x^2$

3 For each of the following find $f'(x)$:

a $f(x) = x^{12}$ **b** $f(x) = 3x^7$ **c** $f(x) = 5x$
d $f(x) = 5x + 3$ **e** $f(x) = 30$ **f** $f(x) = 5x^2 - 3x$
g $f(x) = 10x^5 + 3x^4$ **h** $f(x) = 2x^4 + \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2.$

Examples 28, 29 **4** **a** Find the gradient of the curve with equation $y = x^3 + 1$ at the points:

i $(1, 2)$ **ii** $(a, a^3 + 1)$

b Find the derivative of $x^3 + 1$ with respect to x.

Example 30

5 For each of the following, find $\frac{dy}{dx}$:

a $y = -x$ **b** $y = 10$ **c** $y = 4x^3 - 3x + 2$

d $y = \frac{1}{3}\left(x^3 - 3x + 6\right)$ **e** $y = (x + 1)(x + 2)$ **f** $y = 2x(3x^2 - 4)$

g $y = \frac{10x^5 + 3x^4}{2x^2}, \quad x \neq 0$

6 **a** Given that $y = x^3 - 3x^2 + 3x$, find $\frac{dy}{dx}$. Hence, show that $\frac{dy}{dx} \geq 0$ for all x values, and interpret this in terms of the graph of $y = x^3 - 3x^2 + 3x$.

b Given that $y = \frac{x^2 + 2x}{x}$, for $x \neq 0$, find $\frac{dy}{dx}$.

c Differentiate $y = (3x + 1)^2$ with respect to x.

Examples 28, 31

7 At the points on the following curves corresponding to the given values of x, find the y coordinate and the gradient.

a $y = x^2 - 2x + 1, x = 2$ **b** $y = x^2 + x + 1, x = 0$

c $y = x^2 - 2x, x = -1$ **d** $y = (x + 2)(x - 4), x = 3$

e $y = 3x^2 - 2x^3, x = -2$ **f** $y = (4x - 5)^2, \ x = \frac{1}{2}$

8 For each of the following, find $f'(x)$ and $f'(1)$, if $y = f(x)$ and also find the $\{(x, y)\colon f'(x) = 1\}$; i.e. the coordinates of the points at which the gradient is 1.

a $y = 2x^2 - x$ **b** $y = 1 + \frac{1}{2}x + \frac{1}{3}x^2$

c $y = x^3 + x$ **d** $y = x^4 - 31x$

What is the interpretation of $\{(x, y)\colon \ f'(x) = 1\}$ in terms of the graphs?

Example 32

9 Find:

a $\frac{d}{dt}(3t^2 - 4t)$ **b** $\frac{d}{dx}(4 - x^2 + x^3)$ **c** $\frac{d}{dz}(5 - 2z^2 - z^4)$

d $\frac{d}{dy}(3y^2 - y^3)$ **e** $\frac{d}{dx}(2x^3 - 4x^2)$ **f** $\frac{d}{dt}(9.8t^2 - 2t)$

Example 33

10 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given values:

a $y = x^2$; gradient $= 8$ **b** $y = x^3$; gradient $= 12$

c $y = x(2 - x)$; gradient $= 2$ **d** $y = x^2 - 3x + 1$; gradient $= 0$

e $y = x^3 - 6x^2 + 4$; gradient $= -12$ **f** $y = x^2 - x^3$; gradient $= -1$

9.9 The derived function for rational exponents ($x^{p/q}$)

Consider $y = x^{\frac{1}{n}}$, where n is a non-zero integer, and $x > 0$.

$$\therefore y^n = x$$

$$\therefore \frac{dx}{dy} = ny^{n-1}$$

We can, in fact, take the reciprocals of both sides:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{ny^{n-1}}\\ &= \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}}\\ &= \frac{1}{n}x^{\frac{1}{n}-1}\end{aligned}$$

Thus, if $y = x^{\frac{1}{n}}$, $\dfrac{dy}{dx} = \dfrac{1}{n}x^{\frac{1}{n}-1}$, where n is a non-zero integer, and $x > 0$.

This result may now be extended to any rational power.

Consider the table below. A graphics calculator has been used to generate the values in columns 2 and 3 using **nDeriv(**. Check these calculations. Validate your answers using the $f'(x)$ expression (column 4).

$f(x)$	$f'(1)$	$f'(64)$	$f'(x)$
$x^{\frac{2}{3}}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{2}{3}x^{-\frac{1}{3}}$
$3x^{\frac{2}{3}}$	2	$\frac{1}{2}$	$\frac{2}{3} \times 3x^{-\frac{1}{3}}$
$x^{\frac{3}{2}}$	$\frac{3}{2}$	12	$\frac{3}{2}\, x^{\frac{1}{2}}$
$-4\,x^{\frac{3}{2}}$	$-\,6$	-48	$\frac{3}{2} \times -4\,x^{\frac{1}{2}}$
$\frac{1}{2}\,x^{\frac{4}{3}}$	$\frac{2}{3}$	$\frac{8}{3}$	$\frac{4}{3} \times \frac{1}{2}\,x^{\frac{1}{3}}$
$-\frac{1}{4}\,x^{\frac{3}{4}}$	$-\frac{3}{16}$	$-\frac{3}{32\sqrt{2}}$	$\frac{3}{4} \times -\frac{1}{4}x^{-\frac{1}{4}}$
$x^{-\frac{3}{2}}$	$-\frac{3}{2}$	$-\frac{3}{65536}$	$-\frac{3}{2}x^{-\frac{5}{2}}$

Therefore, we conclude that:

Given $y = x^{\frac{p}{q}}$, $\dfrac{dy}{dx} = \dfrac{p}{q}x^{\frac{p}{q}-1}$, where p and q are non-zero integers, and $x > 0$.

In summary, the derivative of a power function may be stated as:

If $f(x) = k\,x^{a}$, $f'(x) = a\,kx^{a-1}$, for $x > 0$ where a, k are real numbers.

Example 34

Find the derivative of each of the following with respect to x:

a $4\sqrt[3]{x^2}$ **b** $\sqrt[5]{x} - \dfrac{2}{x^3}$ **c** $\dfrac{(1+\sqrt{x})^2}{\sqrt{x}}$

Solution

a Let

$$\begin{aligned} y &= 4x^{\frac{2}{3}} \\ \therefore \frac{dy}{dx} &= \frac{2}{3} \times 4x^{\frac{2}{3}-1} \\ &= \frac{8}{3}x^{-\frac{1}{3}} \end{aligned}$$

b Let

$$\begin{aligned} y &= x^{\frac{1}{5}} - 2x^{-3} \\ \therefore \frac{dy}{dx} &= \frac{1}{5}x^{\frac{1}{5}-1} - 3 \times -2x^{-3-1} \\ &= \frac{1}{5}x^{-\frac{4}{5}} + 6x^{-4} \end{aligned}$$

c Let

$$\begin{aligned} y &= \frac{(1+\sqrt{x})^2}{\sqrt{x}} \\ &= \frac{1+2\sqrt{x}+x}{\sqrt{x}} \\ &= \frac{1}{\sqrt{x}} + 2 + \sqrt{x}, x \neq 0 \\ &= x^{-\frac{1}{2}} + 2 + x^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= -\frac{1}{2}x^{-\frac{1}{2}-1} + \frac{1}{2}x^{\frac{1}{2}-1} \\ &= -\frac{1}{2\sqrt{x^3}} + \frac{1}{2\sqrt{x}} \end{aligned}$$

Using technology

Using the TI-Nspire:

1 In the calculator application, access the derivative operator, as discussed earlier, for each of the steps below.

2 Place an x in the denominator, type **x^(p/q)** in the brackets and then press enter.

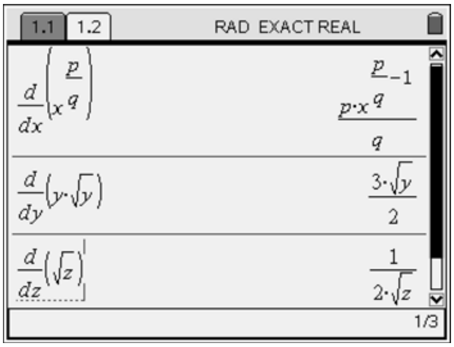
Using the ClassPad:

1 In the main application, access the derivative operator, as discussed earlier, for each of the steps below.

2 Place an x in the denominator, type **x^(p/q)** in the brackets and then press EXE.

3 Place a y in the denominator, type $y\sqrt{y}$ in the brackets and then press enter.

4 Place a z in the denominator, type $\sqrt{z}$ in the brackets and then press enter.



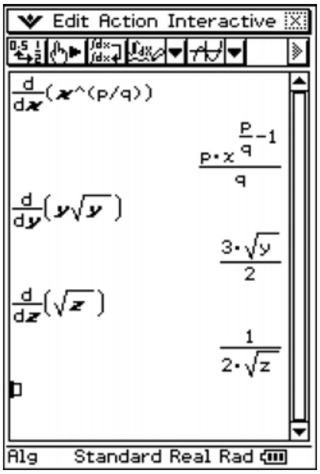
$\frac{d}{dx}\left(x^{\frac{p}{q}}\right)$ $\frac{p \cdot x^{\frac{p}{q}-1}}{q}$

$\frac{d}{dy}\left(y \cdot \sqrt{y}\right)$ $\frac{3 \cdot \sqrt{y}}{2}$

$\frac{d}{dz}\left(\sqrt{z}\right)$ $\frac{1}{2 \cdot \sqrt{z}}$

3 Place a y in the denominator, type $y\sqrt{y}$ in the brackets and then press EXE.

4 Place a z in the denominator, type $\sqrt{z}$ in the brackets and then press EXE.



$\frac{d}{dx}(x\text{^}(p/q))$

$\frac{p \cdot x^{\frac{p}{q}-1}}{q}$

$\frac{d}{dy}\left(y\sqrt{y}\right)$

$\frac{3 \cdot \sqrt{y}}{2}$

$\frac{d}{dz}\left(\sqrt{z}\right)$

$\frac{1}{2 \cdot \sqrt{z}}$

Exercise 9I

Example 34 **1** Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{3}}$ **b** $x^{\frac{3}{2}}, x > 0$

c $x^{\frac{5}{2}} - x^{\frac{3}{2}}, x > 0$ **d** $2x^{\frac{1}{2}} - 3x^{\frac{5}{3}}, x > 0$

e $x^{-\frac{5}{6}}, x > 0$ **f** $x^{-\frac{1}{2}} - 4, x > 0$

2 **a** Find the gradient of $y = x^{\frac{1}{3}}$ at each of the following points:

i $\left(\frac{1}{8}, \frac{1}{2}\right)$ **ii** $\left(-\frac{1}{8}, -\frac{1}{2}\right)$

iii $(1, 1)$ **iv** $(-1, -1)$

b Comment on your results.

3 Consider the graphs of $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ for $x > 0$.

a Solve for x: $x^{\frac{1}{2}} < x^{\frac{1}{3}}$.

b Find the values for x for which the gradient of $y = x^{\frac{1}{2}}$ is greater than the gradient of $y = x^{\frac{1}{3}}$.

Example 34 **4** Differentiate each of the following with respect to x:

a $\left(2 - 5\sqrt{x}\right)^2$ **b** $\left(3\sqrt{x} + 2\right)^2$ **c** $\frac{2 + \sqrt{x}}{x^2}$

d $\frac{x^2 + 2}{\sqrt{x}}$ **e** $3\sqrt{x}\left(x^2 + 2\right)$

MAPS **5** Use a derivative and algebra to solve for x: $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right) = \frac{2}{3}$. Validate your answer using technology.

MAPS **6** Use a graphics calculator to investigate the domain and range of the function $f(x) = \sqrt{-x}$.
Solve $f'(x) = -1$, using a derivative and algebra. Use a calculator to validate your answer.

9.10 Modelling and problem solving

Example 35

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $2 \le x \le 3$

b the average rate of change for the interval $2 \le x \le 2 + h$

c the instantaneous rate of change of f with respect to x when $x = 2$

Solution

a The average rate of change $= \frac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b For the interval the average rate of change

$$\begin{aligned} &= \frac{f(2+h) - f(2)}{2 + h - 2} = \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \frac{6h + h^2}{h} \\ &= 6 + h \end{aligned}$$

c When $x = 2$, the instantaneous rate of change $= f'(2) = 6$. This can also be seen from the result of part **b**.

Example 36

A balloon that develops a microscopic leak will decrease in volume. Its volume $V(cm^3)$ at time t (seconds) is $V = 600 - 10t - \frac{1}{100}t^2, t > 0$.

a Find the rate of change of volume after:

  i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

Solution

a $\dfrac{dV}{dt} = -10 - \dfrac{t}{50}$

i When $t = 10$:

$$\begin{aligned}\frac{dV}{dt} &= -10 - \frac{1}{5}\\ &= -10\frac{1}{5}\end{aligned}$$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$:

$$\begin{aligned}\frac{dV}{dt} &= -10 - \frac{2}{5}\\ &= -10\frac{2}{5}\end{aligned}$$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$.

Consider $V = 0$.

$$\therefore\ 600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore\ \ t = \frac{10 \pm \sqrt{100 + \frac{1}{100} \times 600 \times 4}}{-0.02}$$

$\therefore\ \ t = -1056.78$ or $t = 56.78$ (to 2 decimal places)

$\therefore$ The model may be suitable for $0 < t < 56.78$.

Displacement, velocity and acceleration were introduced for a body moving in a straight line in Section 9.6. Displacement was specified with respect to a reference point O on that line.

For velocity (v m/s):

$$v = \frac{ds}{dt}$$

and acceleration (a m/s^2):

$$a = \frac{dv}{dt}$$

Example 37

A car starts from rest and moves a distance, s metres, in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration and the acceleration when $t = 2$?

Solution

$$\begin{aligned} s &= \frac{1}{6}t^3 + \frac{1}{4}t^2 \\ \frac{ds}{dt} &= \frac{1}{2}t^2 + \frac{1}{2}t \\ \therefore \quad V &= \frac{1}{2}t^2 + \frac{1}{2}t \\ \frac{dv}{dt} &= t + \frac{1}{2} \\ \therefore \quad a &= t + \frac{1}{2} \end{aligned}$$

When $t = 0$, $a = \frac{1}{2}$.

When $t = 2$, $a = 2\frac{1}{2}$.

Hence, the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

Example 38

A point moves along a straight line so that its distance, x cm, from a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$. Find:

a at what times and in what positions the point will have zero velocity

b its acceleration at those instants

c its velocity when its acceleration is zero

Solution

$$\begin{aligned} x &= t^3 - 6t^2 + 9t \\ \frac{dx}{dt} &= 3t^2 - 12t + 9 \\ \therefore \quad v &= 3t^2 - 12t + 9 \end{aligned}$$

a When $v = 0$:

$$\begin{aligned} 0 &= 3t^2 - 12t + 9 \\ 0 &= t^2 - 4t + 3 \\ 0 &= (t - 1)(t - 3) \\ \therefore \quad t &= 1 \text{ or } 3 \end{aligned}$$

When $t = 1$, $x = 4$.

When $t = 3$, $x = 0$.

i.e. Velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

b
$$\begin{aligned} v &= 3t^2 - 12t + 9 \\ \frac{dv}{dt} &= 6t - 12 \\ \therefore \quad a &= 6t - 12 \end{aligned}$$

When $t = 1, a = -6$.

When $t = 3, a = 6$.

$\therefore$ Acceleration $= -6$ m/s^2 when $t = 1$ and 6 m/s^2 when $t = 3$.

c
$$\begin{aligned} a &= 6t - 12 \\ 0 &= 6t - 12 \end{aligned}$$

When $t = 2$:

$$\begin{aligned} v &= 3 \times 2^2 - 12 \times 2 + 9 \\ &= -3 \text{ m/s} \end{aligned}$$

Exercise 9J

1 The resistance of a copper wire is measured at various temperatures with the following results:

Temperature (° C)	10	20	30	40	50	60
Resistance (ohms)	23.4	23.9	24.4	24.9	25.5	26.0

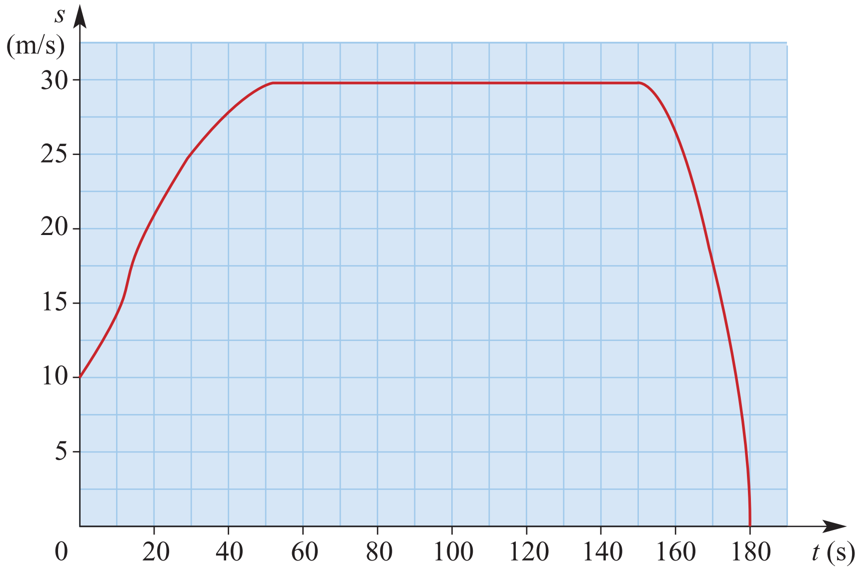
- **a** Plot the data to see whether the resistance rises approximately linearly with temperature.
- **b** If it does, find the rate of increase in ohm per ° C.

Example 35

2 A rock is allowed to fall from the top of a high cliff. It falls y metres in t seconds, where $y = 4.9t^2$. What is the average speed of the rock between:

- **a** **i** $t = 0$ and $t = 2$? **ii** $t = 2$ and $t = 4$?
- **b** **i** How far has the rock fallen in between $t = 4 - h$ and $t = 4$?
  - **ii** What is the average speed between $t = 4 - h$ and $t = 4$?
  - **iii** If it hits the bottom when $t = 4$, find the speed of impact by using the result of part **b ii** and taking $h = 0.2, 0.1, 0.05, 0.01, 0.001$.

3 The graph shows the speed, s (in m/s), of a car travelling along a straight highway at time t seconds, measured from when it crosses an intersection.



a Use the graph to approximate:

  i the average rate at which the car's speed is increasing between times $t = 30$ and $t = 50$

  ii the rate at which the car's speed is increasing when $t = 30$

b Sketch a graph showing the rate of increase of speed against time for $t = 0$ to $t = 180$.

4 The table shows the girth (w cm) of a watermelon, n days after being fed with a fertiliser.

n (days)	1	2	3	4	5	6	7
w (cm)	15	17	20	24	29	35	41

a Use a scale of 2 cm for 1 day and 2 cm for 10 cm of girth to draw the graph illustrating the information.

b Find the gradient of the chord joining the points where $n = 3$ and $n = 7$ and interpret the result.

c From the same graph estimate the gradient of the curve at $n = 4$.

5 A vending machine in a bus terminus contains cans of soft drinks. On a typical day:

- the machine starts $\frac{1}{4}$ full
- no drinks are sold between 1.00 a.m. and 6.00 a.m.
- the machine is filled at 2.00 p.m.

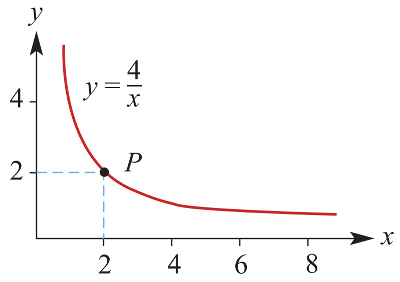
Sketch a graph to show how the number of drinks in the machine may vary between 6.00 a.m. and 12 midnight.

6 **a** $P(a, a^2)$ and $Q(b, b^2)$ are two points on the curve with equation $y = x^2$. Find the gradient of the line joining the points. (Answer in terms of a and b.)

b Use this result to find the gradient of the line for points with $a = 1$ and $b = 2$.

c Use this result to find the gradient of the line if $a = 2$ and $b = 2.01$.

7 The figure shows part of the curve with equation $y = \frac{4}{x}$ and P is the point at which $x = 2$.



a A_1 and A_2 are points on the curve whose x-coordinates are 1.5 and 2.5, respectively. Use your calculator to find their y-coordinates and, hence, the gradient of A_1A_2.

b Repeat for B_1 and B_2, whose x-coordinates are 1.9 and 2.1, respectively.

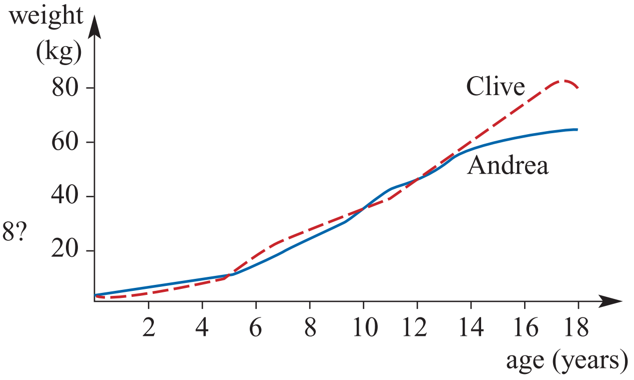
c Repeat for C_1 and C_2, whose x-coordinates are 1.99 and 2.01, respectively.

d Repeat for D_1 and D_2, whose x-coordinates are 1.999 and 2.001, respectively.

e What do the results of parts **a**, **b**, **c** and **d** suggest for the value of the gradient at $x = 2$?

8 Use the method of Question **7** to find the gradient of $y = (2x - 1)^3$ at $x = 1$.

9 The graphs alongside compare the weight of two people over the first 18 years of their lives.



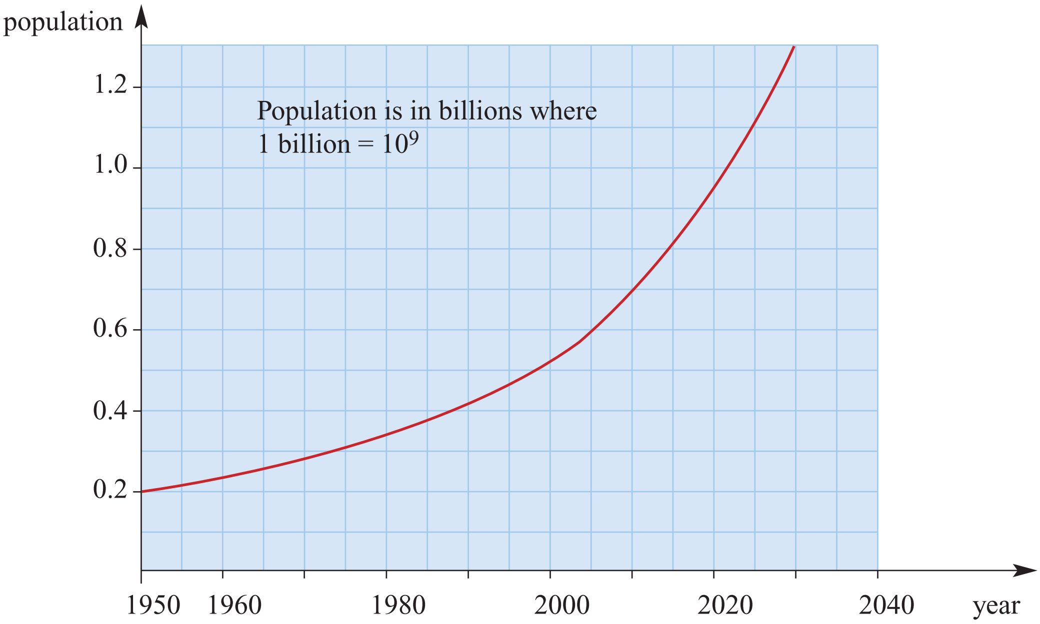
a What was the average rate of change of weight with respect to time for Andrea between the ages of 0 and 18?

b What was the average rate of change of weight with respect to time for Clive between the ages of 0 and 18?

c During which periods did Andrea weigh more than Clive?

d During which periods of time was Clive growing more rapidly than Andrea?

10 The graph below shows an exponential model for Acubaland's population growth. In exponential growth the rate of increase of the population at any time is proportional to the size of the population at that time.



a From the graph find the population of Acubaland in:

i 1960 **ii** 2000

b Calculate the average annual rate of population increase (in billions per year) over the years from 1960 to 2000.

c From the graph estimate the rate of population increase in:

i 1960 **ii** 2000

d How many years do you expect that it will take to double the 2020 population? Explain your reasoning.

11 Draw the graph of $y = 10^x$ and find the gradient of the chord joining the points:

a **i** $x = 2.5$ and $x = 2.8$ **ii** $x = 2.6$ and $x = 2.8$

iii $x = 2.7$ and $x = 2.8$ **iv** $x = 2.75$ and $x = 2.8$

b Validate using the graphics calculator.

c Comment on your result and investigate further.

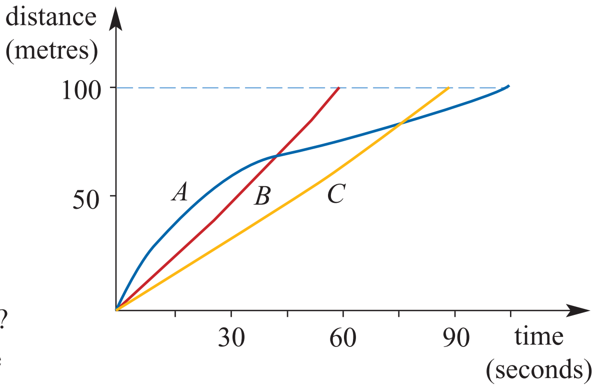
12 **a** Use the result that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ to find an expression for the gradient of the line joining the points $P(a, a^3)$ and $Q(b, b^3)$ on the curve with equation $y = x^3$.

b Find the gradient of the line for $a = 1, b = 2$.

c Find the gradient of the line for $a = 2$ and $b = 2.01$.

d For your expression for the gradient in terms of a and b (part **a**) let $a = b$ and write your new expression in simplest terms. Interpret this result.

13 The rough sketch graph shown below shows what happens when three swimmers compete in a 100-metre race. (The vertical axis shows distance travelled by a swimmer.)

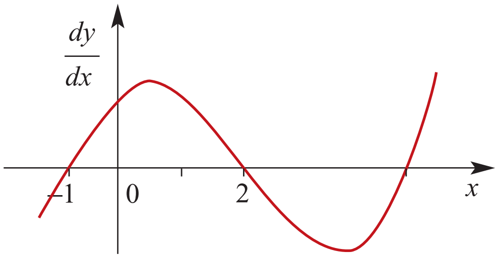


a Who wins the race?
b Who is in front at the 50 m mark?
c What is the approximate distance separating first and third place when the winner finishes?
d What is the approximate time difference between first and third place?
e What is the average speed of each swimmer?
f Describe the race as if you were a commentator.

14 In the following, $f(x)$ is the rule for a well-behaved function f.

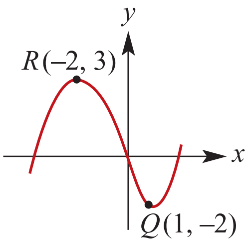
a If for $y = f(x)$ the average rate of change of y with respect to x is m, for $a \leq x \leq b$, then for the same domain the rate of change of y with respect to x where $y = f(x) + c$, is equal to. . . .?
b If for $y = f(x)$ the average rate of change of y with respect to x is m, in the interval $[a, b]$, then for the same interval the rate of change of y with respect to x where $y = cf(x)$, is equal to. . . .?
c If for $y = f(x)$, the average rate of change of y with respect to x is m, for $a \leq x \leq b$, then for the same domain the rate of change of y with respect to x where $y = -f(x)$, is equal to. . . .?
d Give similar statements for the instantaneous rates of change.

15 The diagram below shows part of the graph of $\frac{dy}{dx}$ against x. Sketch a possible shape of y against x over the same interval if:



a $y = -1$ when $x = -1$
b $y = 0$ when $x = 0$
c $y = 1$ when $x = 2$

16 The graph shown is that of a polynomial of the form $P(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.
Note: $Q(1, -2)$ is not a turning point.



17 A body moves in a path described by the equation $y = \frac{1}{5}x^5 + \frac{1}{2}x^4,\ x \geq 0$.

Units are in kilometres, where x and y are the horizontal and vertical axes, respectively.

a What will be the direction of motion (give the answer as angle between direction of motion and the x-axis) when the x value is:

i 1 km? **ii** 3 km?

b Find a value of x for which the gradient of the path is 32.

18 A trail over a mountain pass can be modelled by the curve with equation $y = 2 + 0.12x - 0.01x^3$, where x and y are, respectively, the horizontal and vertical distances measured in kilometres, $0 \leq x \leq 3$.

a Find the gradients at the beginning and the end of the trail.

b Calculate the point at which the gradient is zero, and also calculate the height of the pass.

19 **a** Show that the gradients of the curve $y = x(x - 2)$ at the points (0, 0) and (2, 0) differ only in sign. What is the geometrical interpretation for this?

b If the gradient of the curve $y = x(x - 2)(x - 5)$ at the points (0, 0), (2, 0) and (5, 0) are l, m and n, respectively, show that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 0$.

20 A solid circular cylinder has radius r cm and height h cm. It has a fixed volume of 400cm^3.

a Find h in terms of r.

b Show that the total surface area, A cm^2, of the cylinder is given by $A = 2\pi r^2 + \frac{800}{r}$.

c Find $\frac{dA}{dr}$.

d Solve the equation $\frac{dA}{dr} = 0$ for r.

21 A rectangle has sides of length x cm and y cm and the area of the rectangle is 16 cm^2.

a Find y in terms of x.

b Show that the perimeter, P cm, is given by $P = 2x + \frac{32}{x}$.

c Find the value of P for which the value of $\frac{dP}{dx} = -6$.

22 The curve with equation $y = -2 + \sqrt{x}$ meets the x-axis at A and the y-axis at B.

a Find the coordinates of A and B.

b Find $\frac{dy}{dx}$.

c **i** Find the gradient of the curve where $x = 1$.

ii Find the equation of the tangent at the point where $x = 1$.

d Find the values of x for which $\frac{dy}{dx} < 1$.

23 An open rectangular box of height h cm has a horizontal rectangular base with side lengths x cm and $2x$ cm. If the volume of the box is 36 cm^2:

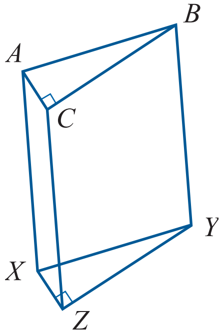
a Express h in terms of x.

b Show that the total surface area of the box is given by $A = 2x^2 + \frac{108}{x}$.

c Calculate the values of x and h for which $\frac{dA}{dx} = 0$.

d Use a graphics calculator to sketch the graph of A against x for $x > 0$.

24 The prism shown in the diagram has a triangular cross-section, as shown. The 'ends' of the prism shown are congruent right-angled triangles with the right angles at C and Z.



$AX = CZ = BY = y$ cm, $AC = XZ = 3x$ cm
and $CB = ZY = 4x$ cm.
The volume of the prism is 1500 cm^2.

a Express y in terms of x.

b Show that the total surface area, S cm^2, is given by $S = 12x^2 + \frac{3000}{x}$.

c Find $\frac{dS}{dx}$.

d Find the value of S for which $\frac{dS}{dx} = 0$.

Example 35 **25** If $y = 35 + 12x^2$:

a Find the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?

b Find the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?

c Find the rate of change of y with respect to x when $x = 2$.

Example 36 **26** According to a business magazine the expected assets, \$$M$, of a proposed new company will be given by $M = 200\,000 + 600t^2 - \frac{200}{3}t^3$, where t is the number of months after the business commences.

a Find the rate of growth of assets at time t months.

b Find the rate of growth of assets at time $t = 3$ months.

c Will the rate of growth of assets be 0 at any time?

Examples 37, 38 **27** The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \frac{1}{3}t^3 - 12t + 6$. Find the:

a rate of change of position with respect to time at $t = 3$

b time at which the velocity is zero

Example 38 **28** Let $s = 10 + 15t - 4.9t^2$ be the height (in metres) of an object at time t (in seconds).

a Find the velocity at time t.

b Find the acceleration at time t.

29 As a result of a survey, the marketing director of a company found that the revenue, $\$R$, from selling n produced items at $\$P$ is given by the rule $R = 30P - 2P^2$.

a Find $\frac{dR}{dP}$ and explain what it means.

b Calculate $\frac{dR}{dP}$ when $P = 5$ and $P = 10$.

c For what selling prices is revenue rising?

30 The population, P, of a new housing estate t years after 30 January, 2005 is given by the rule $P = 100(5 + t - 0.25t^2)$.

What will be the rate of change of the population after:

a 1 year? **b** 2 years? **c** 3 years?

31 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \frac{5}{8}\left(10t^2 - \frac{t^3}{3}\right)$, $0 \le t \le 20$.

a Find the volume of water in the flask at time:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask.

c Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

32 A model aeroplane flying level at 250 m above the ground suddenly dives. Its height, h (m), above the ground at time (t) after beginning to dive is given by
$h(t) = 8t^2 - 80t + 250, 0 \le x \le 10$.

At what rate is the plane losing height at:

a $t = 1$? **b** $t = 3$? **c** $t = 5$?

33 A particle moves along a straight line so that after t seconds its distance from O, a fixed point on the line, is s m, where $s = t^3 - 3t^2 + 2t$.

a When is the particle at O? **b** What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

Example 38 **34** A car starts from rest and moves a distance, s m, in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

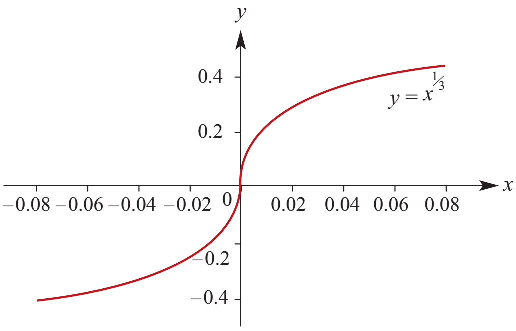
a What is the acceleration when $t = 0$? **b** What is the acceleration when $t = 2$?

35 The figure is the graph of the function $f(x) = x^{\frac{1}{3}}$, where $x, f(x)$ are real numbers.

a Use a calculator and the **nDerive(** feature to determine the gradient of the curve at the origin.

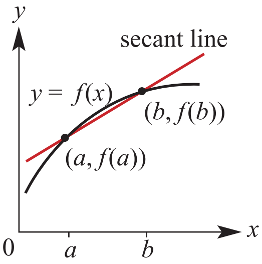
b Use a calculator to determine the gradient of the tangent to the curve at the origin.

c Evaluate $f'(0)$ and compare your answer to parts **a** and **b**. Discuss.

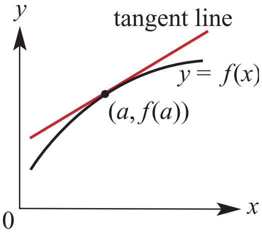


Chapter summary

- Where a real-life situation is modelled by a straight-line graph the gradient represents a rate of change of one quantity with respect to a second.
- Average speed $= \dfrac{\text{Total distance travelled}}{\text{Total time taken}}$
- If $y = f(x)$, then the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line joining $(a, f(a))$ to $(b, f(b))$.



- If $y = f(x)$, then the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



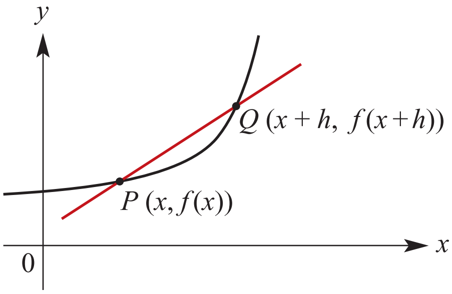
- The notation for limit as h approaches 0 is written as $\lim\limits_{h \to 0}$.
- For the graph of $y = f(x)$ of the function $f\colon R \to R$.

  The gradient of the chord $PQ = \dfrac{f(x+h) - f(x)}{h}$.

  The gradient of the graph at P is given by $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

  This limit gives a rule for the derived function denoted by f', where

  $f'\colon R \to R$ and $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.



- The general rule of the derived function of $f(x) = x^n,\ n = 1, 2, 3, \ldots$

  For $f(x) = x^n,\ f'(x) = nx^{n-1},\ n = 1, 2, 3, \ldots$

  For $f(x) = c,\ f'(x) = 0$, where $c =$ constant

  For example:

  For $f(x) = x^2,\ f'(x) = 2x$.

  For $f(x) = x^3,\ f'(x) = 3x^2$.

  For $f(x) = x^4,\ f'(x) = 4x^3$.
- The derivative of a number multiple is the multiple of the derivative.

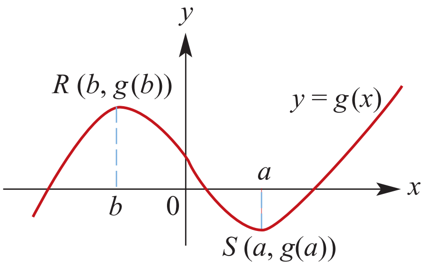
  For $g(x) = kf(x)$, where k is a constant,

  $g'(x) = kf'(x)$

  For example:

  $g(x) = 3x^2,\ g'(x) = 3(2x) = 6x$

- The derivative of a constant is always zero:
  For $g(x) = a$, $g'(x) = 0$
  For example:
  $$f(x) = 27.3,\ f'(x) = 0$$
- The derivative of the sum is the sum of the derivatives:
  $$\begin{aligned} \text{For} \quad f(x) &= g(x) + h(x) \\ \text{then} \quad f'(x) &= g'(x) + h'(x) \end{aligned}$$
  For example:
  $$\begin{aligned} &f(x) = x^2 + x^3,\ f'(x) = 2x + 3x^2 \\ &g(x) = 3x^2 + 4x^3,\ g'(x) = 3(2x) + 4\left(3x^2\right) = 6x + 12x^2 \end{aligned}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$ the gradient is $g'(a)$.
  For: $x < b$ the gradient is positive; i.e. $g'(x) > 0$
  $x = b$ the gradient is zero; i.e. $g'(b) = 0$
  $b < x < a$ the gradient is negative; i.e. $g'(a) < 0$
  $x = a$ the gradient is zero; i.e. $g'(a) = 0$
  $x > a$ the gradient is positive; i.e. $g'(x) > 0$



- The general result for differentiating functions, including powers of x with negative integers:
  For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer. For $f(x) = 1$, $f'(x) = 0$.
  Note that for $n \leq -1$, take the domain of f to be all real values excluding 0, and for $n \geq 1$ we take the domain of f to be all real numbers.
- The general result for any non-zero real power gives:
  For $f(x) = x^a$, $f'(x) = ax^{a-1}$, for $x > 0$ and a is a real number.
- Given $y = x^{\frac{p}{q}}$, $\dfrac{dy}{dx} = \dfrac{p}{q}x^{\frac{p}{q}-1}$, where p and q are non-zero integers, and $x > 0$.

Multiple-choice questions

1 A bushwalker walks 12 km in 2 hours, stops for 45 minutes and then walks a further 8 km in another 1.25 hours. The average walking speed of the bushwalker over the entire walk is:

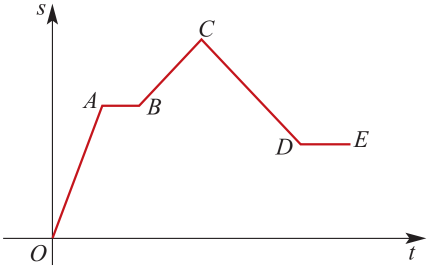
A 10 km/h **B** 9 km/h **C** 5 km/h
D 4 km/h **E** 7.2 km/h

2 Postal workers sort 12 000 letters during the normal day shift of 8 hours and, with a reduced workforce during the 2 hours overtime shift, they sort a further 2500 letters. The average rate of letter sorting per hour is:

A 1375 letters per hour **B** 1450 letters per hour **C** 1300 letters per hour
D 1400 letters per hour **E** 1500 letters per hour

*Questions **3–5** refer to the following information*:

The graph shows the movement of a vehicle over a period of time. It represents the distance (s) from a fixed point at a given time (t).



3 The line segment OA represents a stage of the movement during which the vehicle is:

A speeding up **B** slowing down **C** travelling north

D travelling at a constant speed **E** stationary

4 The line segment AB represents a stage of the movement during which the vehicle is:

A speeding up **B** slowing down **C** travelling east

D travelling at a constant speed greater than zero **E** stationary

5 The section(s) of the graph which represent(s) the vehicle when it is stationary is/are:

A at O **B** at A and C **C** between C and D

D between A and B and D and E **E** at no time

6 The average rate of change of the function $y = 3 \times 2^x$ over the interval [0, 2] is:

A 9 **B** 4.5 **C** 12 **D** 6 **E** 5

7 The population of trout in a trout pond is growing. If the population P, after t weeks is given by $P = 10 \times 1.1^t$, the average rate of growth of the population during the 5th week is closest to:

A 16 trout per week **B** 15 trout per week **C** 1.5 trout per week

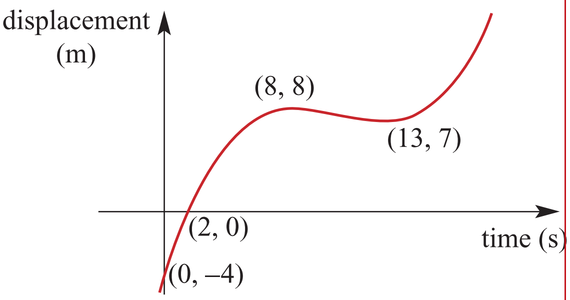
D 4 trout per week **E** 15.35 trout per week

8 Given $f(x) = 2x^3 + 3x$, the average rate of change of $f(x)$ with respect to x for the interval $[-2, 2]$ is:

A 0 **B** -22 **C** -11 **D** 22 **E** 11

*Questions **9–10** refer to the following information*:

A particle moves along a horizontal line. The graph of the particle's displacement relative to the origin over time is shown.



9 The particle has a velocity of zero at:

A 8 s and 13 s **B** 2 s **C** 0 s **D** 8 s and 7 s **E** −4 s

10 The time interval(s) during which the particle has a negative velocity are:

A $8 < t < 13$ **B** $0 < t < 2$ and $8 < t < 13$ **C** $0 < t < 2$
D $7 < t < 8$ **E** $(0, \infty)$

11 The gradient of the curve $y = x^3 + 4x$ at the point where $x = 2$ is:

A 12 **B** 4 **C** 10 **D** 16 **E** 8

12 The gradient of the chord of the curve $y = 2x^2$ between the points where $x = 1$ and $x = 1 + h$ is given by:

A $2(x + h)^2 - 2x^2$ **B** $4 + 2h$ **C** 4 **D** $4x$ **E** $4 + h$

13 If $y = 2x^4 - 5x^3 + 2$, then $\frac{dy}{dx}$ equals:

A $8x^3 - 5x^2 + 2$ **B** $4x^4 - 15x^2 + 2$ **C** $4x^4 - 10x^2$
D $8x^3 - 15x + 2$ **E** $8x^3 - 15x^2$

14 If $f(x) = x^2(x + 1)$, then $f'(-1)$ equals:

A −1 **B** 1 **C** 2 **D** −2 **E** 5

15 If $f(x) = (x - 3)^2$, then $f'(x)$ equals:

A $x - 3$ **B** $x - 6$ **C** $2x - 6$ **D** $2x + 9$ **E** $2x$

16 If $y = \frac{2x^4 + 9x^2}{3x}$, then $\frac{dy}{dx}$ equals:

A $\frac{2x^4}{3} + 6x$ **B** $2x + 3$ **C** $2x^2 + 3$ **D** $\frac{8x^3 + 18x}{3}$ **E** $8x^3 + 18x$

17 Given that $y = x^2 - 6x + 9$, the values of x for which $\frac{dy}{dx} \geq 0$ are:

A $x \geq 3$ **B** $x > 3$ **C** $x \geq -3$ **D** $x \leq -3$ **E** $x < 3$

18 If $y = 2x^4 - 36x^2$, the points at which the tangent to the curve is parallel to the x-axis are:

A 1, 0 and 3 **B** 0 and 3 **C** −3 and 3 **D** 0 and −3 **E** −3, 0 and 3

19 The coordinates of the point on the graph of $y = x^2 + 6x - 5$ at which the tangent is parallel to the line $y = 4x$ are:

A $(-1, -10)$ **B** $(-1, -2)$ **C** $(1, 2)$ **D** $(-1, 4)$ **E** $(-1, 10)$

20 If $y = -2x^3 + 3x^2 - x + 1$, then $\frac{dy}{dx}$ equals:

A $6x^2 + 6x - 1$ **B** $-6x^2 + 6x$ **C** $-6x^2 + 3x - 1$
D $-6x^2 + 6x - 1$ **E** $6x^2 - 6x - 1$

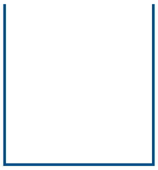
21 If $f(x) = 2x^{\frac{p}{q}}$, where p and q are integers, $f'(x)$ equals:

A $2x^{\frac{(p-q)}{q}}$ **B** $2px^{\frac{p}{q} - 1}$ **C** 2
D $\frac{2p}{q}x^{\frac{(p-q)}{q}}$ **E** $\frac{2p}{q}x$

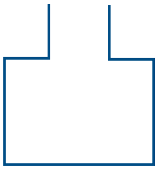
Short-response questions

1 A liquid is poured into each of the containers shown below at a constant rate. For each container, draw a graph to show how the depth of the water varies with time.

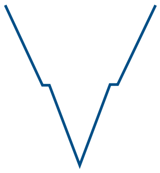
a



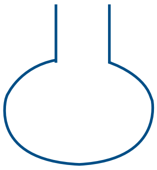
b



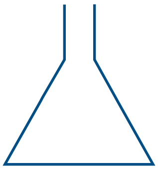
c



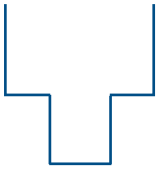
d



e



f



2 **a** A car travels from New Delhi to Agra, a distance of 200 km, in 3 hours. If the car travels at a constant speed, draw a distance–time graph and calculate this constant speed. For the distance–time graph, use minutes as the unit of time.

b A person walks at a constant speed of 1.5 m/s for 20 seconds. Draw a distance–time graph to illustrate this.

c A car travels along a road for $6\frac{1}{2}$ hours. The trip can be described in three sections.

Section A Car travels at a constant speed of 40 km/h for 1 hour.

Section B Car travels at a constant speed of 80 km/h for $3\frac{1}{2}$ hours.

Section C Car travels at a constant speed of 90 km/h for 2 hours.

Draw a distance–time graph that illustrates this motion.

3 The surface area, s cm^2, of a cube is given by the expression $s = 6x^2$, where x cm is the length of each edge of the cube. Find the average rate at which the surface area changes with respect to x, as x increases from $x = 2$ to $x = 4$.

4 For $y = \cos x$:

a Find the average rate at which y changes with respect to x over each of these domains.

i $0 \le x \le 0.1$ **ii** $0 \le x \le 0.01$

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

5 Let $s(t) = 4t - 6t^3$ be the displacement function of a particle moving on a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval $0 \le t \le 1$.

b Find the average velocity for the time interval $0.9 \le t \le 1$.

c Estimate the instantaneous velocity for $t = 1$.

6 Find $\dfrac{dy}{dx}$ when:

a $y = 3x^2 - 2x + 6$ **b** $y = 5$ **c** $y = 2x(2 - x)$

d $y = 4(2x - 1)(5x + 2)$ **e** $y = (x + 1)(3x - 2)$ **f** $y = (x + 1)(2 - 3x)$

7 Find $\frac{dy}{dx}$ when:

a $y = -x$ **b** $y = 10$ **c** $y = \frac{(x+3)(2x+1)}{4}$

d $y = \frac{2x^3 - x^2}{3x}$ **e** $y = \frac{x^4 + 3x^2}{2x^2}$

8 For each of these functions, find the y coordinates and the gradient at the point on the curve for the given value of x.

a $y = x^2 - 2x + 1, x = 2$ **b** $y = x^2 - 2x, x = -1$

c $y = (x+2)(x-4), x = 3$ **d** $y = 3x^2 - 2x^3, x = -2$

9 Find the coordinates of the points on the curves, given by the following equations, at which the gradient has the given value:

a $y = x^2 - 3x + 1;\ \frac{dy}{dx} = 0$ **b** $y = x^3 - 6x^2 + 4;\ \frac{dy}{dx} = -12$

c $y = x^2 - x^3;\ \frac{dy}{dx} = -1$ **d** $y = x^3 - 2x + 7;\ \frac{dy}{dx} = 1$

e $y = x^4 - 2x^3 + 1;\ \frac{dy}{dx} = 0$ **f** $y = x\,(x-3)^2;\ \frac{dy}{dx} = 0$

10 For the function with rule $f(x) = 3(2x-1)^2$, find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) \geq 0$

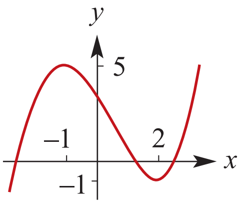
d $f'(x) < 0$ **e** $f(x) > 0$ **f** $f'(x) = 3$

11 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point (1, 1). Find the:

a values of a and b

b coordinates of the points at which the gradient is 0

12 Sketch the graph of $y = f'(x)$. (All details cannot be determined but the axis intercepts and shape of graph can be determined.)

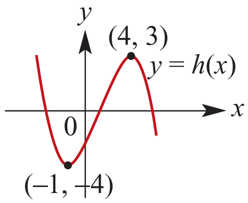


13 For the graph of $y = h(x)$, find:

a $\{x : h'(x) > 0\}$

b $\{x : h'(x) < 0\}$

c $\{x : h'(x) = 0\}$



14 Find the derivative of each of the following with respect to x:

a x^{-4} **b** $2x^{-3}$ **c** $\frac{-1}{3x^2}$

d $\frac{-1}{x^4}$ **e** $\frac{3}{x^5}$ **f** $\frac{x^2 + x^3}{x^4}$

g $\frac{3x^2 + 2x}{x^2}$ **h** $5x^2 - \frac{2}{x}$

15 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{2}}$ **b** $\sqrt[3]{x}$ **c** $\frac{-2}{x^{\frac{1}{3}}}$

d $x^{\frac{4}{3}}$ **e** $x^{-\frac{1}{3}}$ **f** $x^{-\frac{1}{3}} + 2x^{\frac{3}{5}}$

16 Differentiate each of the following with respect to x:

a $(2x+3)^2$ **b** $2(3x+4)^4$

17 Find the gradient of each of the following curves at the given point:

a $y = \sqrt{x}$; $(9, 3)$ **b** $y = \frac{2}{x^2}$; $\left(4, \frac{1}{8}\right)$ **c** $y = 3 + \frac{2}{x}$; $(1, 5)$

18 Find the coordinates of the point(s) on the curve with equation $y = \frac{1}{x}$ for which the gradient is -4.

19 Find the coordinates of the point(s) on the curve with equation $y = \sqrt{x}$ for which the gradient is 2.

20 An object follows a path (in a vertical plane) described by the equation $y = x - 0.01x^2$, where x is the horizontal distance travelled and y the height above ground level when the particle has travelled a distance, x. The object travels from (0, 0), a point at ground level.

a How far horizontally does the particle go before returning to ground level?

b Find $\frac{dy}{dx}$.

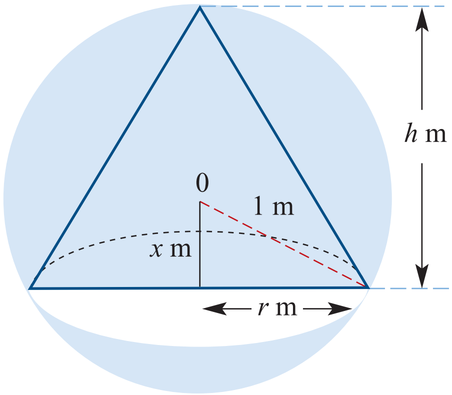
c Find the value of x for which $\frac{dy}{dx} = 0$ and the corresponding y value.

d Sketch the graph of y against x.

e Find the coordinates of the point on the path for which the gradient is:

i $\frac{1}{2}$ **ii** $-\frac{1}{2}$

21 A right circular cone lies inside a sphere of radius 1 m, as shown. The centre of the sphere, *0*, lies x m from the base of the cone.



a The volume of a cone is given by the formula

$$V = \frac{1}{3}\pi r^2 h.$$

Find:

i r in terms of x

ii h in terms of x

b Show that $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

c State a suitable domain for the function with rule $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

d **i** Find $\frac{dV}{dx}$. **ii** Find $\{x : \frac{dV}{dx} = 0\}$.

iii State the maximum possible volume of the cone.

e Sketch the graph of V against x.

22 The number of insects in a colony at time t days after 1 January, 2009 is approximated by the function with rule $P(t) = 1000 \times 2^{\frac{t}{20}}$, where $t = 0$ corresponds to 1 January, 2009. This rule for the population is valid for the entire year.

a Find the approximate number of insects in the colony on 1 January.

b Find the approximate number of insects in the colony on 10 January (i.e. when $t = 9$).

c For what values of t is $P(t)$ equal to:

i 4000? **ii** 6000? (Give your answer correct to 2 decimal places.)

d Find $P(20)$ and $P(15)$ and, hence, calculate the average rate of change of P with respect to time for the interval of time $15 \le t \le 20$, giving your answer correct to 2 decimal places.

e **i** Find the average rate of change of P with respect to t for the interval $15 \le t \le 15 + h$, in terms of h.

ii Explain how the instantaneous rate of change of P with respect to t, for $t = 15$, could be found by numerical methods.

23 The population density (number of residents per unit area) of many cities depends on the distance from the city centre. For a particular city, the population density P (in thousands of people per square kilometre) at a distance of r kilometres from the centre is approximated by $P = 10 + 40r - 20r^2$.

a What is the population density in the centre of the city?

b What are the possible values for r? **c** Sketch a graph of P against r.

d **i** Find $\frac{dP}{dr}$. **ii** Evaluate $\frac{dP}{dr}$ when $r = 0.5$, 1 and 2.

iii Sketch a graph of $\frac{dP}{dr}$ against r.

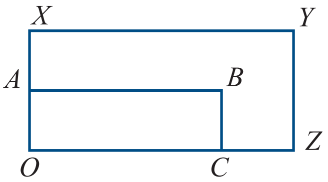
e Use a calculator to determine where the population density is greatest.

24 A tadpole begins to swim vertically upwards in a pond and, after t seconds, it is $(25 - 0.1t^3)$ cm below the surface.

a How long does the tadpole take to reach the surface, and what is its speed then?

b What is the average speed over this time?

25



The area of rectangle $OABC$ is 120 cm^2. Let the length of OC be x cm, $CZ = 5$ cm and $AX = 7$ cm.

a Find the length of OA in terms of x. **b** Find the length of OX in terms of X.

c Find the length of OZ in terms of X.

d Find the area, A cm^2, of rectangle $OXYZ$ in terms of x.

e Find the value of x for which the area, A cm^2, is a minimum $\left(\frac{dA}{dx} = 0\right)$. Validate your answer using technology.

Answers

Chapter 1

Exercise 1A

1 a 57.5 **b** −1.2 **c** 24.25 **d** −16
e −4 **f** 2.75 **g** 6 **h** 1.5
i $\frac{10}{7}$ **j** 3 **k** 7 **l** −4
m −14.4 **n** −5 **o** 9 **p** −1
q −1.2 **r** 1.4 **s** 51 **t** 82
u −11.5 **v** 12 **w** 8 **x** −2.8
y 3.5 **z** 2.4

2 a $-\frac{b}{a}$ **b** $\frac{e-d}{c}$ **c** $\frac{c}{a}-b$ **d** $\frac{b}{a-c}$
e $\frac{ab}{a+b}$ **f** $a+b$ **g** $\frac{b-d}{a-c}$ **h** $\frac{bd-c}{a}$

3 a −18 **b** −78.2 **c** 16.75 **d** 28
e 34 **f** 0.1154

Exercise 1B

1 a $x(x+3)$ **b** $x(x-5)$ **c** $x(x+1)$
d $x(3x-4)$ **e** $x(5x-1)$ **f** $5x(x-3)$
g $3x(2x-5)$ **h** $-x(x+5)$ **i** $-4x(x-4)$

2 a $(x+4)(x+6)$ **b** $(x+1)(x+8)$
c $(x-3)(x-8)$ **d** $(x-6)(x+5)$
e $(x-4)(x-5)$ **f** $(x+3)(x-40)$
g $(x+2)(x-9)$ **h** $(x-3)(x-16)$
i $(x-7)(x+12)$ **j** $(5x+3)(x+4)$
k $(3x-2)(2x-1)$ **l** $(5x-4)(x-3)$
m $(x+4)(6x-5)$ **n** $(3x+2)(5x-7)$
o $(x+1)(15x-14)$

3 a $(x-7)(x+7)$ **b** $(x-4)(x+4)$
c $(1-x)(1+x)$ **d** $(2x-9)(2x+9)$
e $2(5x-7)(5x+7)$ **f** $4(x-3)(x+3)$

4 a $5(x-4)(x+4)$ **b** $6(x-3)(x+3)$
c $2(2x-5)(2x+5)$ **d** $4(x+2)(x+3)$
e $3(x+2)(x+3)$ **f** $2(x-2)(x-7)$
g $5(y+2)(y-6)$ **h** $3(x+3)(2x+5)$
i $2(2x+7)(3x-4)$ **j** $x(x-6)(x+1)$
k $x(5x-6)(x-2)$ **l** $3x(x-4)^2$

Exercise 1C

1 a −3 or 3 **b** −6 or 6
c −8 or 8 **d** −0.5 or 0.5
e 4 or −4 **f** −0.25 or 0.25
g −4.98 or 4.98 **h** 0

2 a 2 or 4 **b** −3 or 11
c −7 or 2 **d** −16 or 4
e −1.5 or −1 **f** 0.5 or 1.5
g −3 or 8 **h** −1.5 or $-\frac{2}{3}$
i −1.5 or 2

3 a −7.606 or −0.394 **b** 0.7085 or 11.29
c −3.245 or 9.245 **d** −6.071 or 1.071
e −5.804 or 0.804 **f** −12.75 or 2.746

4 a −1.5 or −1 **b** 0.134 or 1.866
c −5.266 or 6.266 **d** −5.727 or 1.881
e −13.57 or 1.572 **f** −1.954 or 8.954

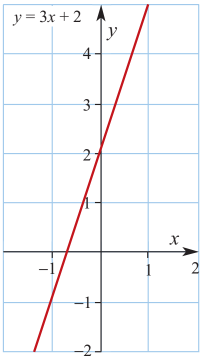
5 a −0.65 or 4.65 **b** −0.41 or 2.41
c −1.23 or 1.90

6 a −8 or −1 **b** −11 or 3
c −4 or 7 **d** −2.243 or 6.243
e −2.281 or −0.219 **f** −1.5 or −0.5
g −2 or 8 **h** 0.5 or $\frac{5}{3}$
i −1.886 or 2.386 **j** $\frac{5}{6}$ or 3
k −1.5 or 3 **l** 0.5 or 0.6
m −0.75 or $\frac{2}{3}$ **n** 0.5
o $\frac{10}{3}$ or 0 **p** 0 or 3
q −5 or −3 **r** 0.2 or 2

7 4 or 9
8 3
9 2 or 2.375
10 1 or $-\frac{m}{4}$

Exercise 1D

1 a



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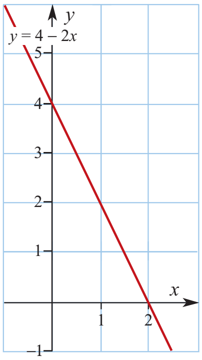

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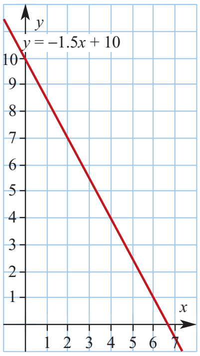

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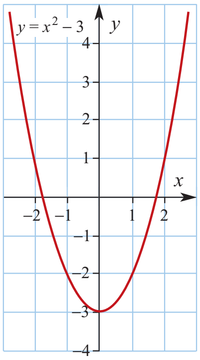

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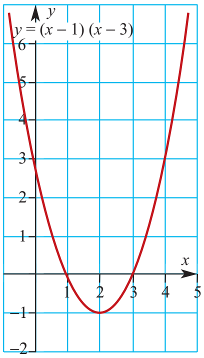

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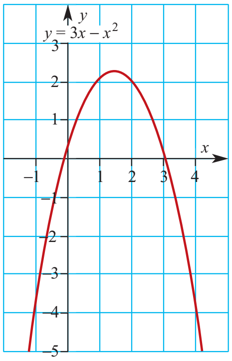

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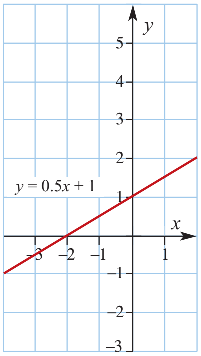

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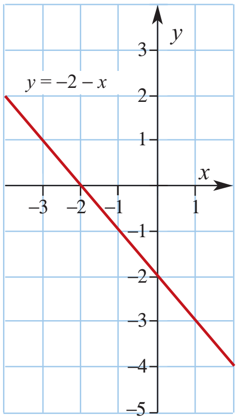

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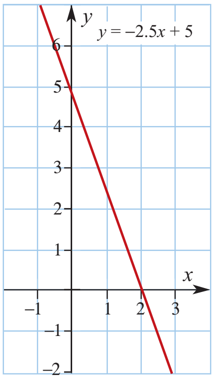

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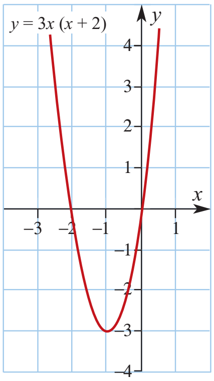

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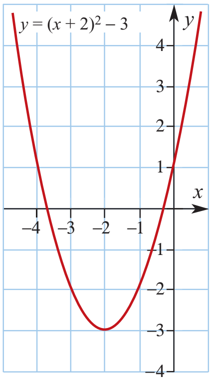

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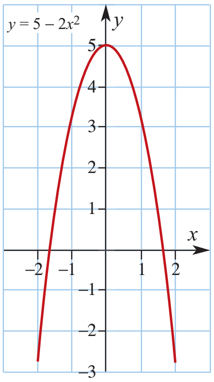
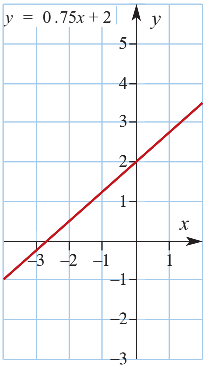


Exercise 1E

1 a



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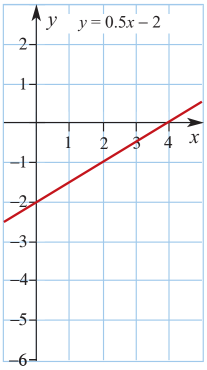

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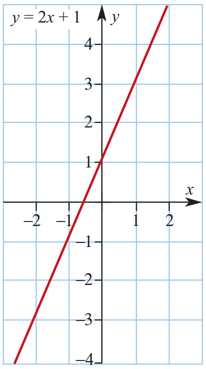

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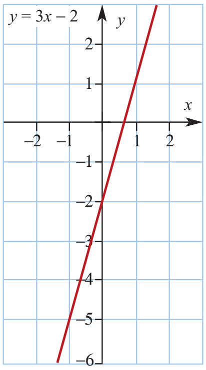

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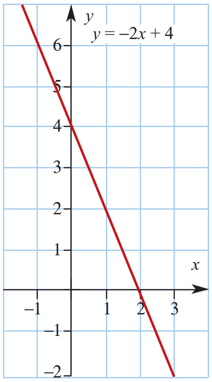

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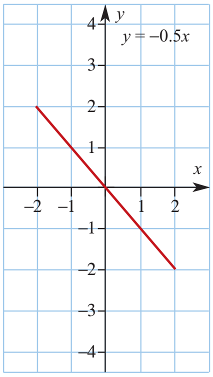

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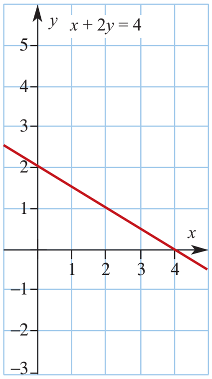

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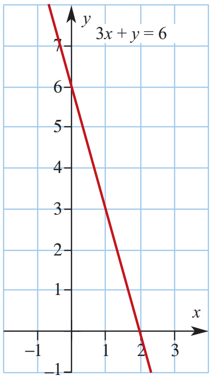

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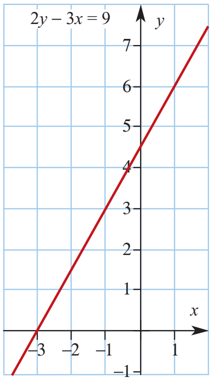

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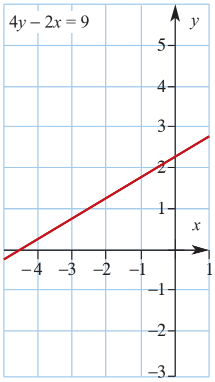

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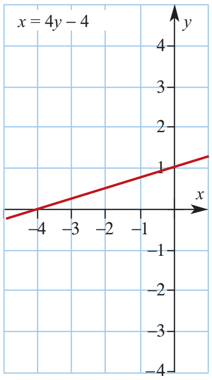

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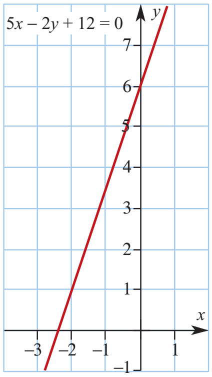

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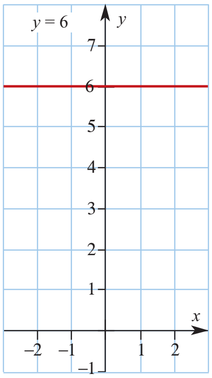

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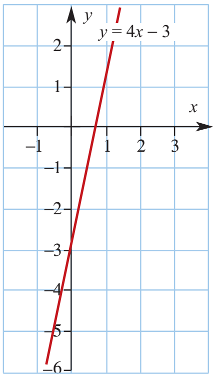

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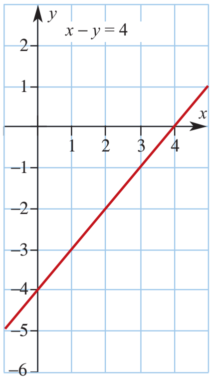

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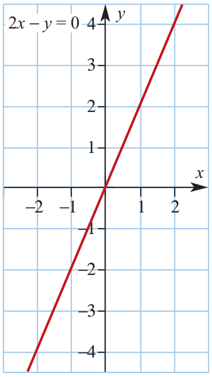

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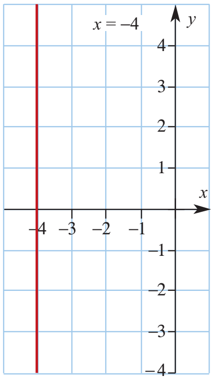

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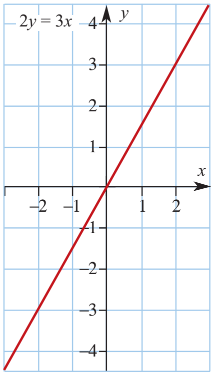

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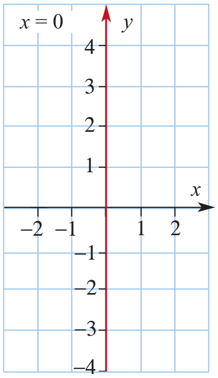

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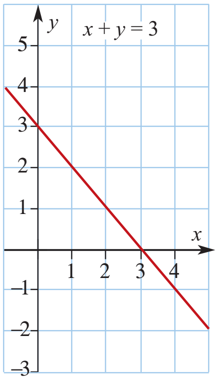

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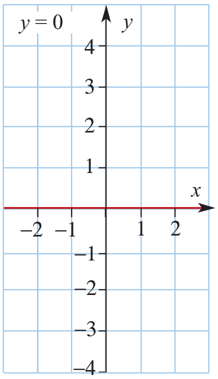

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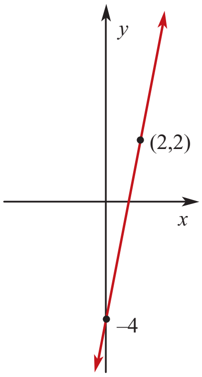

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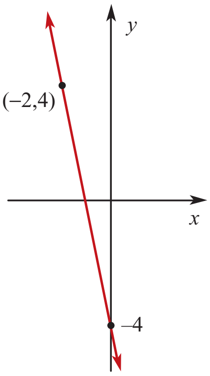

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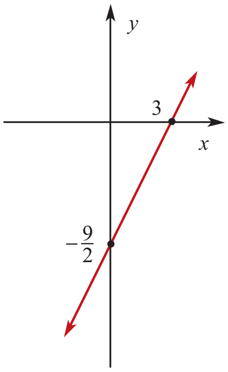

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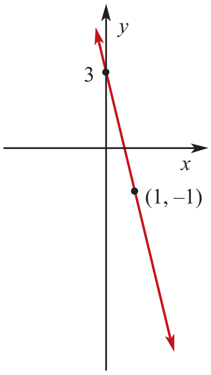

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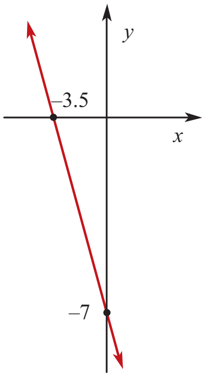

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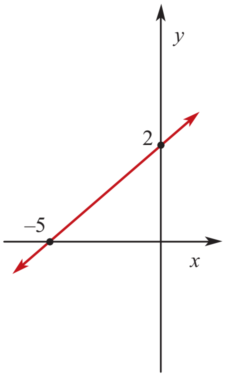

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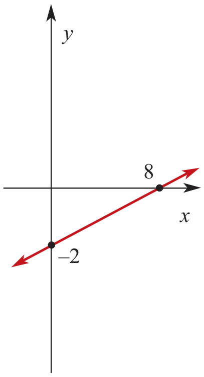

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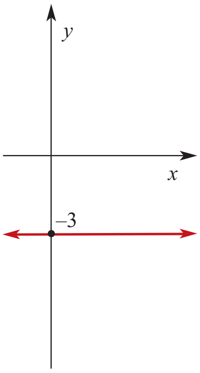

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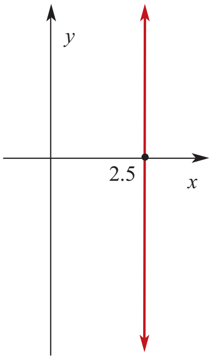

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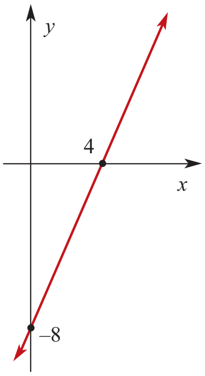

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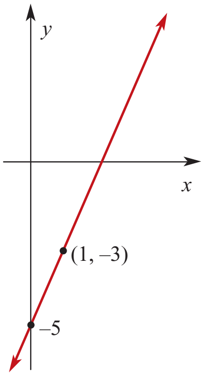

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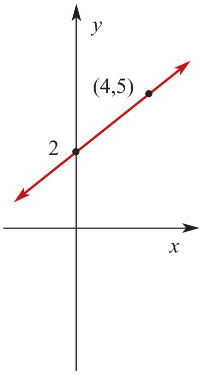
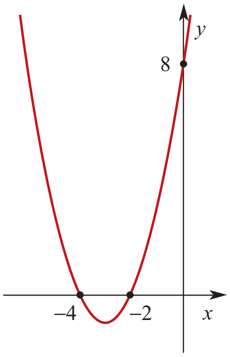


Exercise 1F

1 a



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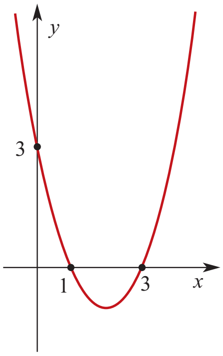

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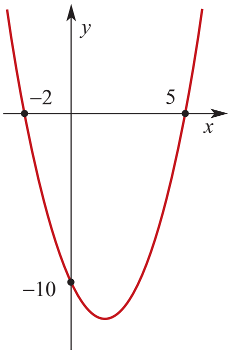

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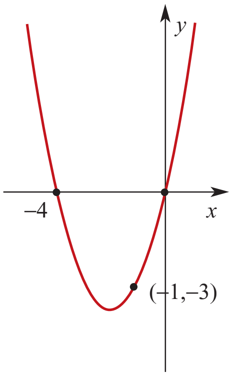

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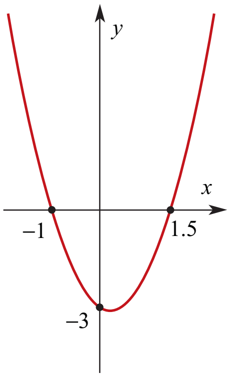

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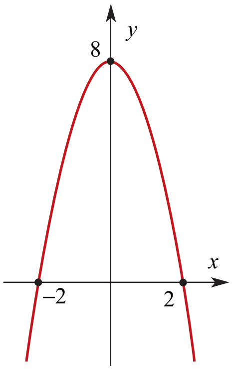
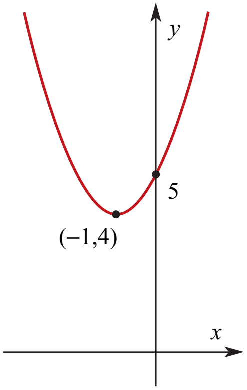
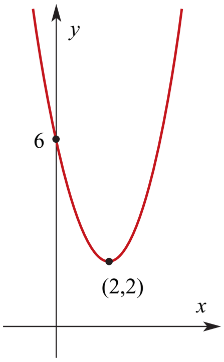


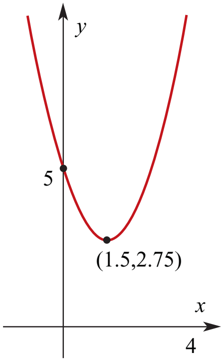
2 a $y = (x + 1)^2 + 4$



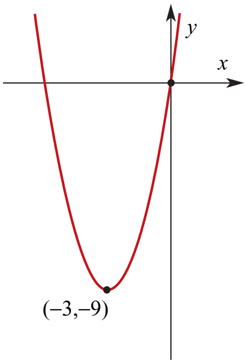
b $y = (x - 2)^2 + 2$



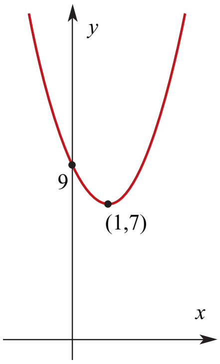
c $y = (x - 1.5)^2 + 2.75$



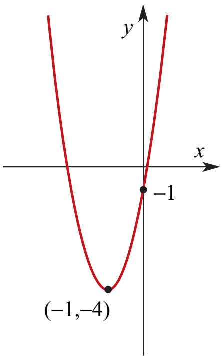
d $y = (x + 3)^2 - 9$



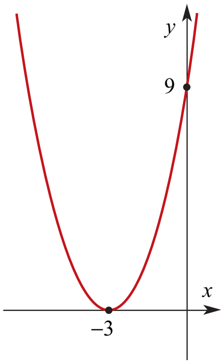
e $y = 2(x - 1)^2 + 7$



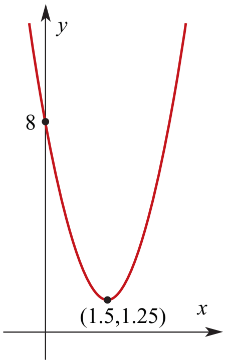
f $y = 3(x + 1)^2 - 4$



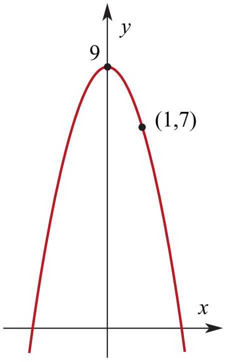
g $y = (x + 3)^2$



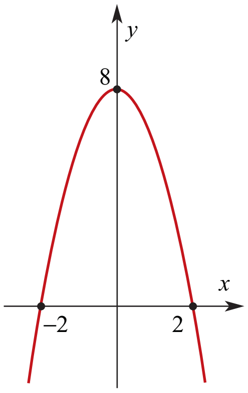
h $y = 3(x - 1.5)^2 + 1.25$



i $y = 9 - 2x^2$



3 a



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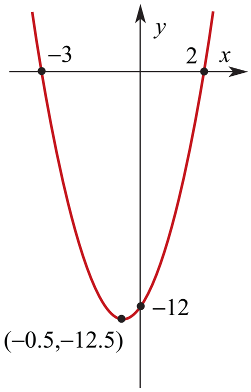

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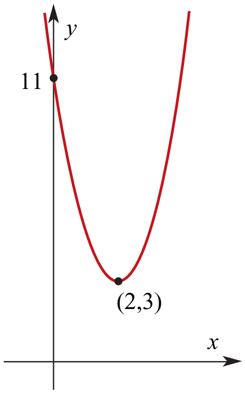

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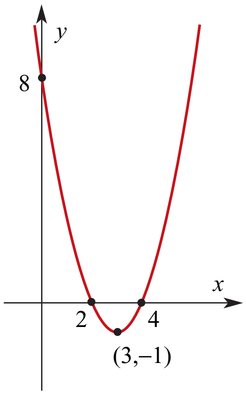

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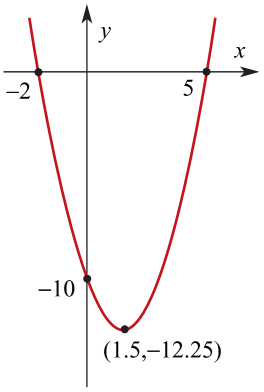

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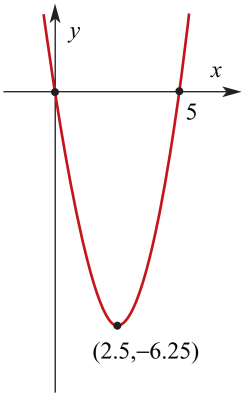

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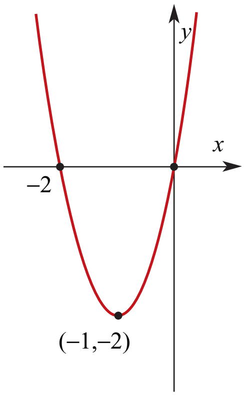

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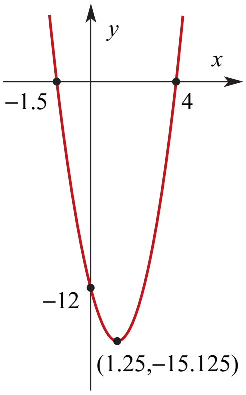

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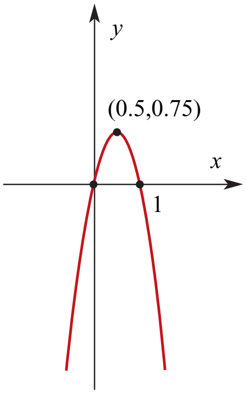
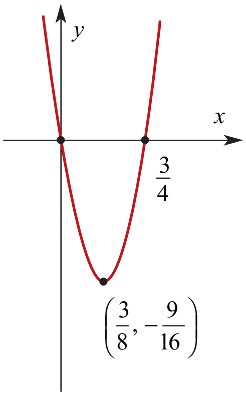


4 a



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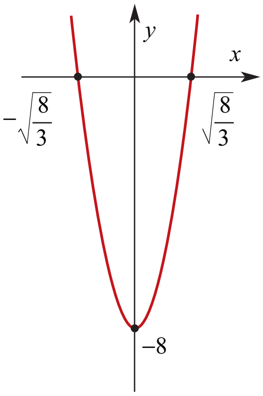

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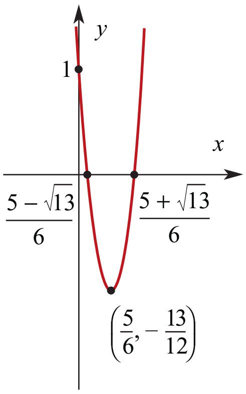


Exercise 1G

1 a $x = -1 \quad y = -1$ **b** $x = 5 \quad y = 21$
c $x = -1 \quad y = 5$ **d** $x = 8 \quad y = -2$
e $x = 3 \quad y = 4$ **f** $x = 7 \quad y = \frac{1}{2}$
2 a $x = 1 \quad y = 5$ **b** $x = 2.5 \quad y = -1$
c $x = 2 \quad y = -1$ **d** $s = -1 \quad t = 4$
e $p = 1 \quad q = -1$ **f** $x = -1 \quad y = 2.5$
g $x = -1 \quad y = 2$ or $x = 3 \quad y = 10$ **h** $x = -3 \quad y = 8$ or $x = 1 \quad y = 0$
i $x = 1 \quad y = 0$ **j** $x = -2 \quad y = -4$ or $x = 5 \quad y = 17$
k $x = 1 \quad y = -1$ **l** $x = 0 \quad y = 0$ or $x = 1 \quad y = 2$
3 a $x = 3 \; y = -2$ **b** $x = 7 \quad y = 2$
c $x = 4 \quad y = -3$ **d** $x = 7 \quad y = 3$
e $x = 4 \quad y = -2$ **f** $x = 3 \quad y = 1$
g $x = 5 \quad y = -2$ **h** $x = 3 \quad y = -2$
i $x = -2 \quad y = -3$
4 a $x = 2 \quad y = 4$
b $x = -1 \quad y = 7$
c $x = 0 \quad y = 8$
d $x = -2.236 \quad y = 0.528$ or $x = 2.236 \quad y = 9.472$
e $x = -2.608 \quad y = 3.804$ or $x = 2.108 \quad y = 1.446$
f $x = -0.781 \quad y = 4.390$ or $x = 1.281 \quad y = 3.360$

Exercise 1H

1 a 20 **b** −12 **c** 16 **d** 9 **e** 41
2 a cross **b** neither **c** touch
d cross **e** neither **f** touch
3 a 2 **b** 0 **c** 1 **d** 2 **e** 1 **f** 0
4 a 1 rational root **b** 2 rational roots
c 2 irrational roots **d** 1 rational root
e 2 irrational roots **f** no real roots
5 $\Delta = m^2 + 8m + 16$
6 a $m = 3$ or -3 **b** $-3 < m < 3$
c $m > 3$ or $m < -3$

Exercise 1I

1 a–d Check with your teacher
2 a $c = -13$ **b** $m = 3$ or 7
c $c = -\frac{1}{4}$
3 a $a = 3$ or -1 **b** $b = 1$
c $c > -\frac{1}{4}$

Exercise 1J

1 a $y = 3x + 5$ **b** $y = -4x + 6$
c $y = 3x - 4$
2 a $y = 3x - 11$ **b** $y = -2x + 9$
3 a $y = 2x + 6$ **b** $y = 2.5x - 2$
4 a $y = 4x + 4$ **b** $y = -\frac{2}{3}x$
c $y = -x - 2$ **d** $y = \frac{1}{2}x - 1$
e $y = 3.5$ **f** $x = -2$
5 a $2x + 3y - 12 = 0$ **b** $2x + y + 6 = 0$
c $x + y - 8 = 0$ **d** $x + 2y - 4 = 0$
e $2x - 3y - 2 = 0$ **f** $x = 3$
g $2x - 2y + 7 = 0$ **h** $y = 5$
i $2x + 4y - 1 = 0$
6 yes
7 AB: $2x - 3y + 1 = 0$
BC: $3x + 2y - 18 = 0$
AC: $x - 8y - 6 = 0$

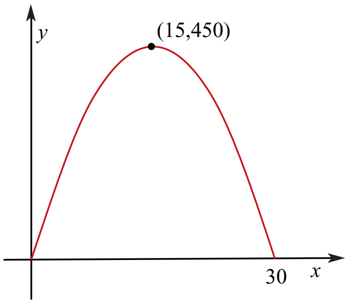
Exercise 1K

1 a $y = -\frac{5}{16}x^2 + 5$ **b** $y = x^2$
c $y = \frac{1}{11}x\,(x + 7)$ **d** $y = (x - 1)(x - 3)$
e $y = -\frac{5}{4}(x + 1)^2 + 5$ **f** $y = (x - 2)^2 + 2$
2 a $y = \frac{10}{7}(x - 2)(x + 4)$ **b** $y = -4x\,(x - 4)$
c $y = \frac{4}{5}(x - 2)(x - 5)$
3 a $y = \frac{1}{2}(x + 1)^2 + 2$
b $y = -\frac{3}{2}(x - 2)^2 + 3$
c $y = \frac{4}{3}x^2$
4 a $y = -2x^2 + x + 5$
b $y = \frac{1}{4}x^2 - \frac{1}{2}x + 3$
c $y = 4x^2 - 7x$
5 $y = \frac{1}{180}(x - 90)^2 + 30$

Exercise 1L

1 a,b Check with your teacher
2 a C = 10, F = 50
b −40° C = −40° F
3 a,b Check with your teacher
4 13
5 a $\dfrac{bc - a}{c - 1}$ **b** $\dfrac{2b}{c - a}$
6 $\dfrac{ab}{a - b + c}$

7 60 km
8 50 km/h
9 6 cm and 2 cm
10 6 cm and 5 cm
11 $A = 60x - 2x^2$



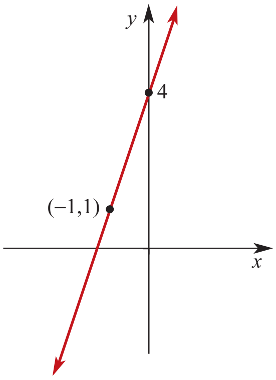
12 $0.226 < x < 0.774$
13 1.5 cm or 1.32 cm
14 0.618
15 a $m = \pm\sqrt{8}$ **b** $m \le -\sqrt{5}$ or $m \ge \sqrt{5}$
16 9.9 cm
17 half-way across the edge of the square

Multiple-choice answers

1 C **2** C **3** A **4** A **5** C **6** E
7 A **8** B **9** E **10** D

Short-response answers

1 a 1 **b** −1.5 **c** $-\frac{2}{3}$ **d** −27
e 12 **f** $\frac{44}{13}$ **g** 6
2 a $a - b$ **b** $\frac{cd - b}{a}$ **c** $c + \frac{d}{a}$
3 a $4(x - 2)$ **b** $x(3x + 8)$
c $3x(8a - 1)$ **d** $(6 - x)(6 + x)$
e $(x + 4)(x - 3)$ **f** $(x + 2)(x - 1)$
g $(2x - 1)(x + 2)$ **h** $(3x + 2)(2x + 1)$
i $(6x + 5)(x - 3)$
4 a $x = \pm 5$ **b** $x = \pm\sqrt{33}$ **c** $x = \pm 7$
5 a −4 or 3 **b** −8 or −3 **c** −6 or 11
d 1.5 or 5 **e** −2.5 or $\frac{4}{3}$ **f** −1.5 or 2
6 a 0.5505 or 5.4495 **b** −2.781 or −0.719
c −3.107 or 0.1073
7 a



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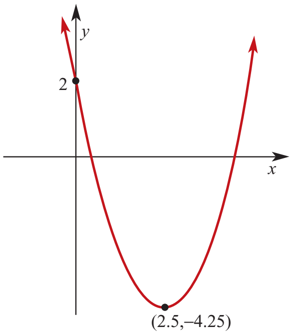

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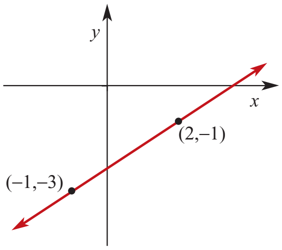
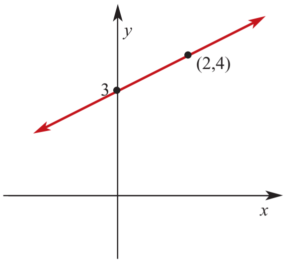


8 a



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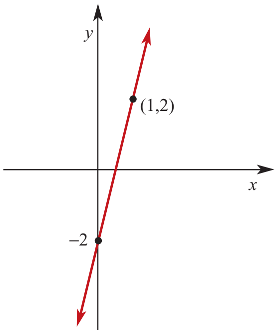

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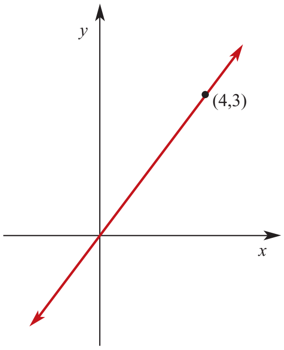
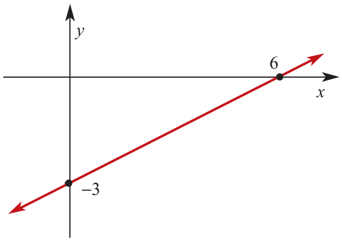


9 a



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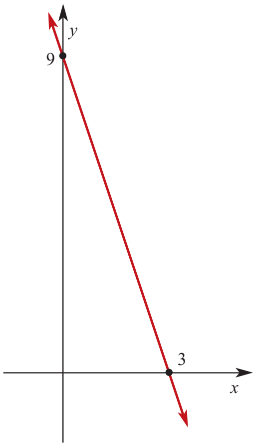

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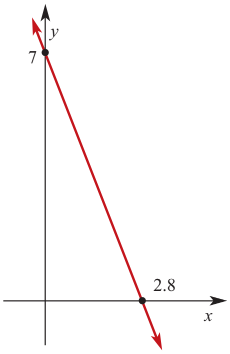
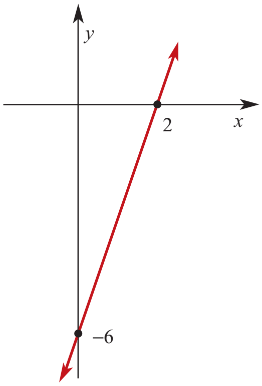


10 a



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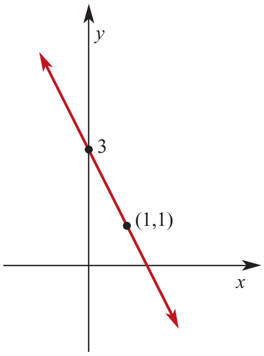

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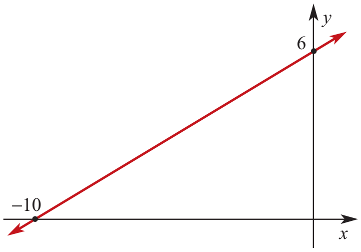
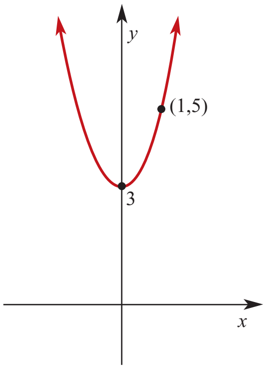


11 a



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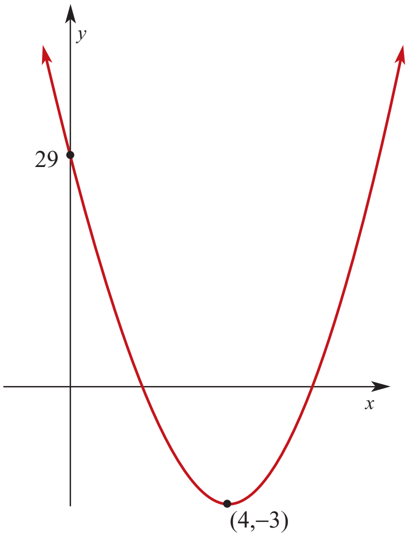

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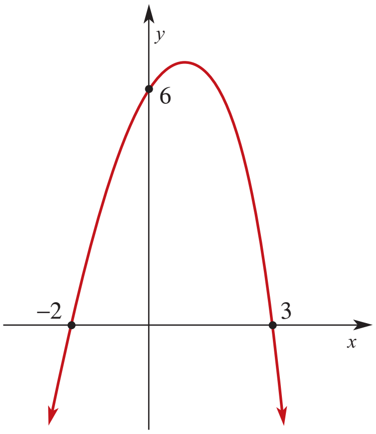
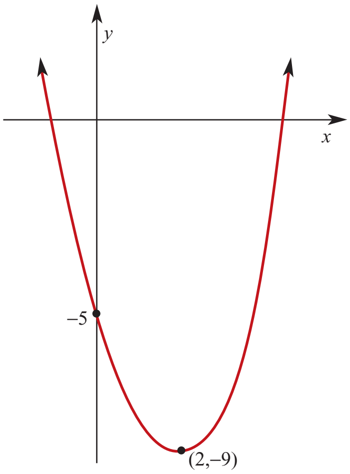
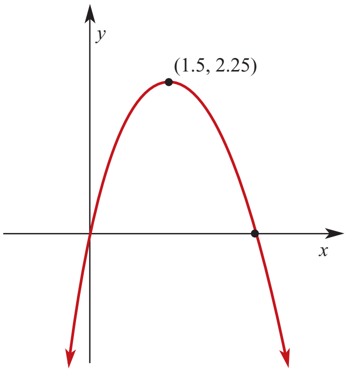


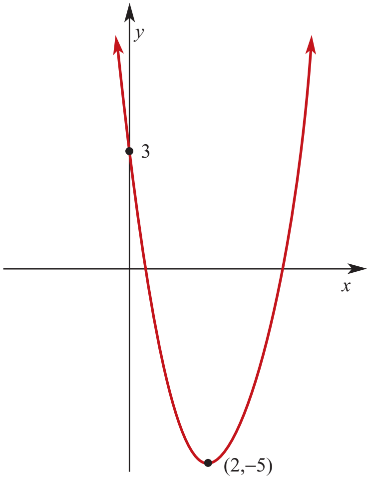
12 a $y = (x - 2)^2 - 9$



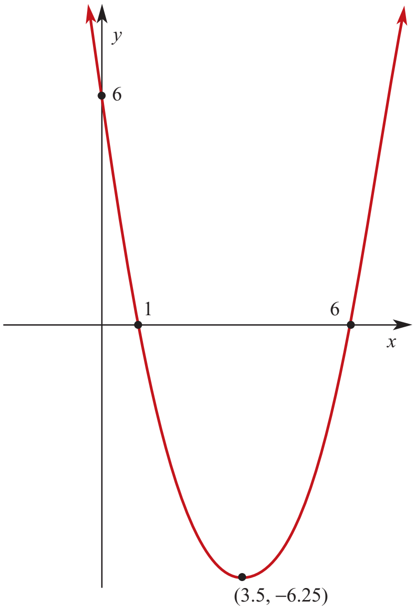
b $y = -(x - 1.5)^2 + 2.25$



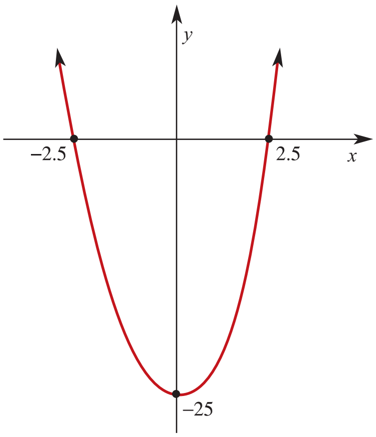
c $y = 2(x - 2)^2 - 5$



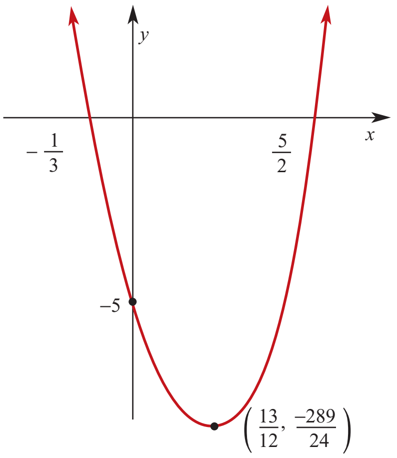
13 a



b



c



14 a $x = -1 \quad y = 2$
b $x = -1 \quad y = 4$
c $x = 4 \quad y = 1$
15 a $x = 2.5 \quad y = -1$
b $x = 4 \quad y = -2$
c $x = 1 \quad y = -0.5$
16 a $x = 0.5 \quad y = 6$
b $x = -2.5 \quad y = -3$
c $x = 2.121 \quad y = -0.257$
or
$x = -2.121 \quad y = -8.742$
17 a $x = 2 \quad y = 4$
b no solution
c $x = 3.0769 \quad y = -4.307$
18 a $(-1, 1)$ and $(3, 9)$
b $(-1.5, 4.5)$ and $(4, 32)$
19 a 2 irrational roots **b** 1 rational root
c no real roots
20 Check with your teacher
21 a $y = (x + 4)(x - 2)$
b $y = 3(x - 1)^2 - 3$
c $y = -\frac{1}{2}(x - 1)^2 + 2$
22 a $y = \frac{4}{3}(x + 1)(x - 3)$
b $y = -2(x - 3)^2 + 2$
c $y = 3x^2 + 4x - 7$

Chapter 2

Exercise 2A

1 a 4.10 **b** 0.87 **c** 2.94
d 4.08 **e** 33.69° **f** 11.92
2 $\frac{40}{\sqrt{3}}$ cm
3 66.42°, 66.42° and 47.16°
4 23 m
5 a 9.59° **b** $\sqrt{35}$ m
6 a 60° **b** 17.32 m
7 a 6.84 m **b** 6.15 m
8 12.51°
9 182.7 m
10 1451 m
11 a $5\sqrt{2}$ cm **b** 45°
12 3.07 cm
13 37.8 cm
14 31.24 m
15 4.38 m
16 57.74 m

Exercise 2B

1 a 8.15 **b** 3.98 **c** 11.75 **d** 9.46
2 a 56.32° **b** 36.22° **c** 49.54°
3 a $A = 48°$, $b = 13.84$ cm, $c = 15.44$ cm
b $a = 7.26$, $C = 56.45°$, $c = 6.26$
c $B = 19.8°$, $b = 4.66$, $c = 8.27$
d $C = 30°$, $a = 5.41$, $c = 15.56$
4 $C = 26.69°$, $A = 24.31°$, $a = 4.18$
5 554.26 m
6 35.64 m
7 1659.86 m
8 a 26.60 m **b** 75.12 m

Exercise 2C

1 5.93 cm
2 $\angle ABC = 97.90°$, $\angle ACB = 52.41°$
3 a 26 **b** 11.74 **c** 49.29° **d** 73
e 68.70 **f** 47.22° **g** 7.59 **h** 38.05°
4 2.626 km
5 3.23 km
6 a 8.23 cm **b** 3.77 cm
7 55.93 cm
8 a 7.326 cm **b** 5.53 cm
9 a 83.62° **b** 64.46°
10 a 87.61 m **b** 67.7 m

Exercise 2D

1 400.10 m
2 34.77 m
3 575.18 m
4 109.90 m
5 16.51 m
6 027°
7 056°
8 a 034° **b** 214°
9 a 3583.04 m **b** 353° or N7°W
10 $\angle ASB = 113°$
11 22.01°
12 a $\angle BAC = 49°$ **b** 264.24 km
13 10.63 km

Exercise 2E

1 a $\frac{\pi}{3}$ **b** $\frac{4\pi}{5}$ **c** $\frac{4\pi}{3}$

d $\frac{11\pi}{6}$ **e** $\frac{7\pi}{3}$ **f** $\frac{8\pi}{3}$

2 a 120° **b** 150° **c** 210° **d** 162°

e 100° **f** 324° **g** 220° **h** 324°

3 a 34.38° **b** 108.29° **c** 166.16° **d** 246.94°

e 213.14° **f** 296.79° **g** 271.01° **h** 343.77°

4 a 0.66 **b** 1.27 **c** 1.87 **d** 2.81

e 1.47 **f** 3.98 **g** 2.38 **h** 5.74

5 a 0.95 **b** 0.75 **c** −0.82 **d** 0.96

e −0.50 **f** −0.03 **g** −0.86 **h** 0.61

i −34.23 **j** 0.36

6 a 0, −1, 0 **b** −1, 0, undefined

c 1, 0, undefined **d** −1, 0, undefined

e −1, 0, undefined **f** 0, 1, 0

g 0, −1, 0 **h** 0, −1, 0

Exercise 2F

1 a $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$ **b** $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$

c 0, −1, 0 **d** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}$

e $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1$ **f** $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$

g $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$ **h** $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$

i $\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$ **j** $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$

k −1, 0, undefined **l** 0, 1, 0

2 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** 0 **d** $-\frac{1}{2}$

e 0 **f** $\sqrt{3}$ **g** $-\frac{\sqrt{3}}{2}$ **h** $\frac{1}{\sqrt{2}}$

i $-\frac{1}{\sqrt{3}}$ **j** −1 **k** −1 **l** undefined

3 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{3}}$ **d** undefined

e 0 **f** $-\frac{1}{\sqrt{2}}$ **g** $\frac{1}{\sqrt{2}}$ **h** −1

Exercise 2G

1 a 13 cm **b** 15.26 cm **c** 31.61° **d** 38.17°

2 a 4 cm **b** 71.57° **c** 12.65 cm

d 13.27 cm **e** 72.45° **f** 266.39 cm^2

3 17.58°

4 1702.55 m

5 10.31° at B, 14.43° at A and C

6 45.04 m

7 a 24.78° **b** 65.22° **c** 20.44°

8 42.40 m

9 1945.54 m

10 a 6.96 cm **b** 16.25 cm^2

11 a 5 km **b** 215.65° **c** 6°33′

12 5 m

13 11.37 m

14 14 ≈ 401 km/h

15 height $= \frac{8}{\sqrt{15}} = 2.07$ m

Multiple-choice answers

1 D **2** C **3** C **4** D **5** B

6 C **7** A **8** C **9** B **10** E

Short-response answers

1 a $\frac{7\pi}{6}$ **b** 0.51^c **c** 2.06^c

2 a 135° **b** 154°42′

3 a $-\frac{1}{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\sqrt{3}$

d 0 **e** $-\frac{1}{2}$ **f** undefined

g $-\frac{1}{2}$ **h** $-\frac{1}{2}$ **i** $-\frac{1}{\sqrt{3}}$

4 35.53 km

5 $\sqrt{91} \approx 9.54$ cm

6 143°

7 a 052.6°

b $\angle TQS = 33.14°$, bearing of T from Q is 105.9°

8 9.4 cm

9 a i 39° **ii** 9°

b 1425.16 m **c** 1083.29 m

10 13.45 km

11 a $AC = 4.16$ km, $BC = 2.4$ km

b 57.6 km/h

12 804 m

13 a $\angle ACB = 12°$, $\angle CBO = 53°$, $\angle CBA = 127°$

b 189.33 m **c** 113.94 m

14 a $\angle TAB = 3°$, $\angle ABT = 97°$, $\angle ATB = 80°$

b 2069.87 m

c 252.25 m

15 a 184.74 m **b** 199.71 m **c** 14.93 m

16 a 370.17 m **b** 287.94 m **c** 185.08 m

17 a $8\sqrt{2}$ cm **b** 10 cm **c** 10 cm **d** 68.90°

Chapter 3

Exercise 3A

1 a x^5 **b** $8x^7$ **c** x^2 **d** $2x^3$
e a^6 **f** 2^6 **g** x^2y^2 **h** x^2y^6
i $\frac{x^3}{y^3}$ **j** $\frac{x^6}{y^4}$ **k** 1 **l** 1
m $\frac{3}{2}$ **n** $\frac{2^3}{5^2}$

2 a x^9 **b** 2^{16} **c** 3^{17} **d** q^8p^9
e $a^{11}b^3$ **f** 2^8x^{18} **g** $m^{11}n^{12}p^{-2}$ **h** $2a^5b^{-2}$

3 a x^2y^3 **b** $8a^8b^3$ **c** x^5y^2 **d** $\frac{9}{2}x^2y^3$

4 a $\frac{1}{n^4p^5}$ **b** $\frac{2x^8z}{y^4}$ **c** $\frac{b^5}{a^5}$ **d** $\frac{a^3b}{c}$
e $a^{n+2}b^{n+1}c^{n-1}$

5 a 1 **b** 3^{17n} **c** $\frac{3^{4n-11}}{2^2}$
d $2^{n+1}3^{3n-1}$ **e** 5^{3n-2} **f** $2^{3x-3} \times 3^{-4}$
g $3^{6-n} \times 2^{-5n}$ **h** $3^3 = 27$ **i** 6
j $\frac{3^{4n}}{2}$ **k** $\frac{3^n}{5^n}$ **l** $\frac{1}{2 \times 3^{5n}}$

6 a $2^{12} = 4096$ **b** $5^5 = 3125$ **c** $3^3 = 27$

7 Check with your teacher

Exercise 3B

1 a 25 **b** 27 **c** $\frac{1}{9}$ **d** 16 **e** $\frac{1}{2}$ **f** $\frac{1}{4}$
g $\frac{1}{25}$ **h** 16 **i** $\frac{1}{10\,000}$ **j** 1000
k 27 **l** $\frac{3}{5}$

2 a $2^{\frac{5}{3}}$ **b** $a^{\frac{1}{6}}b^{-\frac{7}{6}}$ **c** $a^{-6}b^{\frac{9}{2}}$ **d** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$
e $\frac{1}{4}$ **f** x^6y^{-8} **g** $a^{\frac{14}{15}}$

3 a $(2x-1)^{3/2}$ **b** $(x-1)^{5/2}$ **c** $(x^2+1)^{3/2}$
d $(x-1)^{4/3}$ **e** $\frac{x}{\sqrt{x-1}}$ **f** $(5x^2+1)^{4/3}$

Exercise 3C

1 a 3 **b** 3 **c** $\frac{1}{2}$
d $\frac{3}{4}$ **e** $\frac{1}{3}$ **f** 4
g 2 **h** 3 **i** 3

2 a 1 **b** 2 **c** $-\frac{3}{2}$
d $\frac{4}{3}$ **e** -1 **f** 8
g 3 **h** -4 **i** 8
j 4 **k** $3\frac{1}{2}$ **l** 6
m $7\frac{1}{2}$

3 a $\frac{4}{5}$ **b** $\frac{3}{2}$ **c** $5\frac{1}{2}$

4 a 0 **b** 0, -2 **c** 1, 2 **d** 0, 1

5 a 2.32 **b** 1.29 **c** 1.26 **d** 1.75

6 a $x > 2$ **b** $x > \frac{1}{3}$ **c** $x \leq \frac{1}{2}$
d $x < 3$ **e** $x < \frac{3}{4}$ **f** $x > 1$
g $x \leq 3$

7 a $x \leq 1.43$ **b** $x \geq 0.77$
c $x > -1.89$ **d** $x > 2.71$

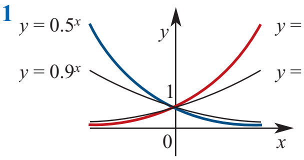
8 0.5

9 $x \leq 2$

10 a 2.5 **b** 2

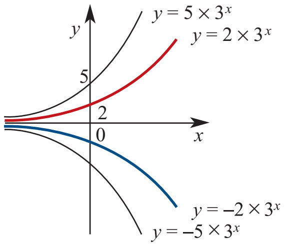
Exercise 3D

1



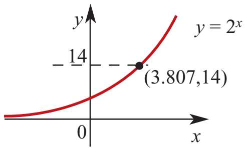
- all pass through (0, 1)
- base > 1, increasing
- base < 1, decreasing
- horizontal asymptote, $y = 0$

2 For $y = a \times b^x$

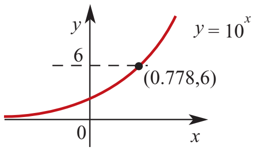


- y-axis intercept $(0, a)$
- c and d are reflections of a and b in the x-axis
- horizontal asymptote, $y = 0$

3 $x = 3.807$



4 $x = 0.778$



5 a–f Check with your teacher
6 a–d Check with your teacher

Exercise 3E

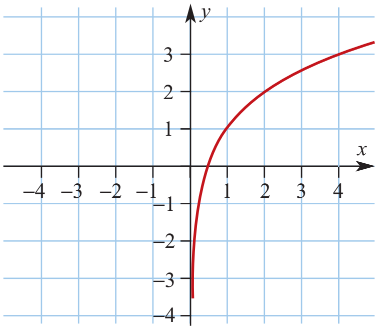
1 a 3 **b** 4 **c** −7 **d** −3 **e** 4
f −3 **g** 4 **h** −6 **i** −9 **j** −1
k 4 **l** −2
2 a $\log_2(10a)$ **b** 1 **c** $\log_2\left(\frac{9}{4}\right)$
d 1 **e** $-\log_5 6$ **f** −2
g $3\log_2 a$ **h** 9
3 a 2 **b** 7 **c** 9 **d** 1
e $\frac{5}{2}$ **f** $5\log_x a$ **g** 3 **h** 1
4 a 2 **b** 27 **c** $\frac{1}{125}$ **d** 8
e 30 **f** $\frac{2}{3}$ **g** 8 **h** 64
i 4 **j** 10
5 a 5 **b** 32.5 **c** 22 **d** 20
e $\frac{3 \pm \sqrt{17}}{2}$ **f** 3 or 0
6 $2 + 3a - \frac{5c}{2}$
7 Check with your teacher
8 10
9 a 4 **b** $\frac{6}{5}$ **c** 3
d 10 **e** 9 **f** 2

Exercise 3F

1 a 2.81 **b** −1.32 **c** 2.40 **d** 0.79 **e** −2.58
f −0.58 **g** −4.30 **h** −1.38 **i** 3.10 **j** −0.68
2 a $x > 3$ **b** $x < 1.46$ **c** $x < -1.15$
d $x \leq 2.77$ **e** $x \geq 1.31$
3 a–f Check with your teacher
4 $m = 0.094$, $d_0 = 41.92$

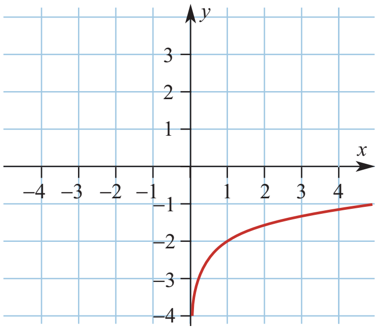
Exercise 3G

1 a



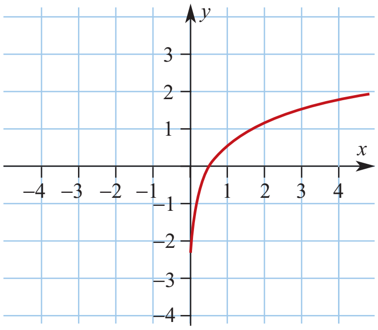
intercept: $x = \frac{1}{2}$
asymptote: $x = 0$

b



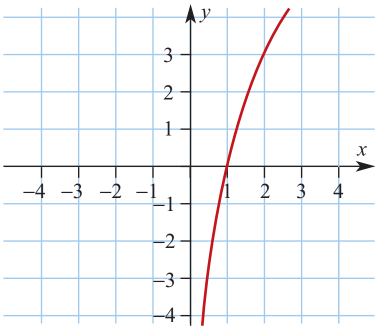
intercept: $x = 25$
asymptote: $x = 0$

c



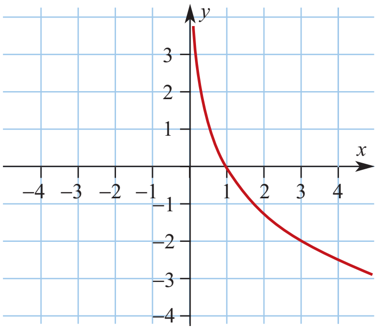
intercept: $x = \frac{1}{\sqrt{3}}$
asymptote: $x = 0$

d



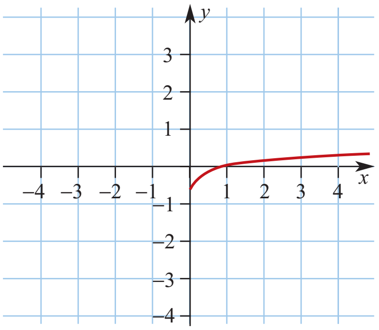
intercept: $x = 1$
asymptote: $x = 0$

e



intercept: $x = 1$
asymptote: $x = 0$

f



intercept: $x = 1$
asymptote: $x = 0$

2 a i $y = 2\log_{10} x$ **ii** $y = \frac{1}{3}\log_{10} x$

b i $y = 10^{\frac{1}{3}x}$ **ii** $y = \frac{1}{3}10^{\frac{1}{2}x}$

3 a $y = \log_3(x - 2)$ **b** $y = 2^x + 3$

c $y = \log_3\left(\frac{x-2}{4}\right)$ **d** $y = \log_5(x + 2)$

e $y = \frac{1}{3} \times 2^x$ **f** $y = 3 \times 2^x$

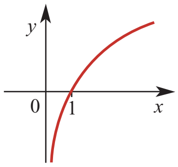
g $y = 2^x - 3$ **h** $y = \log_3\left(\frac{x+2}{5}\right)$

4 a–f Check with your teacher

5 a 0.64 **b** 0.40

6 Check with your teacher

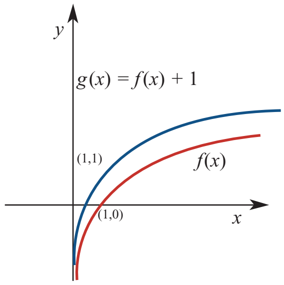
7 $y = \log_{10}(\sqrt{x}) = \frac{1}{2}\log_{10} x$ for $x \in (0, 10]$



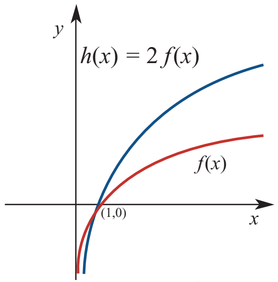
8 Check with your teacher

9 $a = \dfrac{6}{\left(\frac{10}{3}\right)^{2/3}}$ and $k = \frac{1}{3}\log_{10}\left(\frac{10}{3}\right)$

10 a



b



11 Shift (translation) of $\frac{c}{b}$ from y-axis parallel to x-axis, followed by a $\frac{1}{b}$ stretch (dilation) also from the y-axis parallel to the x-axis.

Exercise 3H

1 $y = 1.5 \times 0.575^x$

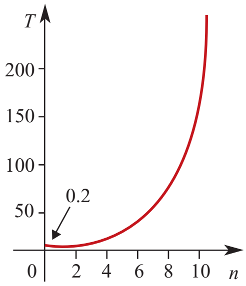
2 $p = 2.5 \times 1.35^t$

3 a

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

b $T = 0.2(2)^n$

c



d 214 748.4 m

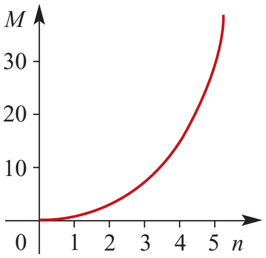
4 a

n	0	1	2	3	4
M	0	1	3	7	15

b $M = 2^n - 1$

n	5	6	7
M	31	63	127

c



d

Three discs	1	2	3
Times moved	4	2	1

Four discs	1	2	3	4
Times moved	8	4	2	1

5 $n = 2$

6 a $\left(\frac{1}{2}\right)^{3n}$ **b** $\left(\frac{1}{2}\right)^{5n-2}$ **c** $n = 3$

7 a $729\left(\frac{1}{4}\right)^{n}$ **b** $128\left(\frac{1}{2}\right)^{n}$ **c** 4 times

8 11.21% (2009)

9 $H = 1.26 \times 10^{-11}$, given $H < 3 \times 10^{-10}$; hence, the department should be concerned.

10 max = 3.6 years, min = 32.4 years after which total slowing increases

11 $t = -1.42$; hence, prime suspect should be Associate Professor.

12 a 9 years
b $k = 1.080$
c 8%

13 0.78125 cm^2

14 $I_{MC} = 10 I_{SF}$

15 approx. 42.5. Shape is similar to an exponential.

Multiple-choice answers

1 C **2** A **3** A **4** B
5 A **6** A **7** A **8** A
9 D **10** B

Short-response answers

1 a a^4 **b** $\frac{1}{b^2}$ **c** $\frac{1}{m^2n^2}$ **d** $\frac{1}{ab^6}$
e $\frac{3a^6}{2}$ **f** $\frac{5}{3a^2}$ **g** a^3 **h** $\frac{n^8}{m^4}$
i $\frac{1}{p^2q^4}$ **j** $\frac{8}{5a^{11}}$ **k** $2a$ **l** $a^2 + a^6$

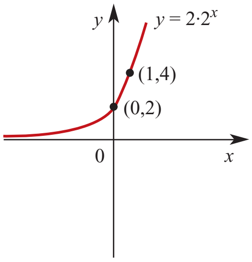
2 a 2.81 **b** 1.40 **c** 0.30 **d** 0.56
e 2.04 **f** 3.00 **g** 1.33 **h** –3.32

3 a 6 **b** 2 **c** 7 **d** 2
e 0 **f** 3 **g** -2 **h** -3
i 4

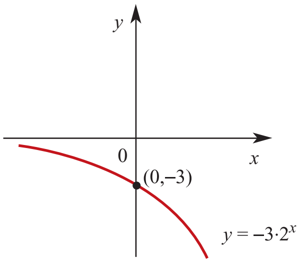
4 a $\log_{10} 6$ **b** $\log_{10} 6$ **c** $\log_{10}\left(\frac{a^2}{b}\right)$
d $\log_{10}\left(\frac{a^2}{25\,000}\right)$ **e** $\log_{10} y$ **f** $\log_{10}\left(\frac{a^2b^3}{c}\right)$

5 a $x = 3$ **b** $x = 3$ or $x = 0$
c $x = 1$ **d** $x = 2$ or $x = 3$

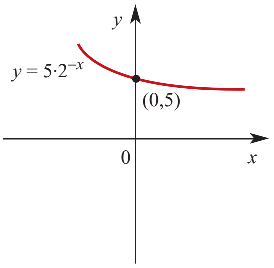
6 a



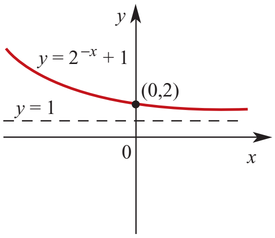
b



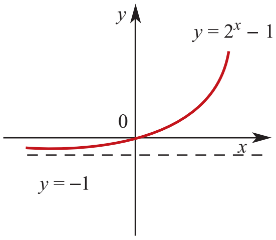
c



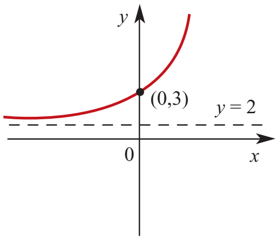
d



e



f



7 $x = 1$

8 Check with your teacher

9 3

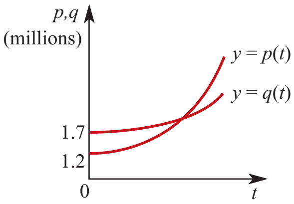
10 a $k = \frac{1}{7}$ **b** $q = \frac{3}{2}$

11 a $a = \frac{1}{2}$ **b** $y = -4$ or $y = 20$

12 a batch 1 $= 15(0.95)^n$, batch 2 $= 20(0.94)^n$

b 32 years

13 a



b i $t = 12.56$ (i.e. mid 1962)

ii $t = 37.56\ldots$(i.e. mid 1987)

14 a company X \$1.82, company Y \$1.51, company Z \$2.62

b company X \$4.37, company Y \$4.27, company Z \$3.47

c intersect at $t = 21.784$ and $t = 2.090$; hence, February 2006 until September 2007

d February 2007 until September 2007, approx. 8 months

15 a 13.81 years **b** 7.38 years

16 a Temperature $= 87.065 \times 0.94^t$

b i 87.1° C **ii** 18.6° C

c Temperature $= 85.724 \times 0.94^t$

d i 85.7° C **ii** 40.8° C

e 28.2 min

17 a $a = 0.2$ and $b = 5$

b i $z = x \log_{10} b$

ii $a = 0.2$ and $k = \log_{10} 5$

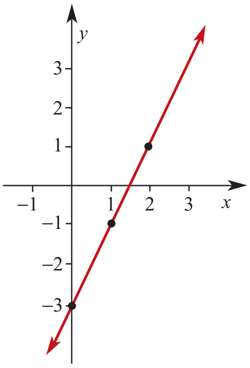
18 a $y = 2 \times 1.585^x$ **b** $y = 2 \times 10^{0.2x}$

c $x = 5 \log_{10}\left(\frac{y}{2}\right)$

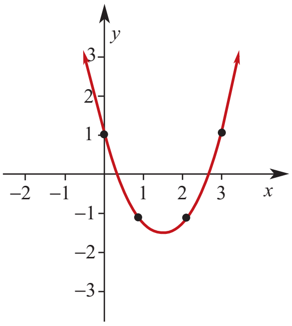
Chapter 4

Exercise 4A

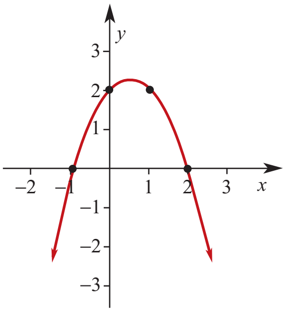
1 a



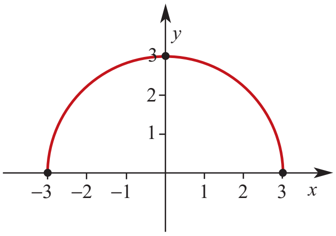
b



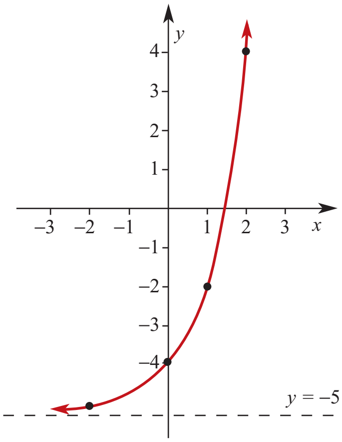
c



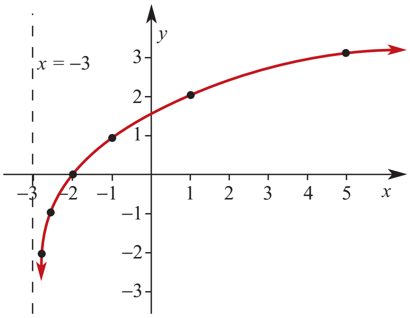
d



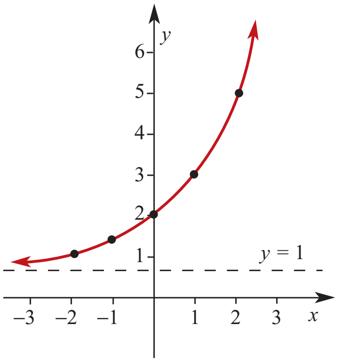
2 a


y
4
3
2
1
−3 −2 −1 1 2 3 x
−1
−2
−3
−4
y = −5

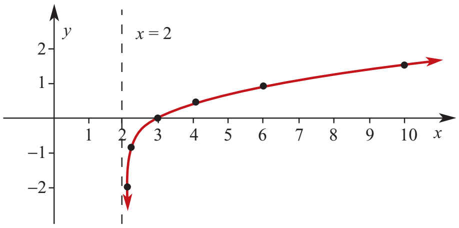
b


y
x = −3
3
2
1
−3 −2 −1 1 2 3 4 5 x
−1
−2
−3

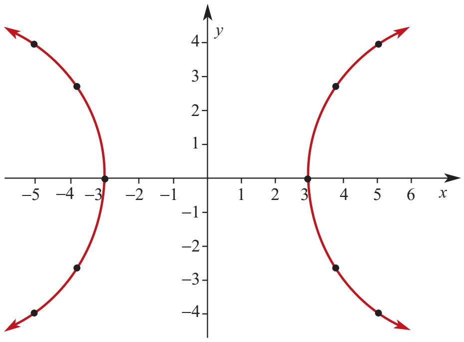
c


y
6
5
4
3
2
1
y = 1
−3 −2 −1 1 2 3 x

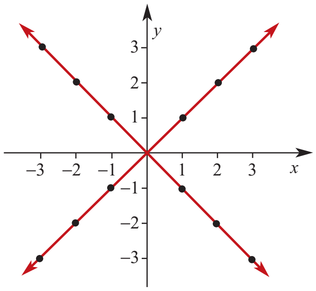
d


y
x = 2
2
1
1 2 3 4 5 6 7 8 9 10 x
−1
−2

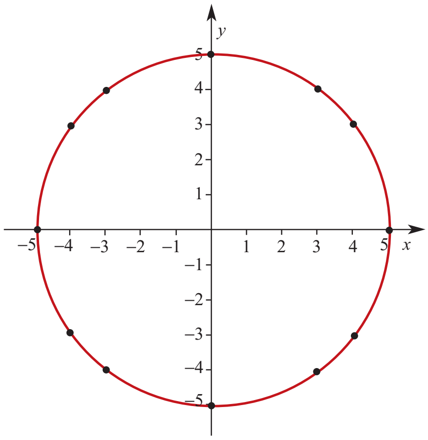
3 a


y
4
3
2
1
−5 −4 −3 −2 −1 1 2 3 4 5 6 x
−1
−2
−3
−4

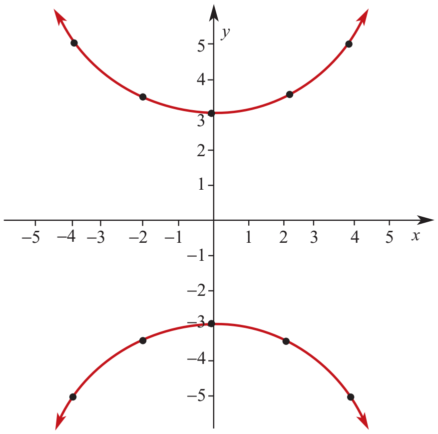
b


y
3
2
1
−3 −2 −1 1 2 3 x
−1
−2
−3

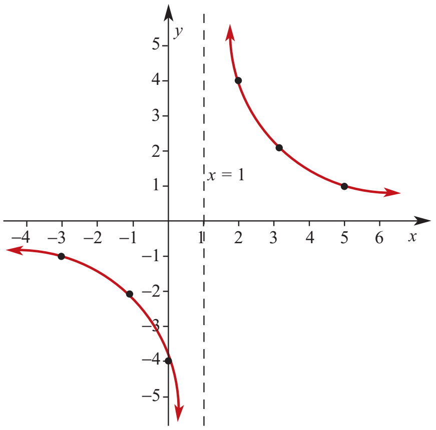
c


y
5
4
3
2
1
−5 −4 −3 −2 −1 1 2 3 4 5 x
−1
−2
−3
−4
−5

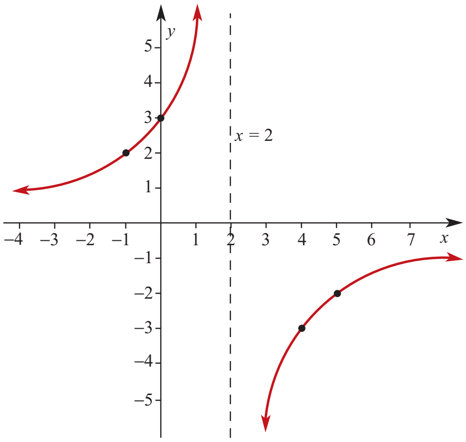
d


y
5
4
3
2
1
−5 −4 −3 −2 −1 1 2 3 4 5 x
−1
−2
−3
−4
−5

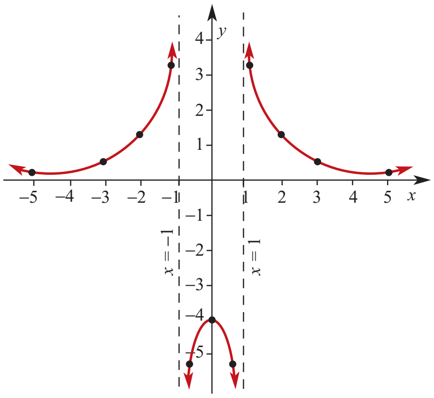
4 a



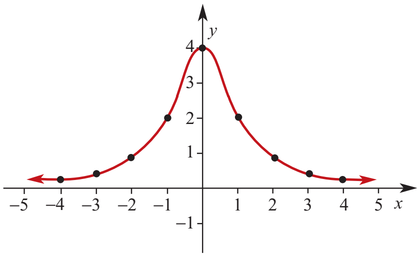
b



c

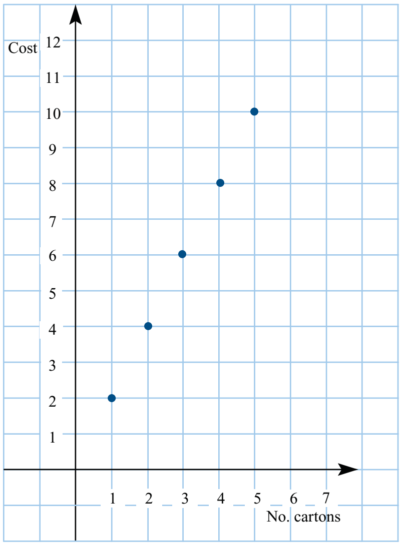


d

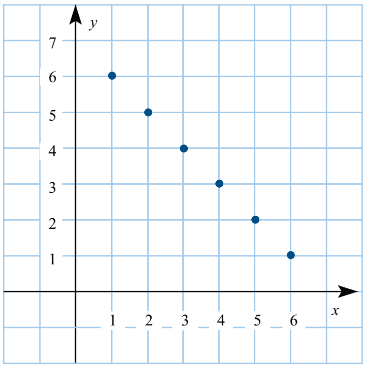


Exercise 4B

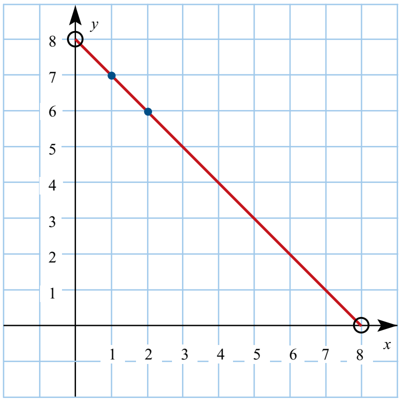
1



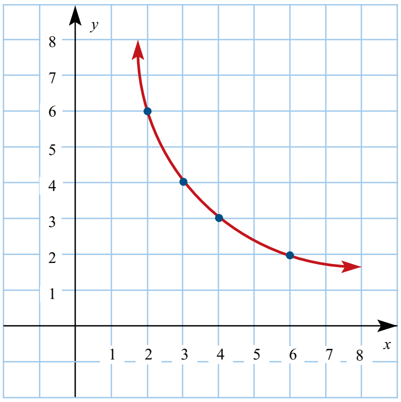
2



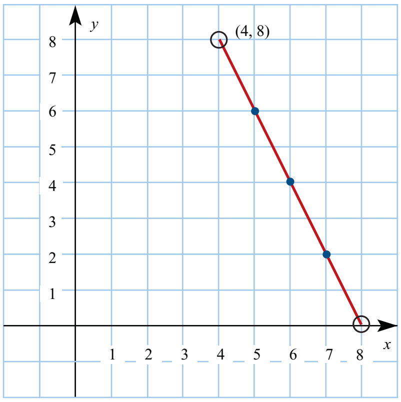
3 a



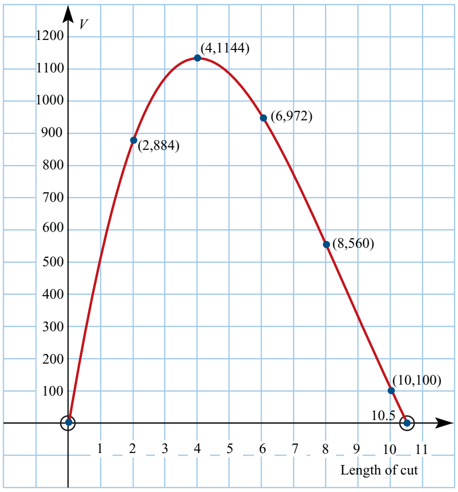
b



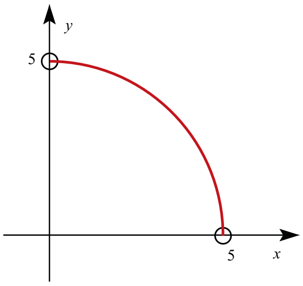
4



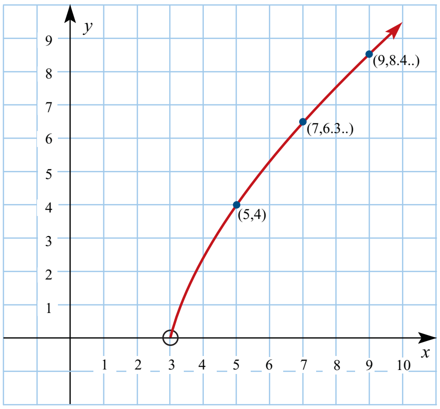
5



6 a



b



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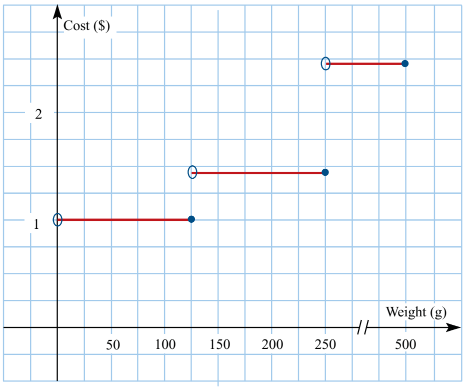

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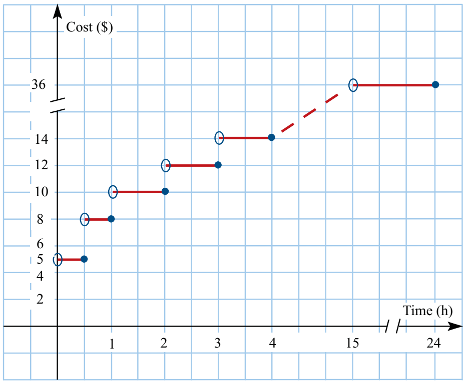
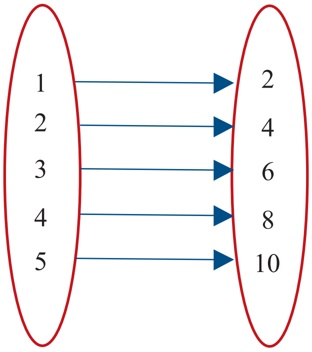


Exercise 4C

1 a Independent: Number of cartons
Dependent: Cost

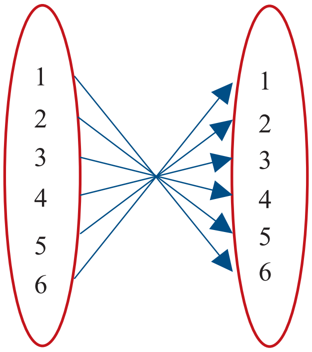
b Domain: {1, 2, 3, 4, 5}
Range: {2, 4, 6, 8, 10}

c



2 a Independent: Number on uppermost face
Dependent: Number on the table
b Domain: $\{1, 2, 3, 4, 5, 6\}$
Range: $\{1, 2, 3, 4, 5, 6\}$

c



3 a i Independent: Height
Dependent: Base
ii Domain: $0 < x < 8$
Range: $0 < y < 8$
iii $x + y = 8$
b i Independent: Height
Dependent: Base
ii Domain: $x > 0$
Range: $y > 0$
iii $xy = 12$

4 a Independent: Slant height
Dependent: Base
b Domain: $4 < x < 8$
Range: $0 < x < 8$
c $2x + y = 16$

5 a Independent: Length of cut
Dependent: Volume
b Domain: $0 < x < 10.5$
Range: $0 < V < 1144$ approx.
c $V = x(30 - 2x)(21 - 2x)$

6 a i Independent: Height
Dependent: Base
ii Domain: $0 < x < 5$
Range: $0 < y < 5$
iii $x^2 + y^2 = 25$
b i Independent: Height
Dependent: Base
ii Domain: $x > 5$
Range: $y > 0$
iii $x^2 = y^2 + 25$

7 a Independent: Weight
Dependent: Cost
b Domain: $0 < x < 500$
Range: $\{1.00, 1.45, 2.45\}$

8 a Independent: Time
Dependent: Cost
b Domain: $0 < x \leq 24$
Range: $\{5, 8, 10, 12, 14, \ldots 36\}$

9 1 Ind: Discrete
Dep: Discrete
2 Ind: Discrete
Dep: Discrete
3 a Ind: Continuous
Dep: Continuous
b Ind: Continuous
Dep: Continuous
4 Ind: Continuous
Dep: Continuous
5 Ind: Continuous
Dep: Continuous
6 a Ind: Continuous
Dep: Continuous
b Ind: Continuous
Dep: Continuous
7 Ind: Continuous
Dep: Discrete
8 Ind: Continuous
Dep: Discrete

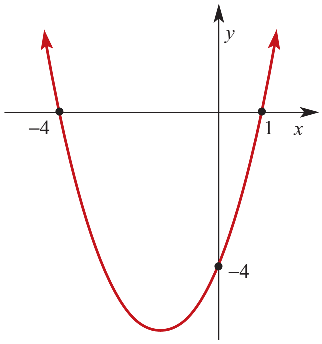
Exercise 4D

1 a function **b** function **c** not
d not **e** function **f** function
2 a function **b** function **c** not
d function **e** function **f** not
3 a function **b** function **c** function
d not **e** function **f** not
g function **h** not
4 a not **b** function **c** function
d not **e** not **f** function

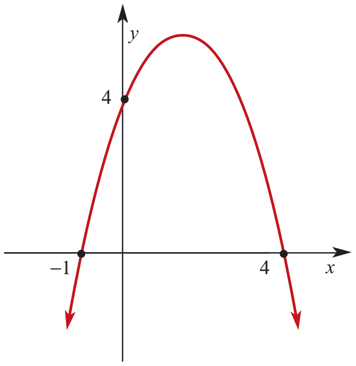
Exercise 4E

1 1 discrete
2 discrete
3 a continuous **b** continuous
4 continuous
5 continuous
6 continuous
7 discontinuous
8 discontinuous

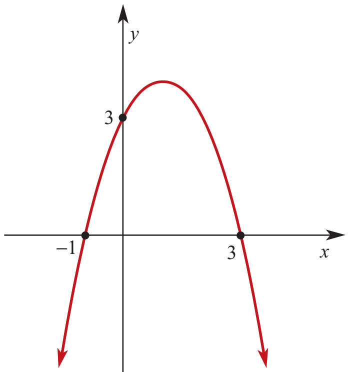
2 a i 7 **ii** 15 **iii** -3 **iv** 10
b i $2a + 5$ **ii** $a^2 + 2a + 7$
iii $-6b - 1$ **iv** $b^2 - 6b + 15$
c i 9.5 **ii** $\pm\sqrt{14}$
3 a i -5 **ii** 5 **iii** 4 **iv** 15
b i $7 - 3b$ **ii** $2a^2 + 8a + 5$
iii $-6a - 5$ **iv** $8b^2 + 24b + 15$
c i $2\frac{1}{3}$ **ii** ± 5
4 a i -24 **ii** 4
b i $5a + 5h - a^2 - 2ah - h^2$ **ii** 4
c i no solution **ii** 1 or 4
5 a i



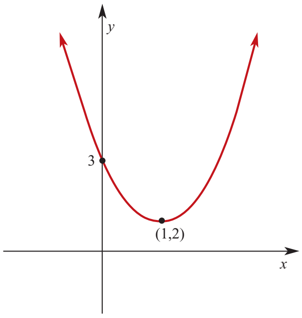
ii



b i Domain: All real numbers
Range: $y \geq -6\frac{1}{4}$
ii Domain: All real numbers
Range: $y \leq 6\frac{1}{4}$
6 a i



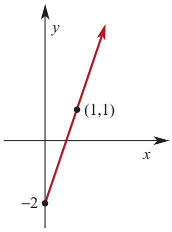
ii



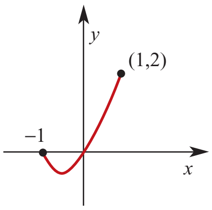
b i Domain: All real numbers
Range: $y \leq 4$
ii Domain: All real numbers
Range: $y \geq 2$

Exercise 4F

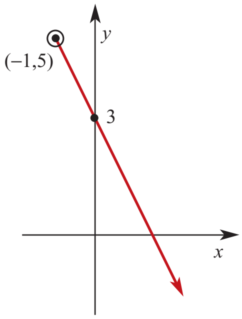
1 a Domain: All real numbers
Range: All real numbers
b Domain: $x \geq 0$
Range: $y \geq 0$
c Domain: All real numbers
Range: $y \geq 1$
d Domain: $-3 \leq x \leq 3$
Range: $-3 \leq y \leq 0$
e Domain: $x > 0$
Range: $y > 0$
f Domain: All real numbers
Range: $y \leq 3$
g Domain: $x \geq 2$
Range: $y \geq 0$
h Domain: $x \geq -1$
Range: All real numbers
i Domain: $x \leq 1\frac{1}{2}$
Range: $y \geq 0$
j Domain: $x \neq -2$
Range: $y \neq 0$
k Domain: $x > 5$
Range: All real numbers
l Domain: $x \neq \frac{1}{2}$
Range: $y \neq 0$
m Domain: $x \neq -2$
Range: $y \neq -4$
n Domain: $x > 3$
Range: $y > 0$
o Domain: $x > -1$
Range: $y > 0$
2 a



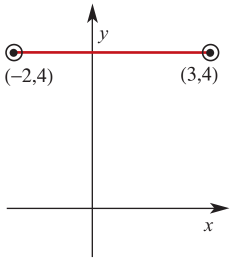
b



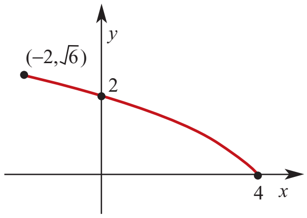
c



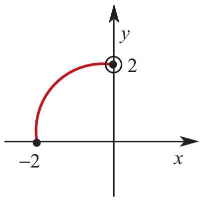
d



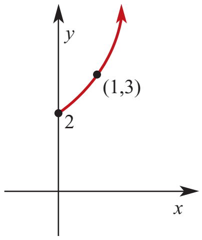
e



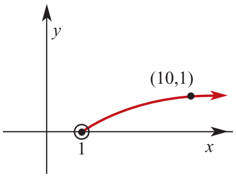
f



g

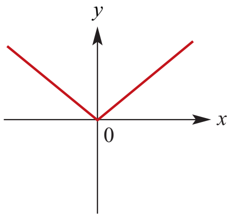


h



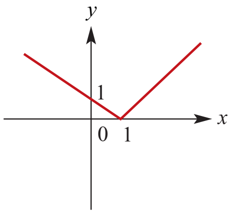
Exercise 4G

1 a



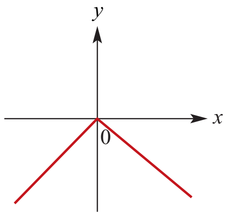
Range = $[0, \infty)$

b



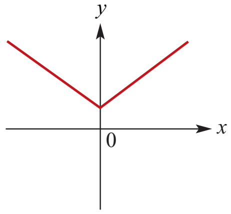
Range = $[0, \infty)$

c



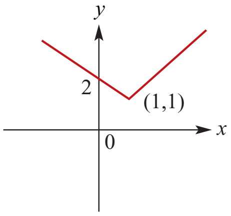
Range = $(-\infty, 0]$

d



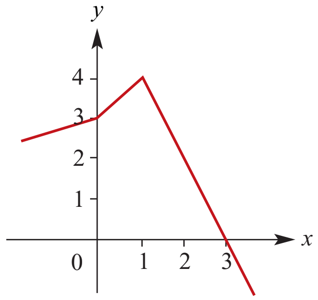
Range = $[1, \infty)$

e



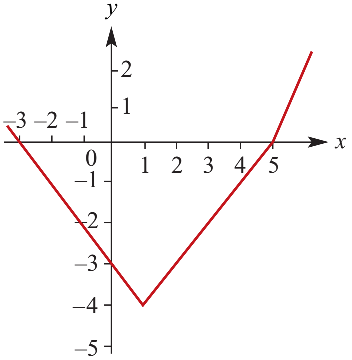
Range = $[1, \infty)$

2 a

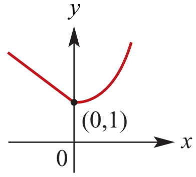


b Range $= (-\infty, 4]$

3

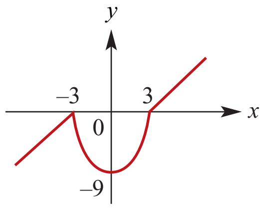


4 a



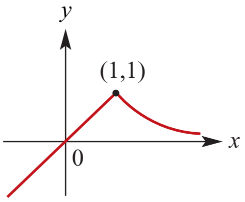
b Range $= [1, \infty)$

5 a



b Range $= R$

6 a



b Range $= (-\infty, 1]$

7
$$f(x) = \begin{cases} x+3, & -3 \le x \le -1 \\ -x+1, & -1 < x \le 2 \\ -\frac{1}{2}x, & 2 < x \le 4 \end{cases}$$

Exercise 4H

1 a $\{(3, 1), (6, -2), (5, 4), (1, 7)\}$
b $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
c $f^{-1}(x) = 3 - \frac{1}{2}x$ **d** $f^{-1}(x) = \frac{1}{3}x - 1$
e $f^{-1}(x) = \frac{3}{2}x + 9$ **f** $f^{-1}(x) = \log_2 x$
g $f^{-1}(x) = 3^{\frac{x}{2}}$ **h** $f^{-1}(x) = x^2 - 2, \quad x \ge 0$
i $f^{-1}(x) = 2 - x^2, \quad x \ge 0$

2 a $f^{-1}(x) = \frac{1}{4} - \frac{1}{8}x, \quad x \le 2$
b $f^{-1}(x) = 2x - 6, \quad x \le 3$
c $f^{-1}(x) = x - 4, \quad x \ge 0$
d $f^{-1}(x) = \frac{1}{3}x + 2, \quad -6 \le x \le 0$
e $f^{-1}(x) = -\sqrt{x}$
f $f^{-1}(x) = \begin{cases} \frac{1}{2}x, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

3 a $f^{-1}(x) = \sqrt{x+1}$ or $f^{-1}(x) = -\sqrt{x+1}$
b $f^{-1}(x) = \sqrt[3]{x-1}$
c $f^{-1}(x) = \sqrt{9-x^2}, x \ge 0$ or
$f^{-1}(x) = -\sqrt{9-x^2}, x \ge 0$
d $f^{-1}(x) = 3 + \sqrt{x-1}$ or
$f^{-1}(x) = 3 - \sqrt{x-1}$

Exercise 4I

1 a $f^{-1}(x) = \sqrt{a^2 - x^2}, \quad x \ge 0$ or
$f^{-1}(x) = -\sqrt{a^2 - x^2}, \quad x \ge 0$
b $f^{-1}(x) = \sqrt{x^2 + a^2}, \quad x \ge 0$ or
$f^{-1}(x) = -\sqrt{x^2 + a^2}, \quad x \ge 0$
c $f^{-1}(x) = \sqrt{x^2 - a^2}, \quad x \ge 0$ or
$f^{-1}(x) = -\sqrt{x^2 - a^2}, \quad x \ge 0$

2 $2a + h - 3$

3 All real numbers

4 $V(r) = 4\pi r^2, \quad r > 0$

5 $A(r) = 64 - \pi r^2, \quad 0 < r \le 4$

6 a $A(x) = x(8 - x), \quad 0 < x < 8$
b $P(x) = 2\left(x + \frac{20}{x}\right), \quad x > 0$

7 $A(x) = \frac{1}{4}x^2, \quad x > 0$

8 $y = 16 - 2x, \quad 4 < x < 8$

9 a $y = \sqrt{25 - x^2}, \quad 0 < x < 5$
b $y = \sqrt{x^2 - 9}, \quad x > 3$

10 $SA(r) = 2\pi\left(r^2 + \frac{200}{r}\right), \quad r > 0$

11 $V(x) = x(21 - 2x)(15 - x), \quad 0 < x < 10.5$

12 $f(x)$ and $h(x)$ are even

$g(x)$ is odd

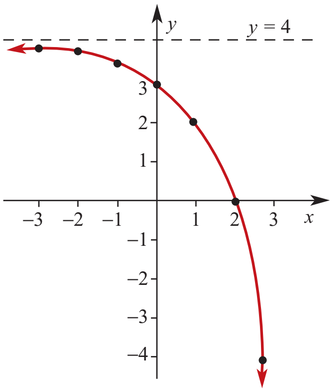
13 Offer A is cheaper if average number of calls is < 65. Offers A and B are the same if the average is 65, otherwise offer B is cheaper.

Multiple-choice answers

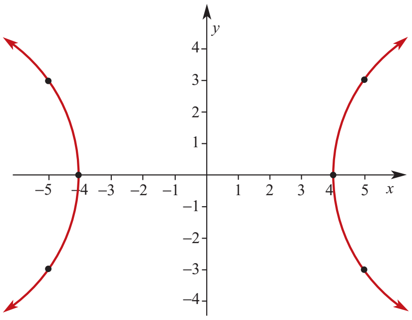
1 D **2** E **3** B **4** B **5** A
6 D **7** C **8** E **9** B **10** D

Short-response answers

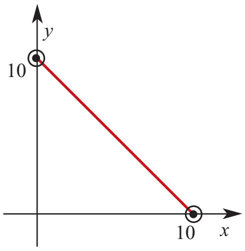
1 a



b



2 a



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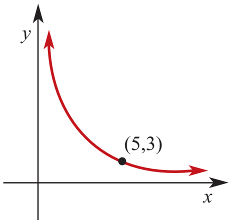
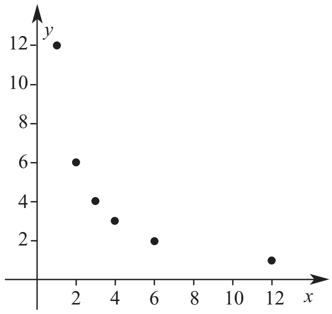


c



3 i a Assuming independent is x
Dependent: y

b Assuming independent is x
Dependent: y

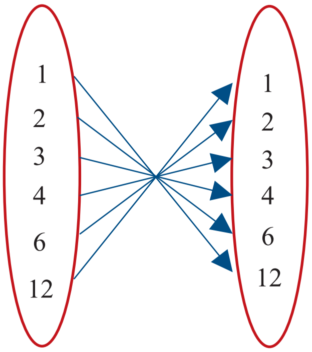
c Assuming independent is x
Dependent: y

ii a Domain: $0 < x < 10$
Range: $0 < y < 10$

b Domain: $x > 0$
Range: $y > 0$

c Domain: $\{1, 2, 3, 4, 6, 12\}$
Range: $\{1, 2, 3, 4, 6, 12\}$

iii



iv a $y = 10 - x, \quad 0 < x < 10$

b $y = \dfrac{15}{x}, \quad x > 0$

c $y = \dfrac{12}{x}, \ x \in \{1, 2, 3, 4, 6, 12\}$

v a independent: continuous
dependent: continuous

b independent: continuous
dependent: continuous

c independent: discrete
dependent: discrete

vi a continuous

b continuous

c discrete

4 a function

b not

c function

d not

5 a i 11

ii 6

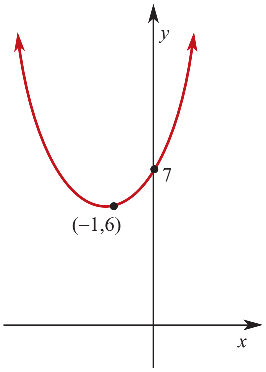
b **i** $2a + 11$

ii $b^2 - 4b + 6$

c **i** 7.5

ii $x = \pm\sqrt{22}$

d



6 **a** Domain: $x \geq -2$

Range: $y \geq 0$

b Domain: $x \neq \dfrac{1}{2}$

Range: $y \neq 0$

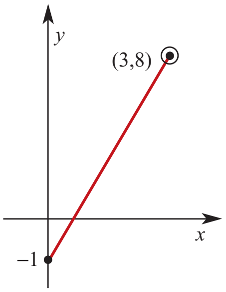
c Domain: $x > 8$

Range: All real numbers

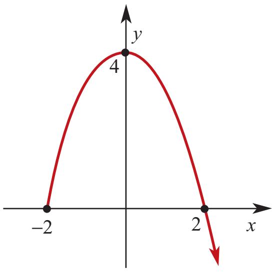
d Domain : All real numbers

Range: All real numbers

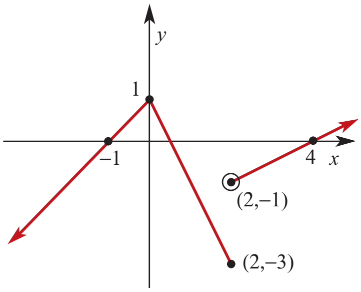
7 **a**



b



c



8 **a** $\{(3, 1), (4, 3), (6, 4), (8, 7)\}$

b $f^{-1}(x) = \dfrac{1}{2}x - \dfrac{3}{2}$

c $g^{-1}(x) = \dfrac{3}{4} - \dfrac{1}{4}x, \quad x \leq 3$

9 **a** $f^{-1}(x) = \sqrt{x + 3}$ or

$f^{-1}(x) = -\sqrt{x + 3}$

b $f^{-1}(x) = \sqrt{x^2 + 9}, \quad x \geq 0$ or

$f^{-1}(x) = -\sqrt{x^2 + 9}, \quad x \geq 0$

10 $A(x) = 25\pi - x^2, \quad 0 < x < \sqrt{50}$

11 $V(x) = x(30 - 2x)(21 - 2x), \quad 0 < x < 10.5$

Chapter 5

Exercise 5A

1 **a** nominal **b** discrete

c nominal **d** continuous

e ordinal **f** ordinal

2 **a** numerical **b** categorical

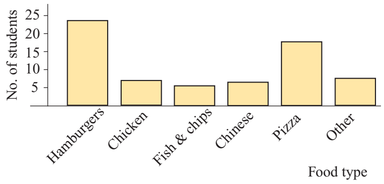
3 **a** discrete **b** discrete

c continuous **d** continuous

e discrete

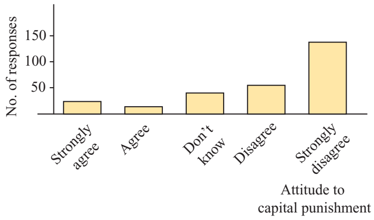
Exercise 5B

1 **a**



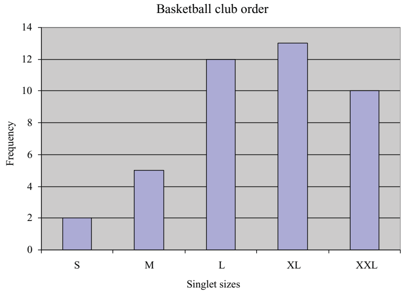
b hamburgers

2 **a**



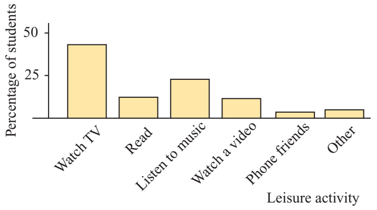
b 32

3 **a**



b XL

4 a



b watching TV

Exercise 5C

1

Number	0	1	2	3	4	5	6
Frequency	4	4	4	4	3	2	1

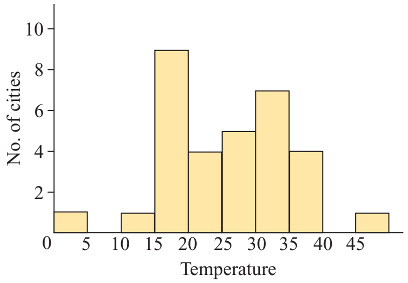
2 a 4 **b** 2 **c** 5 **d** 28

3 a 0 **b** 48 **c** 60–69 **d** 33

4 a, b

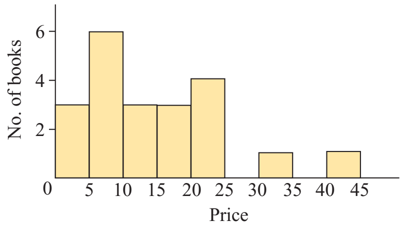
Temperatures (°C)	Frequency	Relative frequency
0 –	1	0.03
5 –	0	0
10 –	1	0.03
15 –	9	0.28
20 –	4	0.13
25 –	5	0.16
30 –	7	0.22
35 –	4	0.13
40 –	0	0
45 –	1	0.03

c



d 47%

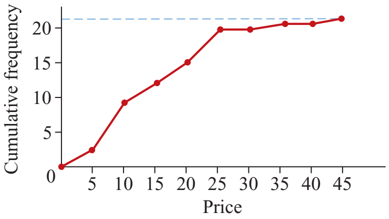
5 a



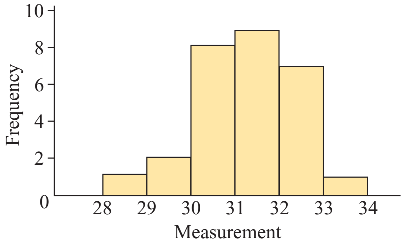
b $5.00–$5.99

c

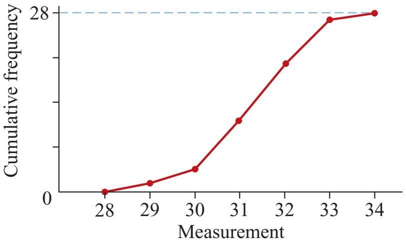
Prices ($)	Cumulative frequency
less than 5	3
less than 10	9
less than 15	12
less than 20	15
less than 25	19
less than 30	19
less than 35	20
less than 40	20
less than 45	21



6 a

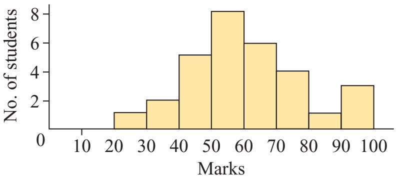


b

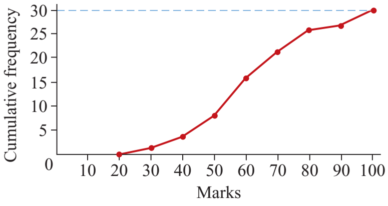


c The students' estimates ranged from 28.9 cm to 33.3 cm, with most students (89%) overestimating the 30 cm measure.

7 a

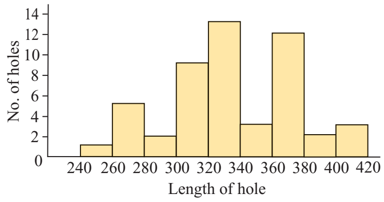


b

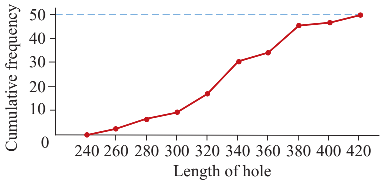


c The students' marks ranged from 21 to 99, with most students (over 70%) scoring more than 50% on the test.

8 a



b



c **i** below 300 m $= \dfrac{4}{25}$

ii no. of holes $\geq$ 360 m = 17,

proportion $= \dfrac{17}{50}$

iii approx. 280 m

Exercise 5D

1 **a** centre **b** neither **c** both

2 **a** positively skewed
b negatively skewed **c** symmetrical

3 symmetrical **4** symmetrical
5 approximately symmetrical

Exercise 5E

1 **a**

4	0 2 4 5
5	8 9
6	5 8
7	7
8	4 8
9	4

7 | 7 represents 7.7 days

b four months

2 **a**

0	4
1	
1	6 8 9
2	1 1 3
2	5 5 5 6 7 7 9 9
3	1 1 2 3 3
3	6 9
4	1
4	6

2 | 5 represents 25 hours (truncated)

b nine batteries

3 a

0	0
1	0 0 4 5 5 6 9
2	0 0 1 3 7 8 9
3	3 7 9
4	6
5	
6	3 7
7	0

4 | 6 represents 46 minutes

b three students
c positively skewed

4 a

2	5 8
3	5 6 9
4	5 6 9
5	2
6	8
7	5 5 6 8 9
8	2 4
9	5
10	
11	
12	
13	
14	9
15	
16	4
17	
18	
19	
20	
21	0

16 | 4 represents $164 (truncated)

b approximately symmetrical

5 a

Father's age		Mother's age
	3	7 8 8 9
4 4 4 3 3 3 1 1 0	4	0 0 0 1 2 3 3 3 3 3 4 4
9 8 8 8 8 7 7 6 6 6 5	4	5 6 7 8 9 9 9
4 2 1 1 0	5	0 0 0
5	5	

0 | 4 represents 40 years

4 | 0 represents 40 years

b Both distributions are approximately symmetrical. Fathers, with ages centred in the late forties, tend to be older than mothers, with ages centred in the early forties. The spread is similar for both distributions.

6 a

Class B		Class A
3 2	1	9
	2	2
	3	9
	4	5 7 8
	5	5 8
9	6	5 8
6 4 3 3 2 2 1 0 0	7	1 6 7 9 9
8 8 4 4 3 2 1 1 0 0	8	0 1 2 2 5 5 9
8 1	9	1 9

9 | 6 represents 69 marks

7 | 1 represents 71 marks

b six students in class A and two in class B
c Class B performed better as more students scored in the higher values of 70s to 90s.

Exercise 5F

1 **a** 800 **b** 400 **c** 14 **d** 9
2 **a** 56 000 **b** 1988–89 **c** 1995–96 **d** 72%
e 1996–97

3 **a** 51% **b** Jan 2005 **c** Aug 2007 **d** 128 000
e Wivenhoe

4 **a** 47 000 000 **b** 100 000 000
c 2004/05 **d** 21%
e Check with your teacher

5 **a** \$8 billion **b** 4.7%

6 **a** 0.029 **b** 0.26 **c** 53

Exercise 5G

1 **a** mean = 18.36, median = 14
b mean = 9.19, median = 10
c mean = 7.41, median = 7.65
d mean = 1.62, median = 1.15

2 **a** mean = 3.24, median = 3
b mean = −0.38, median = 0

3 mean = \$193 386, median = \$140 000; the median is a better measure of centre as it is typical of more house prices.

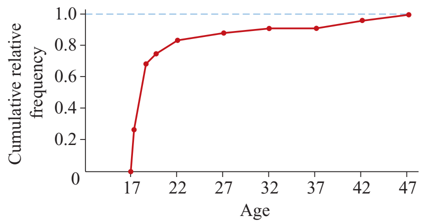
4 mean = 4.06, median = 4; both are reasonable measures of centre in this example.

5 **a** range = 602, IQR = 455
b range = 5.3, IQR = 3.2
c range = 0.57, IQR = 0.21
d range = 7, IQR = 3.5

6 **a** 145 **b** 42

7 **a** 2.4 kg **b** 1.0 kg

8 **a**



median = 18, IQR = 2
b mean = 20.97, $s = 7.37$ **c** 92%

9 **a** 12.39 **b** 1.33 **c** 281.24 **d** 3.04

10 **a** **i** mean = 17.61, $s = 15.96$
ii mean = 195.3, $s = 52.9$
b **i** 94% **ii** 100%

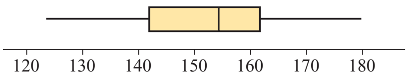
11 **a** **i** mean = 6.79, median = 6.75
ii IQR = 1.8, $s = 0.93$
b **i** mean = 13.54, median = 7.35
ii IQR = 1.81, $s = 18.79$
c The error does not affect the median or interquartile range very much. It doubles the mean and increases the standard deviation by a factor of 20.

12 Approximately 95% of share prices lie between \$44 and \$56.

13 About 95% of days lie in this interval.

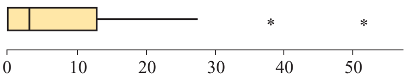
Exercise 5H

1 **a** $m = 154$, $Q_1 = 141.5$, $Q_3 = 161.5$, min = 123, max = 180
b



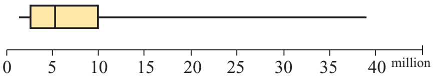
c The distribution of heights is slightly negatively skewed, centred at 154 cm, with the middle 50% of heights ranging from 141.5 cm to 161.5 cm.

2 **a** $m = 3$, $Q_1 = 0$, $Q_3 = 13$, min = 0, max = 52
b 38, 52
c



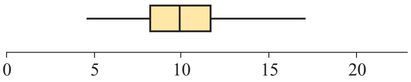
d The distribution of number of books borrowed is positively skewed, centred at 3. While 75% of people borrowed 13 books or less, one person borrowed 38 books and another borrowed 52.

3 **a**



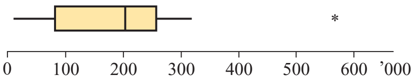
b The distribution of winnings is positively skewed with a median value of \$5.3 million. The middle 50% of players won between \$2.9 million and \$10.0 million. Roger Federer is the outlier, winning \$39 million.

4 **a**



b The distribution is symmetrical, centred at \$10.00. The middle 50% of students earn between \$8.15 and \$11.85 per hour.

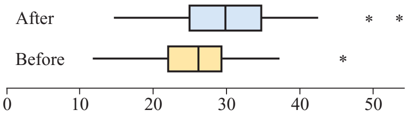
5 **a**



d The distribution is approximately symmetrical, centred at about 210 000, with an outlier at 570 000. The middle 50% of papers have circulations from about 88 000 to 270 000.

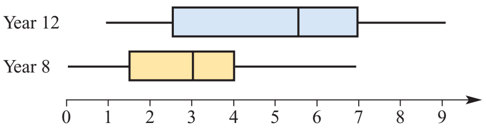
Exercise 5I

1 **a**



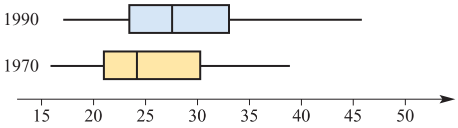
b The distribution for the number of sit-ups is negatively skewed before the course, centred at 26. After the course, the distribution is more symmetric, centred at 30, indicating that the course has been effective. The distribution after the course is more variable than before the course, showing the course has not had the same effect on all participants. There is one outlier in the before group, who can achieve 46 sit-ups, and two in the after group, recording 50 and 54 sit-ups, respectively.

2



a Year 12 **b** Year 12

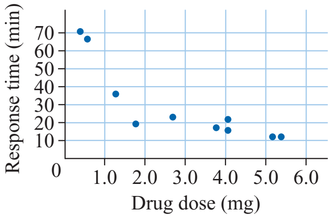
3 a



b The distributions of ages in both groups are slightly positively skewed, with the mothers in 1970 (median = 24.5) generally younger than the mothers in 1990 (median = 28). The variability in both groups is the same (IQR = 10 for both groups).

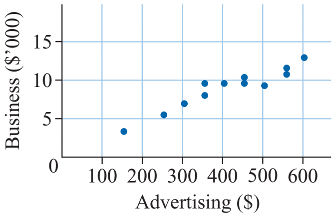
Exercise 5J

1 a



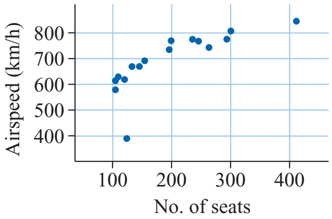
b negative association **c** no outliers

2 a



b positive association **c** no outliers

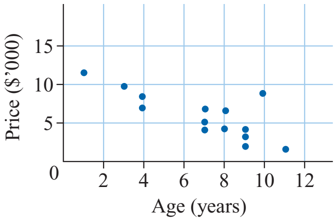
3 a



b positive association

c (122, 378) is an outlier

4 a



b negative association

c (10, 8700) is an outlier

Exercise 5K

1 a no correlation
b weak negative correlation
c strong negative correlation
d weak positive correlation
e strong positive correlation
f strong negative correlation
g strong positive correlation
h no correlation
i strong negative correlation
j weak positive correlation
k strong positive correlation
l moderate negative correlation

2 a 0.71 **b** 0.78 **c** 0.82 **d** 0.92

3 a −0.6 **b** moderate negative correlation

4 a 0.67 **b** moderate positive correlation

5 a 1 **b** strong positive correlation

6 a −0.43 **b** weak negative correlation

Exercise 5L

1 a no linear relationship
b weak negative linear relationship
c strong negative linear relationship
d weak positive linear relationship
e strong positive linear relationship
f strong negative linear relationship
g strong positive linear relationship
h no linear relationship
i moderate negative linear relationship
j weak positive linear relationship

k perfect positive linear relationship
l perfect negative linear relationship

2 a 0.8 **b** 0.8 **c** 0.7
d 0.8 **e** −0.7 **f** −0.2

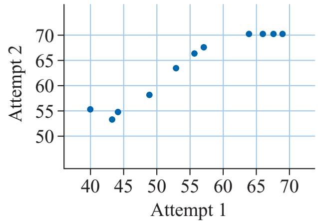
3 a −0.86
b strong negative linear relationship

4 a 0.95
b strong positive linear relationship

5 a 0.77
b strong positive linear relationship

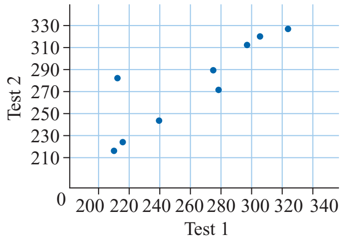
6 a −0.77
b strong negative linear relationship

7



a a strong positive relationship
b Yes, the data are numerical and the relationship is linear. There are no outliers.

8

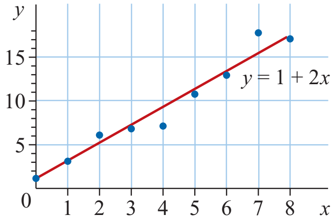


a There is a strong positive linear relationship between the scores on Test 1 and Test 2.
b Yes, the data are numerical and the relationship is linear.
c $q = 0.71, r = 0.87$
d q: moderate positive linear relationship
r: strong positive linear relationship
e i $q = 0.43, r = -0.004$
ii The error in the data has a much greater effect on Pearson's correlation coefficient.

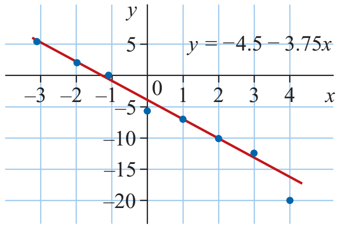
Exercise 5M

Note: Answers will vary for lines drawn by eye.

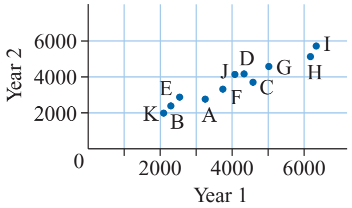
1



2

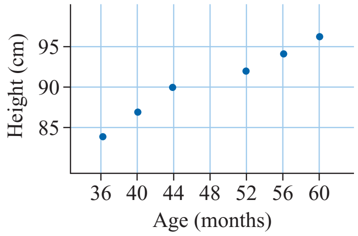


3 a



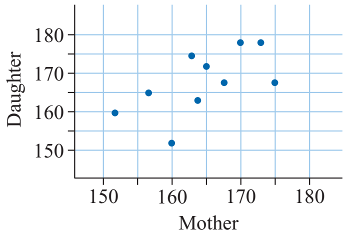
b $y = 424 + 0.794x$
c The positive slope indicates that districts with high rates in Year 1 also had high rates in Year 2.

4 a



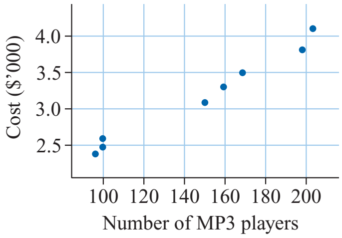
b $y = 72 + 0.4x$
c The intercept (72 cm) is the predicted height at age 0. The slope predicts an increase of 0.4 cm in height each month.
d i 89 cm **ii** 158 cm
e Part **i** is reasonable as it is a value close to the data. Part **ii** is not reliable as the relationship may no longer be linear here.

5 a



b $y = 18.3 + 0.91x$ **c** 173 cm

6 a



b $y = 1300 + 13x$ **c** $\approx$ \$1300

d $\approx$ \$13

7 **a** $y = 70 - 14x$

b The intercept is the predicted time taken to experience pain relief if no drug is given. From the slope we predict a reduction of 14 minutes in time taken to experience pain relief for each mg of drug administered.

c -14 min, which is not a realistic answer

8 **a** $y = 18.2x + 1000$

b Intercept predicts \$1000 of sales if nothing is spent on advertising. The slope means that, on average, each \$1 spent on advertising is associated with an increase of \$18.20 in sales.

c **i** \$19 200 **ii** \$1000

Exercise 5N

1 **a** $y = 68.2 + 0.46x$

b The y-intercept is the predicted height at birth. From the slope, we predict an increase in height of 0.46 cm each month.

c **i** 88 cm **ii** 168 cm

d The height at 42 months is reliable since this is within the range of data given. The height at 18 years is less reliable since this is outside the range of data given.

2 $y = 487.6 + 0.77x$

3 **a** $y = 50.2 + 0.72x$

b An increase of 1 cm in the mother's height is associated with an increase of 0.72 cm in the daughter's height, on average.

c 172 cm (to the nearest cm)

4 **a** $y = 1330 + 12x$

b \$1330

c \$12

5 **a** response time $= 57.0 - 10.2 \times$ drug dose

b The intercept of 57.0 minutes is the predicted time for pain relief when no drug is given. From the slope, we predict a 10.2 minute decrease in response time for each 1 mg of drug given.

c -4.2 min, which is not a realistic answer.

6 **a** business $= 1123.8 + 18.9 \times$ advertising

b Intercept is the volume of business with no advertising. From the slope we predict an increase in business of \$18.90 for every dollar spent on advertising.

c **i** \$20 044 (to the nearest dollar)

ii \$1124 (to the nearest dollar)

Exercise 5O

1 72.667

2 8.5 years

3 \$34 000

4 \$12 000

5 \$21 000

6 9% p.a.

7 **a–d** Check with your teacher

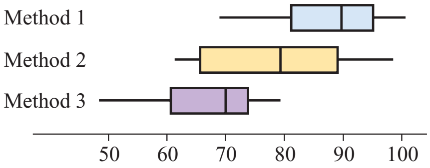
8 Check with your teacher

9 102

10 Check with your teacher

11 Check with your teacher

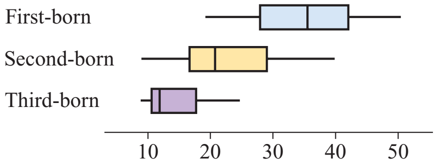
12 **a**



b The distributions of scores are negatively skewed for methods 1 and 3 and symmetrical for method 2. The scores for method 1 are higher than for methods 2 and 3 (90, 79 & 70, respectively), and are also less variable than method 2. They show similar variation to the scores for method 3.

c Thus training method 1 would be recommended, as it consistently produces higher scores.

13 **a**



b The distribution for the first-born is symmetrical, while for the second and third-born the distributions are positively skewed. The centre for the first born is higher than for the second, which is higher than the third (35, 21 & 12, respectively), whilst the variability is most for the first-born, followed by the second-born and then the third-born.

14 **a** yes

b 60; this is reliable because it is an interpolation within the data.

15 4750; this is not reliable as it is an extrapolation far beyond the data.

Multiple-choice answers

1 D	**2** B	**3** D	**4** C	**5** D
6 C	**7** A	**8** D	**9** C	**10** A
11 D	**12** E	**13** A	**14** B	**15** B
16 E	**17** C	**18** C	**19** A	**20** D

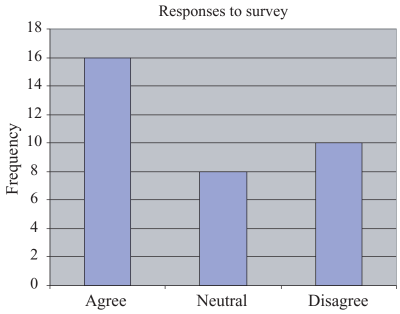
Short-response answers

1 **a** continuous **b** ordinal
c discreate **d** nominal

2 **a** numerical **b** categorical

3 **a**

b



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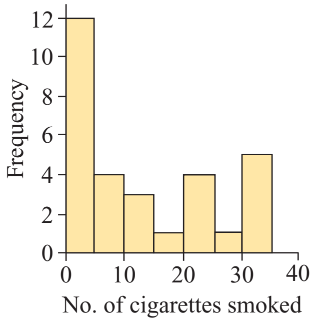


5 **a**

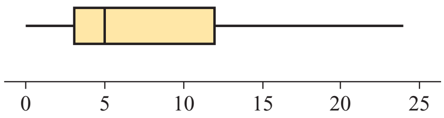
2	2	
3	9	4 | 7 represents 47 minutes
4	3 4 5 7 9	
5	0 1 1 2 2 4 5 6 6 7 9	
6	5 8 9	
7	2	

b $m = 52$, $Q_1 = 47$, $Q_3 = 57$

6 $\overline{x} = \$283.57$, $m = \$267.50$

7 **a** 92.9%
b yes, it is close to 95%

8



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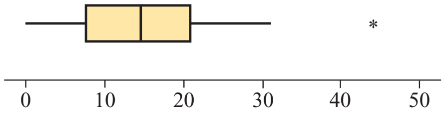
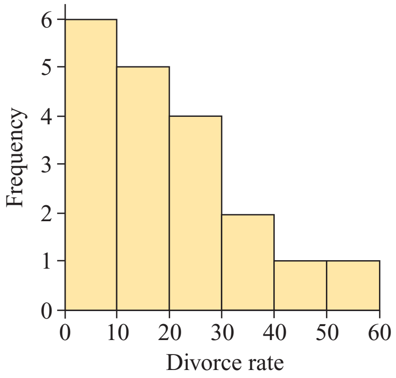


10 **a** numerical

b

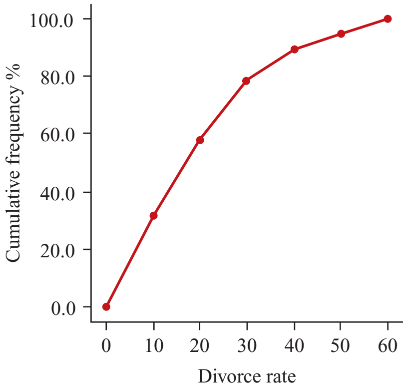
0	0 5 6 6 8 9	
1	4 4 5 8 9	3 | 2 represents 32%
2	5 6 7 8	
3	2 2	
4	4	
5	3	

c positively skewed
d 21.1%
e $\overline{x} = 20.05$, $m = 18$
f



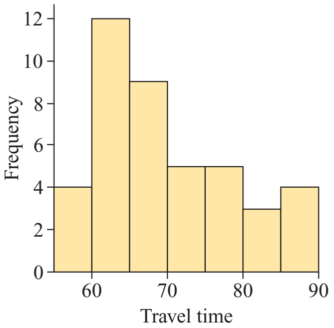
i positively skewed
ii 5

g



i 58%
ii ≈17%

11 a

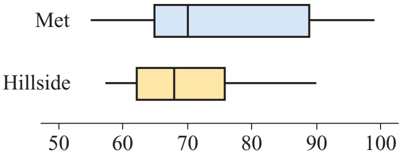


i 21.4% **ii** positively skewed
iii 38.1%

b $\bar{x} = 69.60$, $s = 9.26$, min $= 57$,
$Q_1 = 62$, $m = 68$, $Q_3 = 76$, max $= 90$

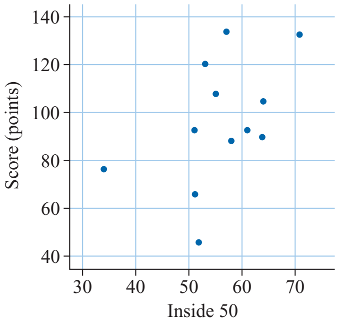
c **i** 69.60 **ii** 68
iii 33, 14 **iv** 76
v 9.26 **vi** 51.08, 88.12

d



e The distributions of travel times are both positively skewed. The travel times for the Met (median = 70) tend to be longer than the travel times for Hillside trains (median = 68). The spread of times is also longer for the Met (IQR = 24) than the travel times for Hillside trains (IQR = 14).

12 a



b positive **c** 0

13 0.927

14 weight ≈ −200 + 2 × height

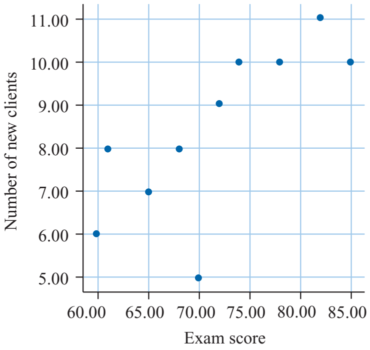
15 errors = 14.9 − 0.533 × time

16 a Intercept: no sensible interpretation. Slope: For each additional second taken to complete the task, on average, the number of errors is reduced by about $\frac{1}{2}$.

b 10

17 a IV = Exam score
DV = Number of new clients

b



c positive **d** 1, strong positive

e 0.748, moderate positive

f number of new clients = −4.00 + 0.173 × exam score

g Intercept: no sensible interpretation. Slope: On average, each extra 1 mark in the final exam is associated with an increase of 0.173 clients.

h 13

i Not very reliable as it is outside the range of the data.

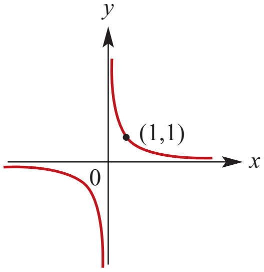
Chapter 6

Exercise 6A

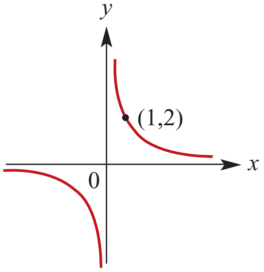
1 a $k = 2.6$ **b** $k = 9$
2 a $k = 1.3$ **b** $k = 4.8$
3 \$55.44 **4** \$6.40/kg **5** $k = 38.88$
6 a $w\,(ox) = \frac{4}{5}ox$ **b** 60 moles **c** 10.8 moles
7 3.75 barrels
8 $1 + \frac{\pi}{2} \approx 2.57$
9 Check with your teacher
10 $k = 14$
11 Check with your teacher
12 a $A = klb, k = 1$ **b** $V = kr^2h, k = \frac{\pi}{3}$
c $A = kd^2, k = \frac{\pi}{8}$
13 a $F = \frac{km_1m_2}{d^2}$
b new force = old force × $\frac{4}{9}$

Exercise 6B

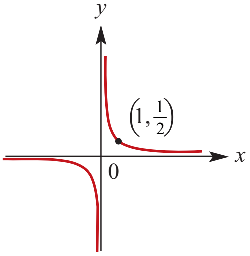
1 a



b



c



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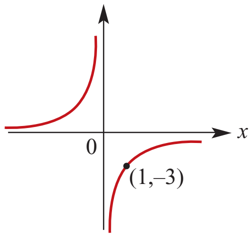

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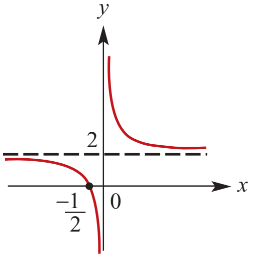

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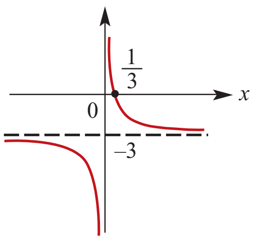
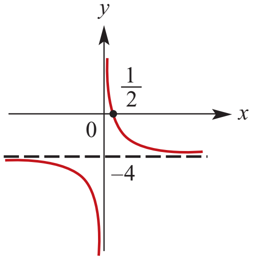


g



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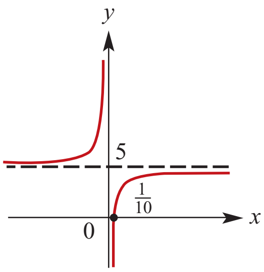
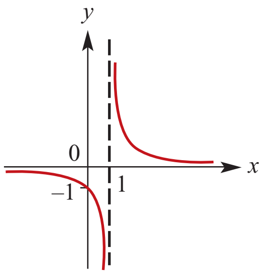


i



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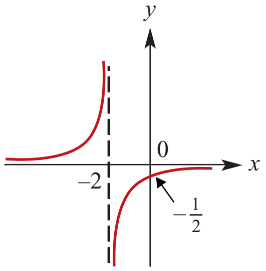
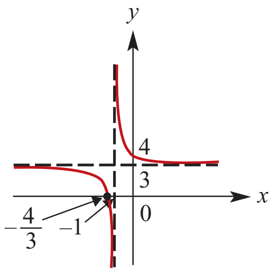
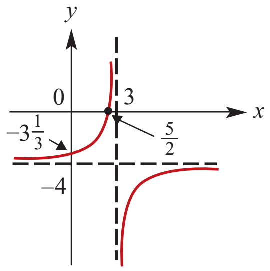


k



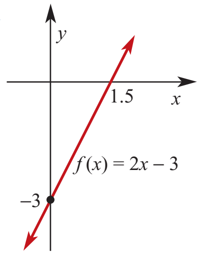
l



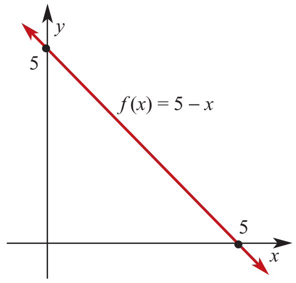
2 a $y = 0, x = 0$
b $y = 0, x = 0$
c $y = 0, x = 0$
d $y = 0, x = 0$
e $y = 2, x = 0$
f $y = -3, x = 0$
g $y = -4, x = 0$
h $y = 5, x = 0$
i $y = 0, x = 1$
j $y = 0, x = -2$
k $y = 3, x = -1$
l $y = -4, x = 3$

Exercise 6C

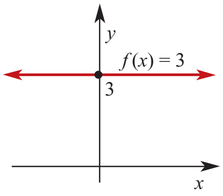
1 a



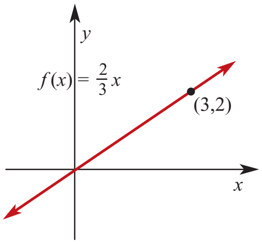
b



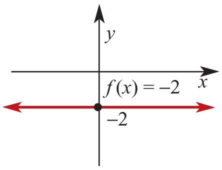
c



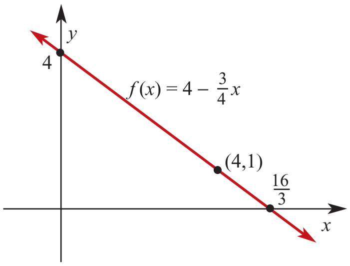
d



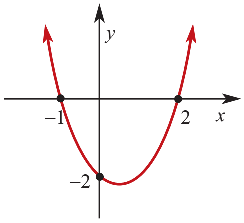
e



f



2 a



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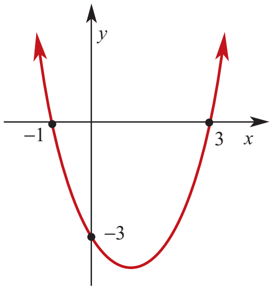

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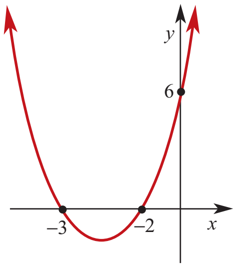

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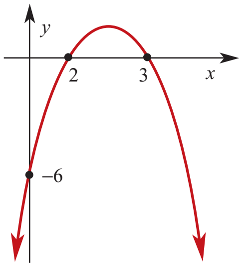

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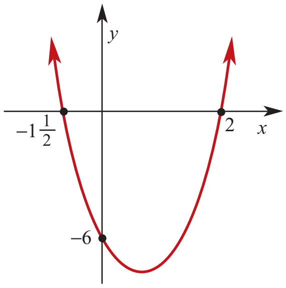

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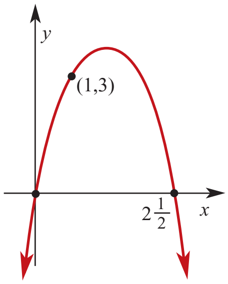
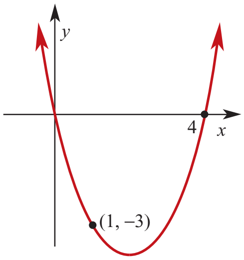


3 a



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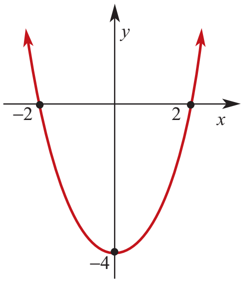

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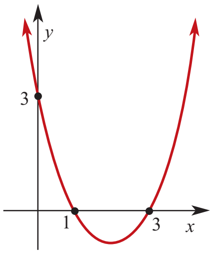

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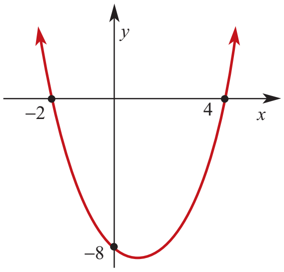

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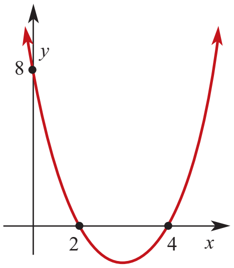

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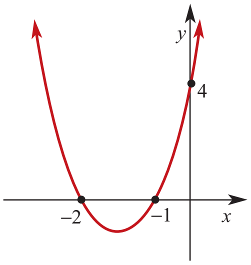

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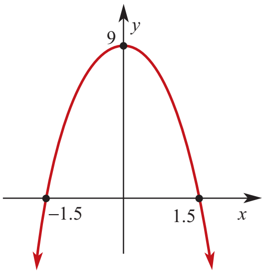

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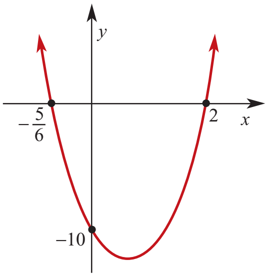


Exercise 6D

1 a $x^2 + 2x + \dfrac{3}{x-1}$ **b** $2x^2 - x - 3 + \dfrac{6}{x+1}$

c $3x^2 - 10x + 22 - \dfrac{43}{x+2}$

d $x^2 - x + 4 - \dfrac{8}{x+1}$

e $2x^2 + 3x + 10 + \dfrac{28}{x-3}$

f $2x^2 - 5x + 37 - \dfrac{133}{x+4}$ **g** $x^2 + x + \dfrac{2}{x+3}$

2 a $\dfrac{1}{2}x^2 + \dfrac{7}{4}x - \dfrac{3}{8} + \dfrac{103}{8(2x+5)}$

b $x^2 + 2x - 3 - \dfrac{2}{2x+1}$

c $\dfrac{1}{3}x^2 - \dfrac{8}{9}x - \dfrac{8}{27} + \dfrac{19}{27(3x-1)}$

d $x^2 - x + 4 + \dfrac{13}{x-2}$

e $x^2 + 2x - 15$

f $\dfrac{1}{2}x^2 + \dfrac{3}{4}x - \dfrac{3}{8} - \dfrac{5}{8(2x+1)}$

Exercise 6E

1 a $(x-1)(x+1)(2x+1)$ **b** $(x+1)^3$

c $(x-1)(6x^2 - 7x + 6)$

d $(x-1)(x+5)(x-4)$

e $(x+1)^2(2x-1)$ **f** $(x+1)(x-1)^2$

g $(x-2)(4x^2 + 8x + 19)$

h $(x+2)(2x+1)(2x-3)$

2 a $(x-1)(x^2+x+1)$

b $(x+4)(x^2-4x+16)$

c $(3x-1)(9x^2+3x+1)$

d $(4x-5)(16x^2+20x+25)$

e $(1-5x)(1+5x+25x^2)$

f $(3x+2)(9x^2-6x+4)$

g $(4m-3n)(16m^2+12mn+9n^2)$

h $(3b+2a)(9b^2-6ab+4a^2)$

3 a $(x+2)(x^2-x+1)$

b $(3x+2)(x-1)(x-2)$

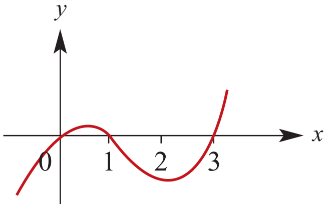
c $(x-3)(x+1)(x-2)$

d $(3x+1)(x+3)(2x-1)$

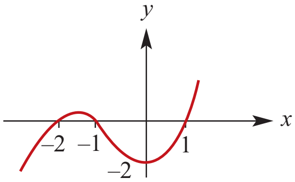
4 $a=3$, $b=-3$, $P(x)=(x-1)(x+3)(x+1)$

Exercise 6F

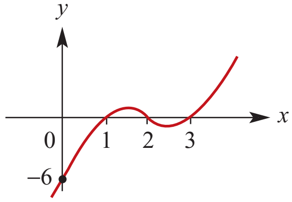
1 a



b



c



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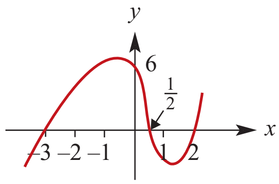

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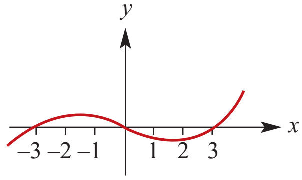

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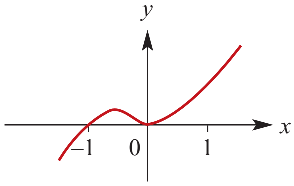

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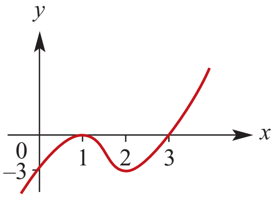

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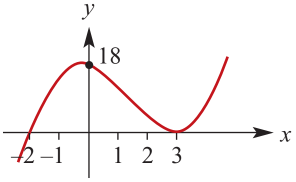
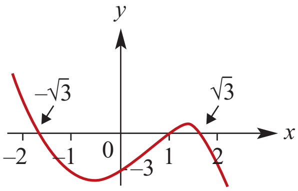
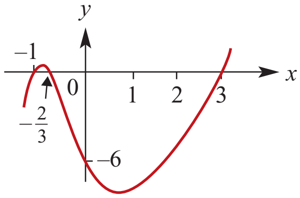


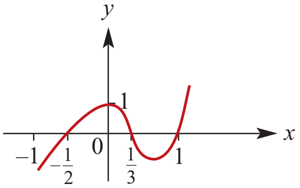
i



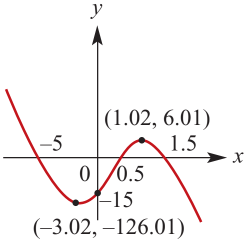
j



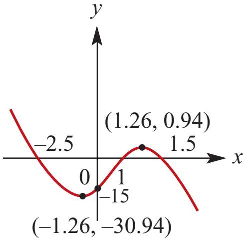
k



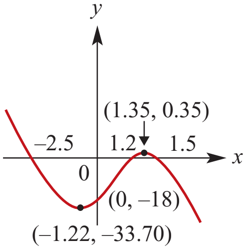
2 a



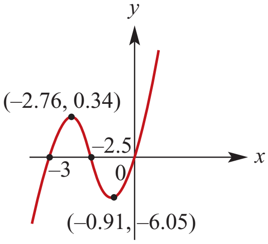
b



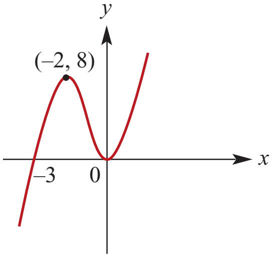
c



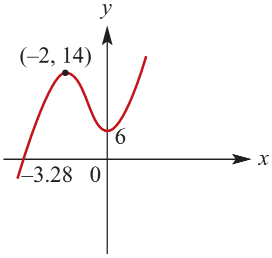
d



e



f



3 $(x + 1)(x + 1)(x - 3) = 0$,
$\therefore$ graph just touches the x-axis at $x = -1$ and cuts it at $x = 3$.

4 a $y = -\frac{1}{8}(x + 2)^3$ **b** $y - 2 = -\frac{1}{4}(x - 3)^3$

5 $y = 2x(x - 2)^2$

6 $y = -2x(x + 4)^2$

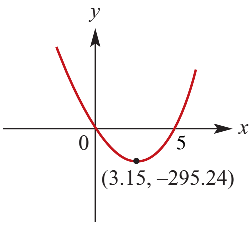
7 a $y = (x - 3)^3 + 2$

b $y = \frac{23}{18}x^3 + \frac{67}{18}x^2$ **c** $y = 5x^3$

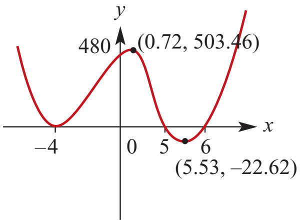
Exercise 6G

1 a $x = 0$ or $x = 3$
b $x = 2$ or $x = -1$ or $x = 5$ or $x = -3$
c $x = 0$ or $x = -2$ **d** $x = 0$ or $x = 6$
e $x = 0$ or $x = 3$ or $x = -3$
f $x = 3$ or $x = -3$
g $x = 0$ or $x = 4$ or $x = -4$
h $x = 0$ or $x = 4$ or $x = 3$
i $x = 0$ or $x = 4$ or $x = 5$
j $x = 2$ or $x = -2$ or $x = 3$ or $x = -3$
k $x = 4$ **l** $x = -4$ or $x = 2$

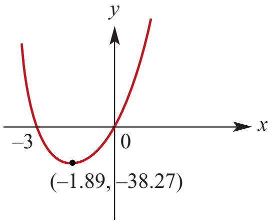
2 a



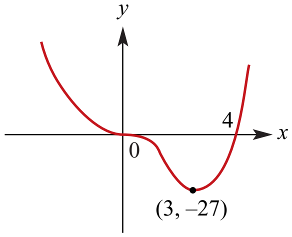
b



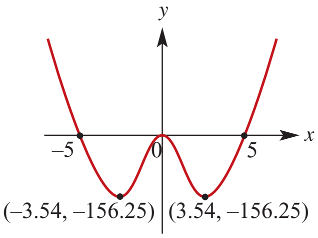
c



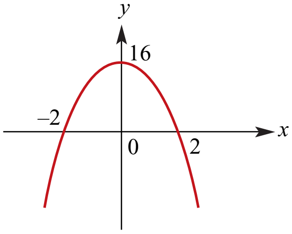
d



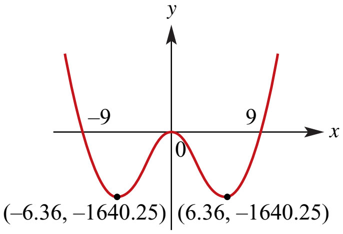
e



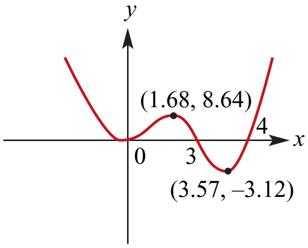
f



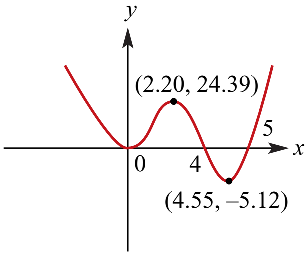
g



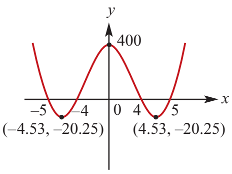
h



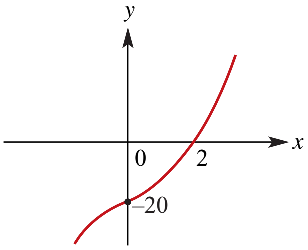
i



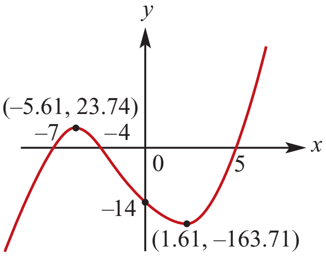
j



k

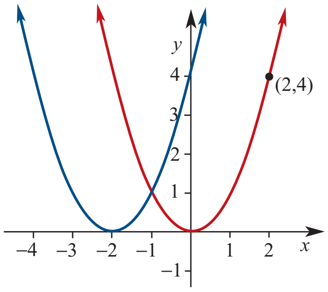


l



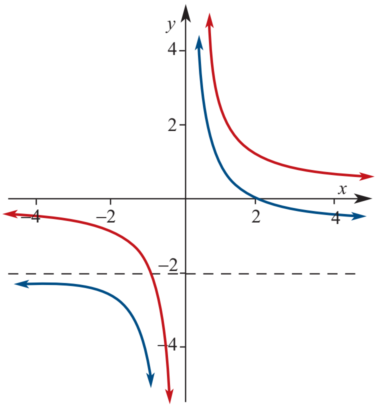
Exercise 6H

1 a



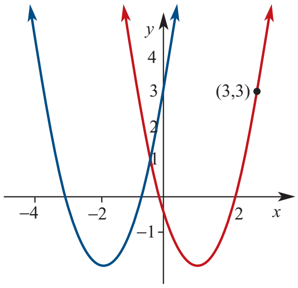
shift left 2

b



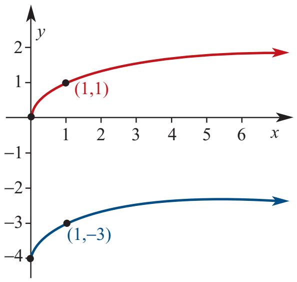
shift down 2

c



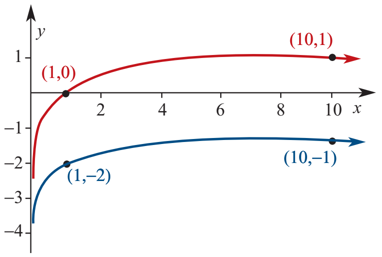
shift left 3

d



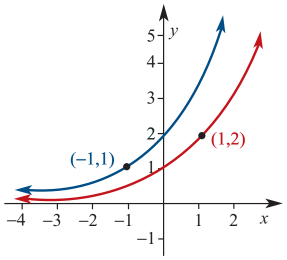
shift down 4

e



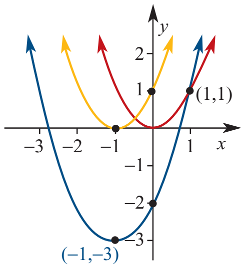
shift down 2

f



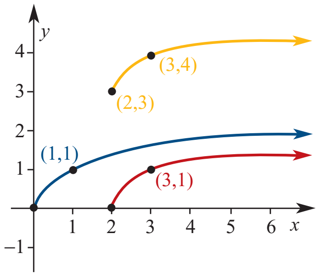
shift left 1

2 a



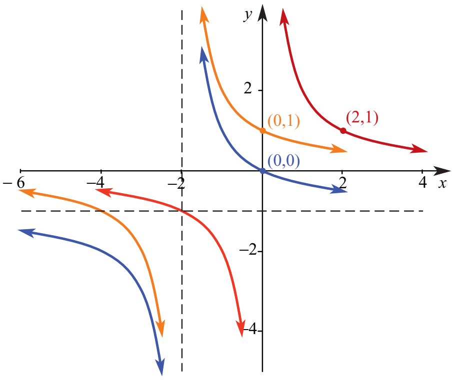
shift left 1, shift down 3

b



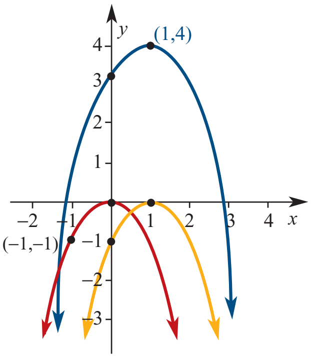
shift right 2, shift up 3

c



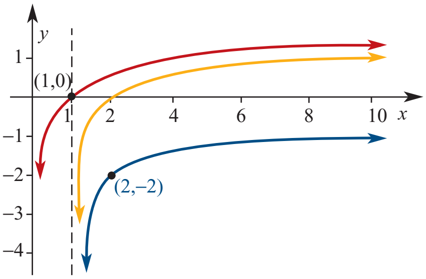
shift left 2, shift down 1

d



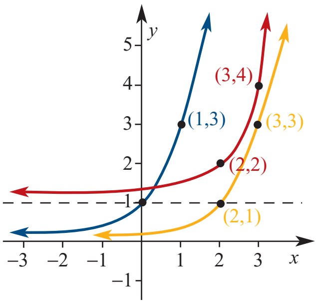
shift right 1, shift up 4

e



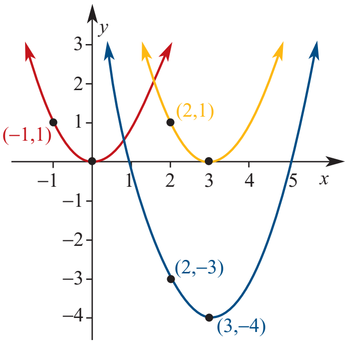
shift right 1, shift down 2

f



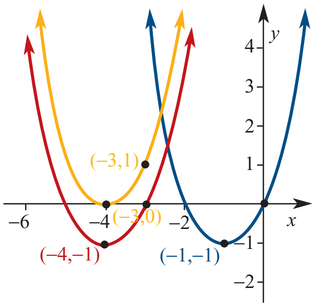
shift right 2, shift up 1

3



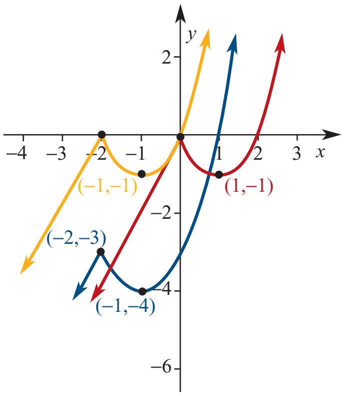
shift right 3, shift down 4

4



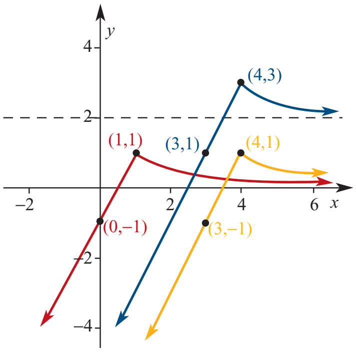
shift left 3, shift up 1

5



shift left 2, shift down 3

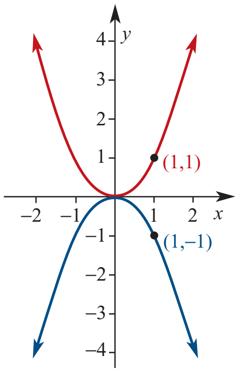
6



shift right 3, shift up 2

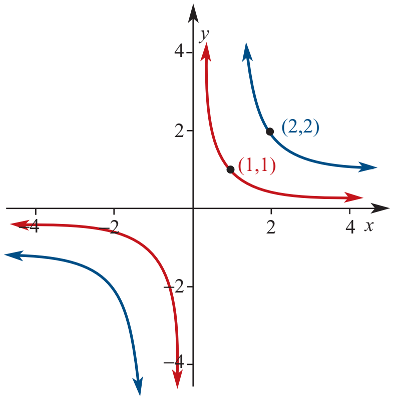
Exercise 6I

1 a



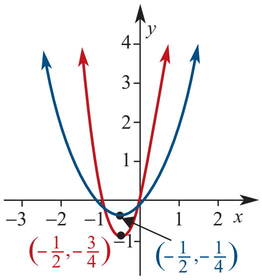
reflect about the x-axis

b



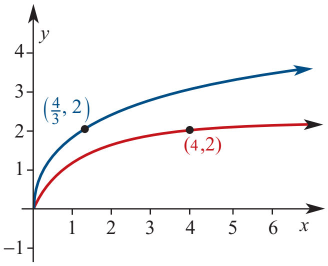
dilate by 4 from the x-axis

c



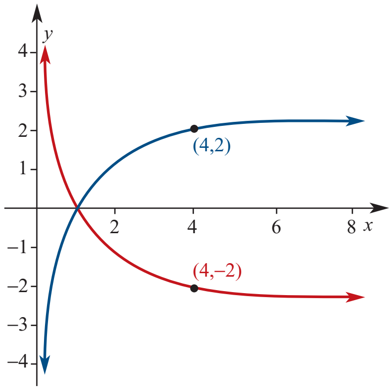
dilate by 3 from the x-axis

d



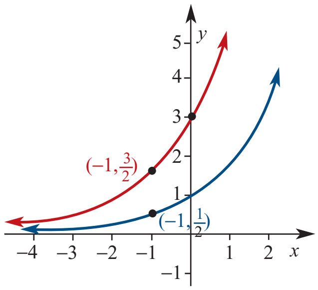
dilate by $\frac{1}{3}$ from the y-axis

e



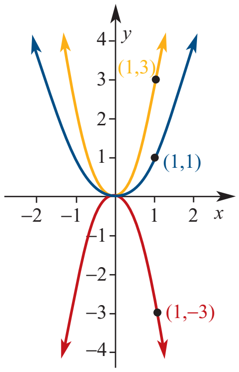
reflect about the x-axis

f



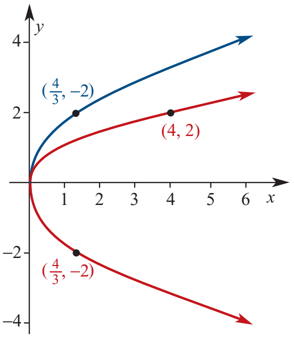
dilate by 3 from the x-axis

2 a



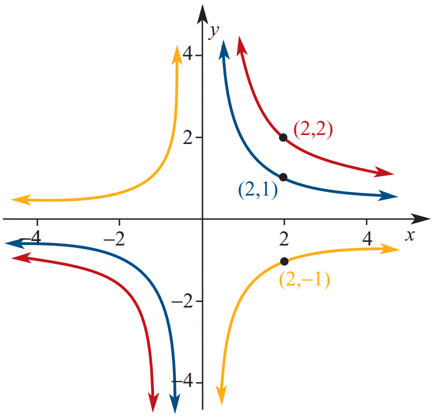
dilate by 3 from the x-axis, reflect about the x-axis

b



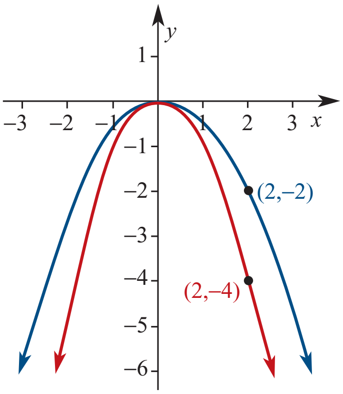
dilate by $\frac{1}{3}$ from the y-axis, reflect about the x-axis

c



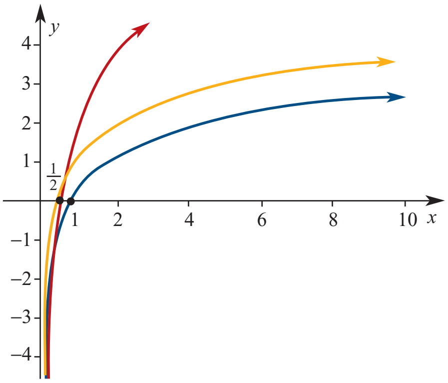
dilate by $\frac{1}{2}$ from the y-axis, reflect about the x-axis

d



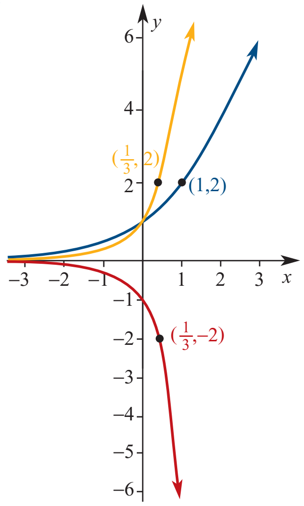
dilate by $\frac{1}{2}$ from the x-axis

e



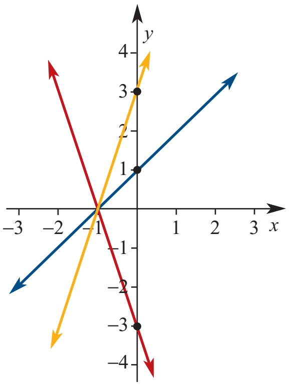
dilate by 3 from the x-axis, dilate by $\frac{1}{2}$ from the y-axis

f



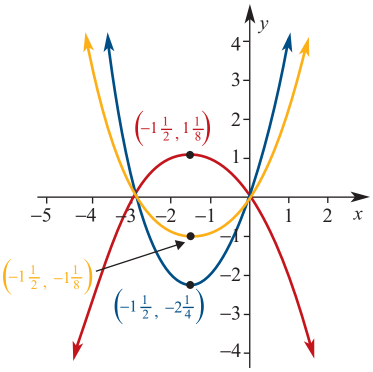
dilate by $\frac{1}{3}$ from the y-axis, reflect about the x-axis

3



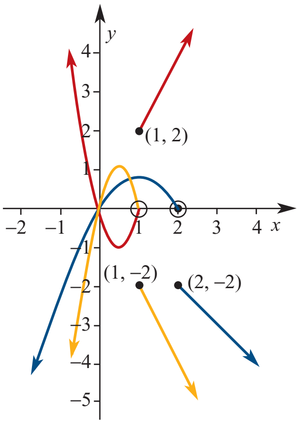
dilate by 3 from the x-axis, reflect about the x-axis

4



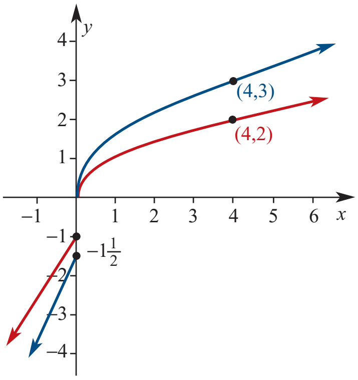
dilate by $\frac{1}{2}$ from the x-axis, reflect about the x-axis

5



dilate by $\frac{1}{2}$ from the y-axis, reflect about the x-axis

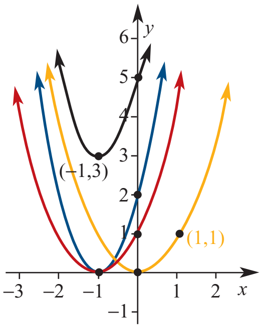
6



dilate by $1\frac{1}{2}$ from the x-axis

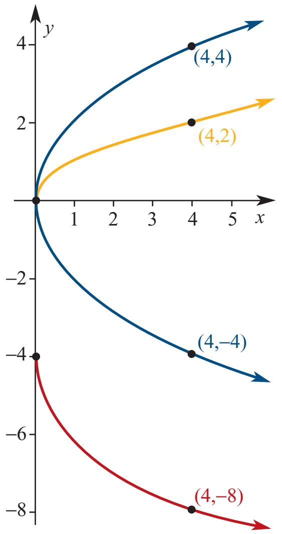
Exercise 6J

1 a



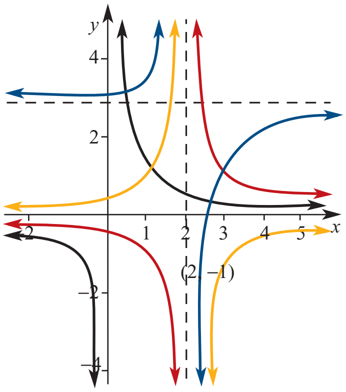
shift left 1, dilate by 2 from the x-axis, shift up 3

b



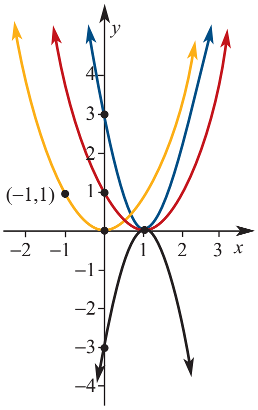
dilate by 2 from the x-axis, reflect about the x-axis, shift down 4

c



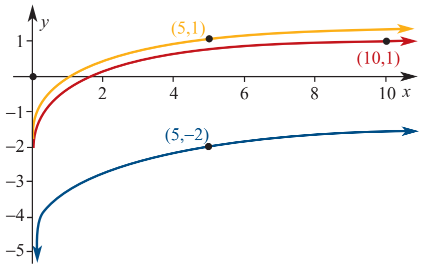
shift right 2, reflect about the x-axis, shift up 3

d



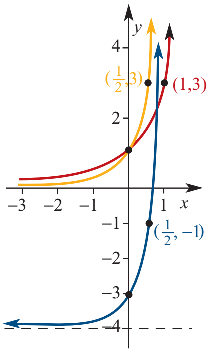
shift right 1, dilate by 3 from the x-axis, reflect about the x-axis

e



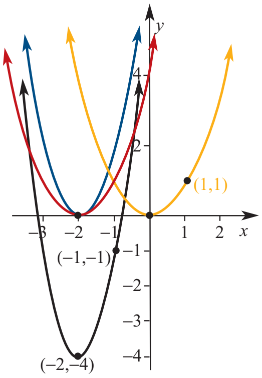
dilate by $\frac{1}{2}$ from the y-axis, shift down 3

f



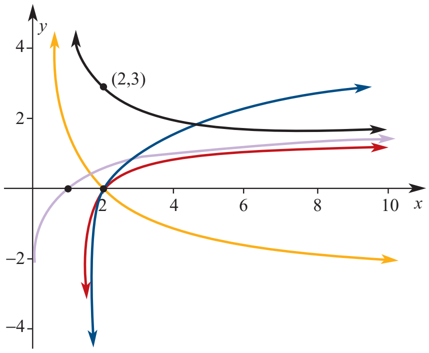
dilate by $\frac{1}{2}$ from the y-axis, shift down 4

2



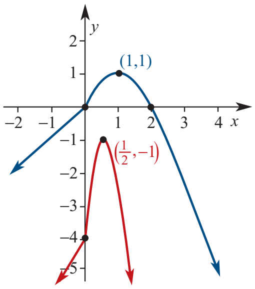
shift left 2, dilate by 3 from the x-axis, shift down 4

3



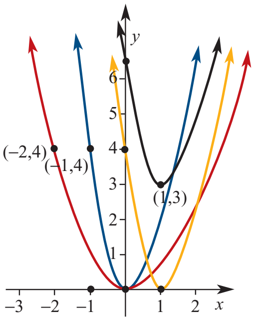
shift right 1, dilate by 2 from the x-axis, reflect about the x-axis, shift up 3

4



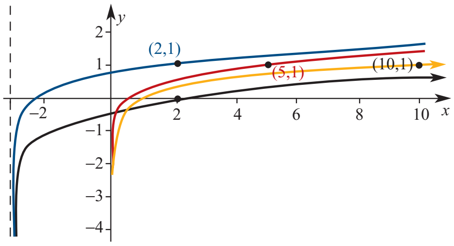
dilate by $\frac{1}{2}$ from the y-axis, dilate by 3 from the x-axis, shift down 4

5 a



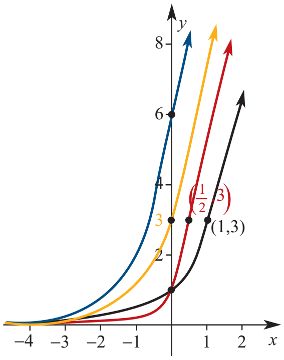
dilate by $\frac{1}{2}$ from the y-axis, shift right 1, shift up 3

b



dilate by $\frac{1}{2}$ from the y-axis, shift left 3, shift down 1

c



dilate by $\frac{1}{2}$ from the y-axis, shift left $\frac{1}{2}$, dilate by 2 from the x-axis

6 a Dilate from y-axis by a factor of $\frac{1}{3}$; translate right 2 units; dilate from x-axis by a factor of 4; translate upwards 1 unit

b Dilate from y-axis by a factor of $\frac{1}{2}$; translate left 5 units; dilate from x-axis by a factor of 8; reflect about the x-axis; translate downwards 7 units

c Dilate from y-axis by a factor of $\frac{1}{6}$; translate right 1 unit; dilate from x-axis by a factor of 2; translate downwards 5 units

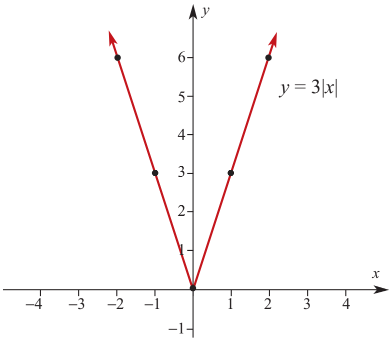
Exercise 6K

1 a 7 **b** 6 **c** 5

d 6 **e** 11 **f** −4

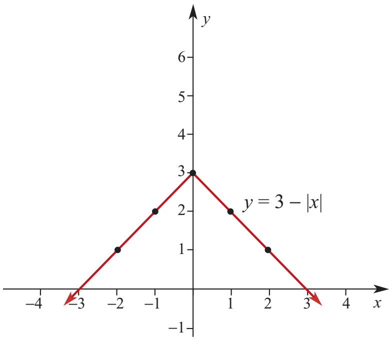
2 a

x	−2	−1	0	1	2
y	6	3	0	3	6

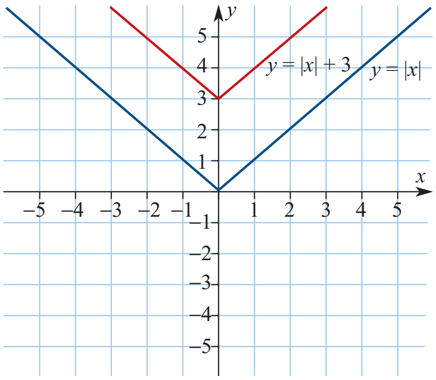


b

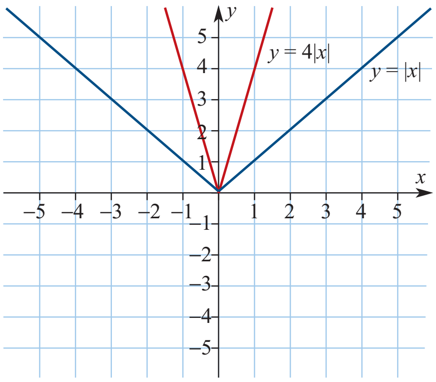
x	−2	−1	0	1	2
y	1	2	3	2	1



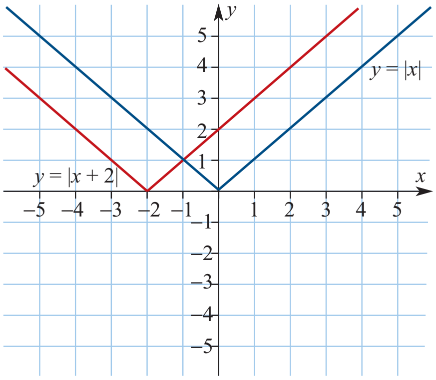
3 a



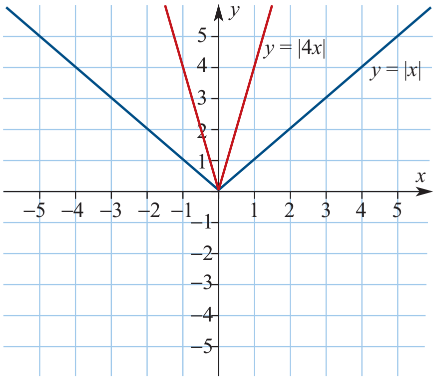
b


y
x
y = 4|x|
y = |x|
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

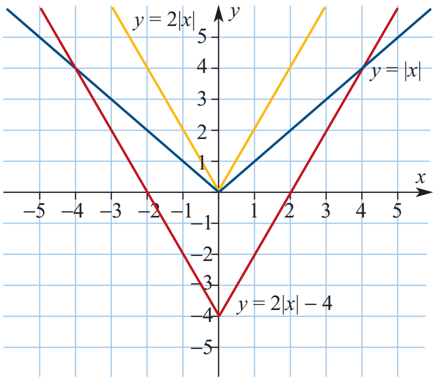
c


y
x
y = |x|
y = |x + 2|
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

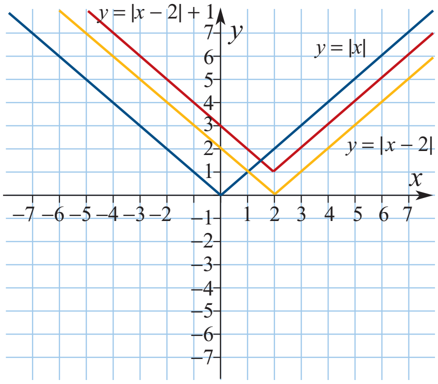
d


y
x
y = |4x|
y = |x|
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

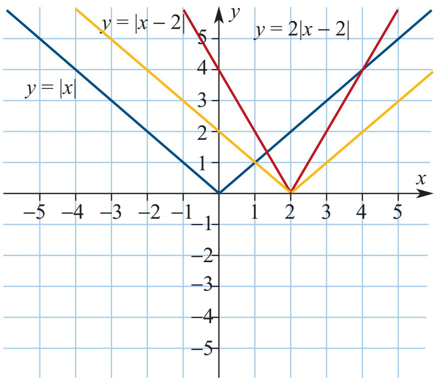
e


y
x
y = 2|x|
y = |x|
y = 2|x| – 4
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

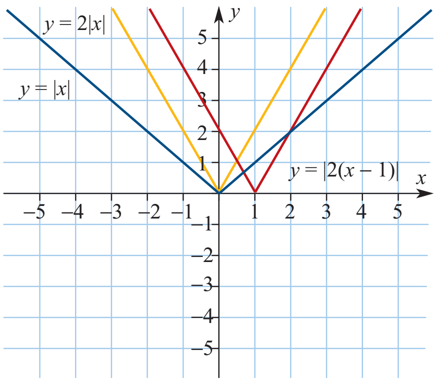
f


y = |x – 2| + 1
y
y = |x|
y = |x – 2|
x
–7 –6 –5 –4 –3 –2 –1 1 2 3 4 5 6 7
7 6 5 4 3 2 1 –1 –2 –3 –4 –5 –6 –7

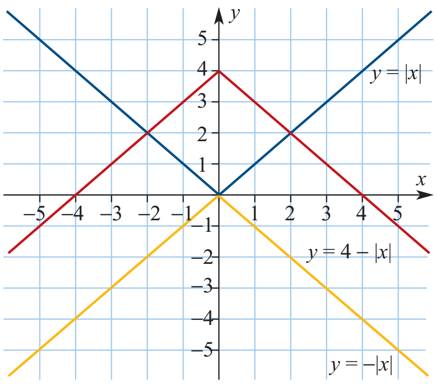
g


y
y = |x – 2|
y = 2|x – 2|
y = |x|
x
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

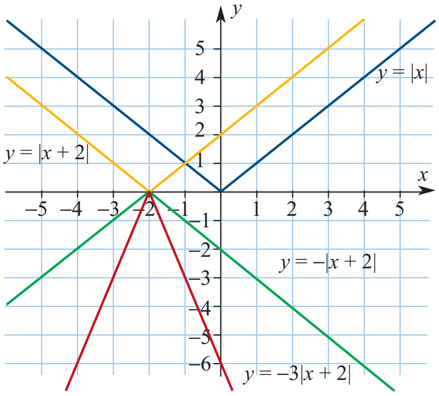
h


y
y = 2|x|
y = |x|
y = |2(x – 1)|
x
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

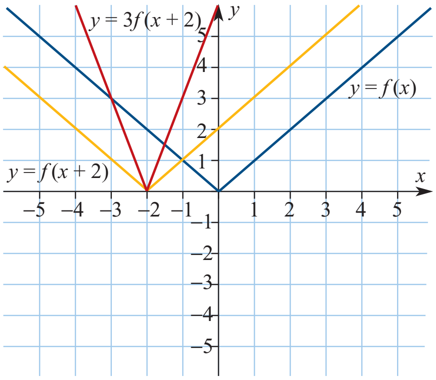
i


y
y = |x|
x
y = 4 – |x|
y = –|x|
–5 –4 –3 –2 –1 1 2 3 4 5
5 4 3 2 1 –1 –2 –3 –4 –5

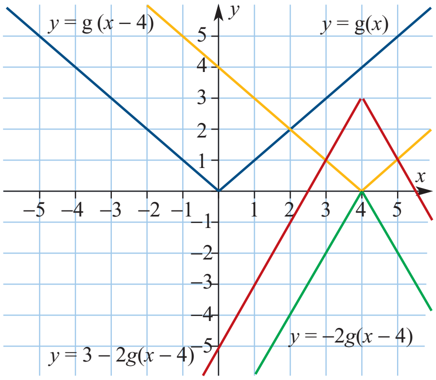
j



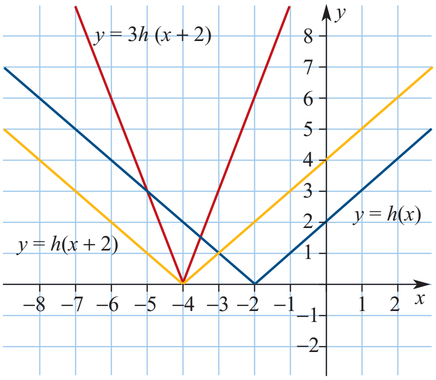
4 a



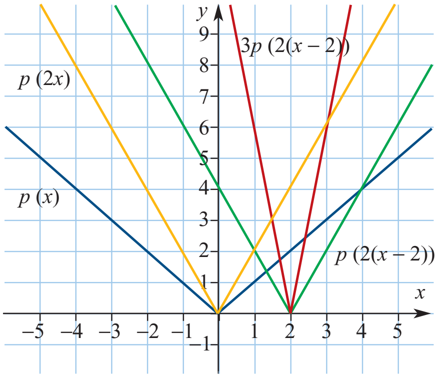
b



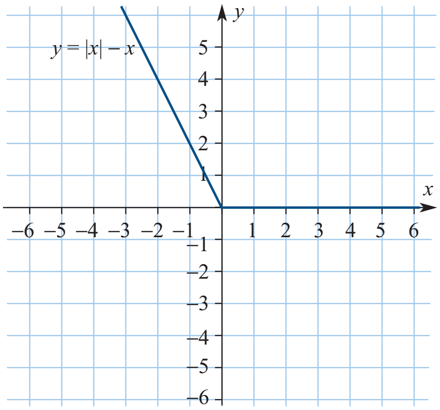
c



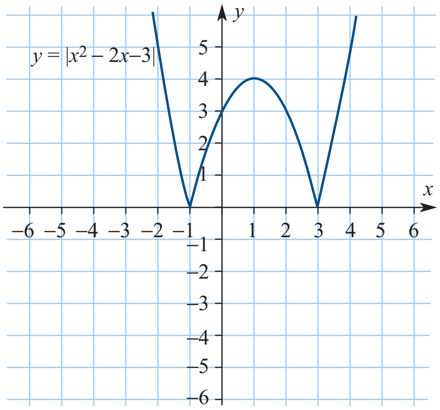
d



5 a



b



Exercise 6L

1 3 cm or 0.838 cm **2** 3 cm or 6.066 cm

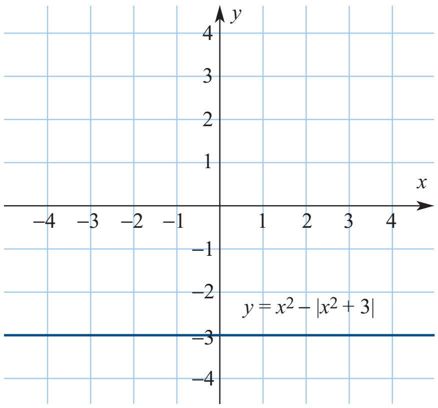
3 $V = \dfrac{1}{32\,400}(t - 900)^2, \quad 0 < t < 900$

4 $V = 0.08\,(x - 5)^3 + 10, \quad x \geq 0$

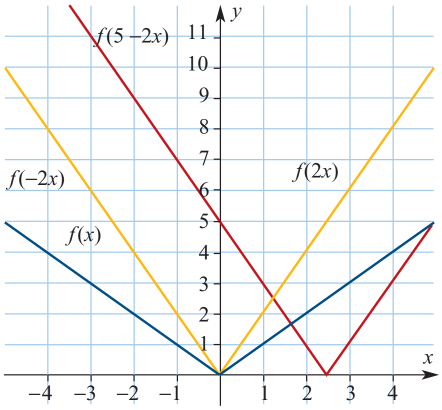
5 12 cm or 22.87 cm

6 30 cm or 71.10 cm **7** 16 384 cm^3

8



9 Dilate by $\frac{1}{2}$ from the y-axis, then reflect in the y-axis, followed by a translation of $2\frac{1}{2}$ to the left. So, plot for $f(x) = |\ x\ |$ transformed to $f(5 - 2x)$ is:



Multiple-choice answers

1 E **2** C **3** E **4** D **5** B

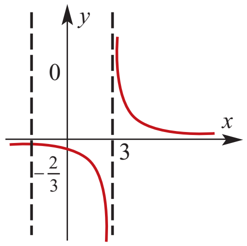
6 D **7** C **8** A **9** C **10** E

Short-response answers

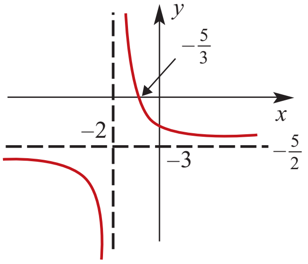
1 a $k = 2.7$ **b** $k = 50$

2 a \$55.90 **b** $k = 22$

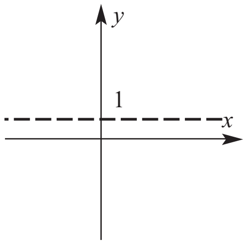
3 a



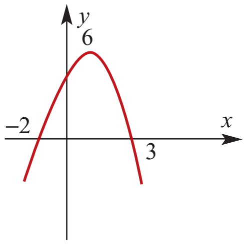
b



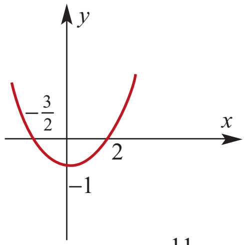
4 a



b



c



5 a $x^2 + 3x + 4 + \dfrac{11}{x - 2}$

b $2x^2 + 6x + 14 - \dfrac{39}{x + 3}$

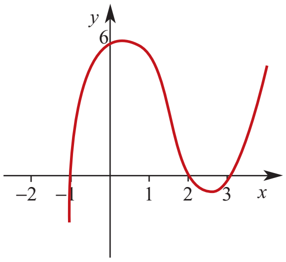
6 a $(x - 1)(x^2 + 8x + 1)$

b $(x - 3)(x - 1)(x + 1)$

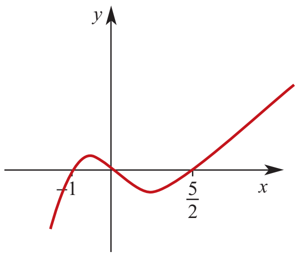
c $2(x - 2)(x^2 + 2x + 4)$

d $(x + 4)(x^2 - 4x + 16)$

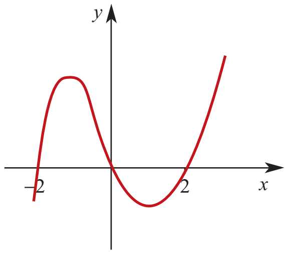
7 a



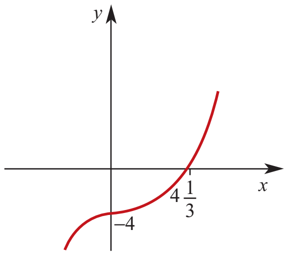
b



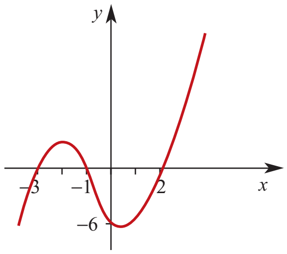
c



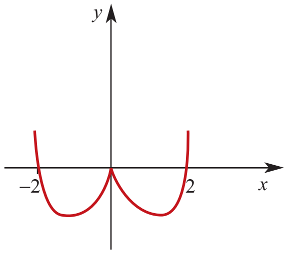
d



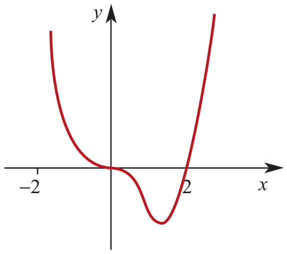
e



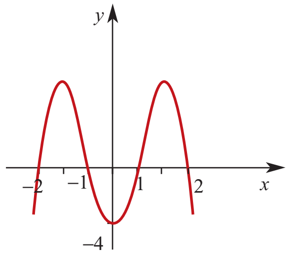
8 a



b



c



9 a translate right by 2, up by 1

b translate left by 1, down by 3

10 a dilate by 2 from x-axis, reflect about x-axis

b dilate by $\frac{1}{2}$ from y-axis

11 a translate left by 1, dilate by 2 from x-axis

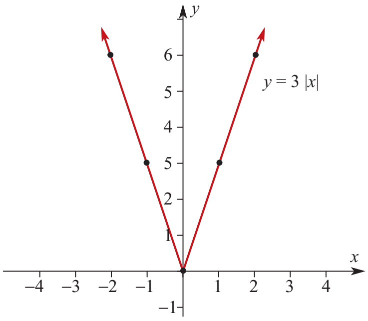
b dilate by $\frac{1}{2}$ from y-axis, translate left by $\frac{1}{2}$

c dilate by $\frac{1}{2}$ from y-axis, translate left by $\frac{5}{2}$, and down by 1

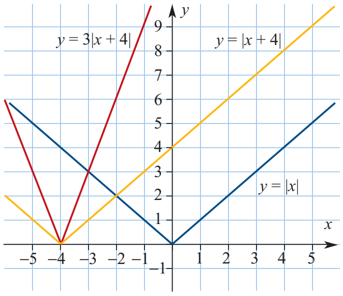
12 a 6 **b** 5 **c** 3

13

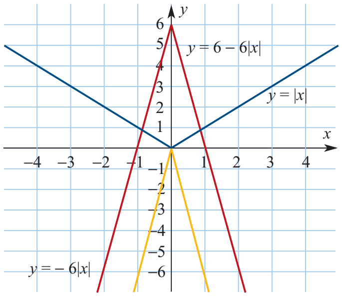
x	–2	–1	0	1	2
y	4	2	0	2	4



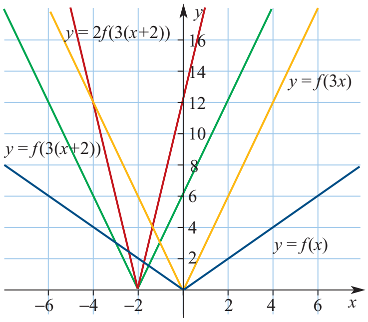
14 a



b



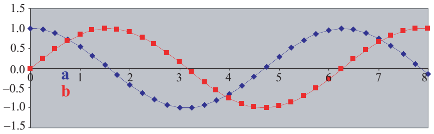
c



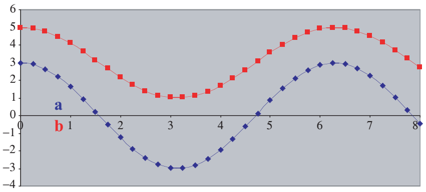
Chapter 7

Exercise 7A

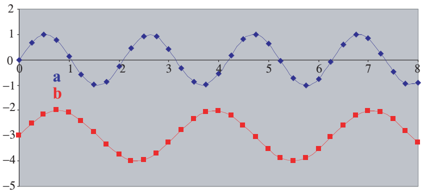
1



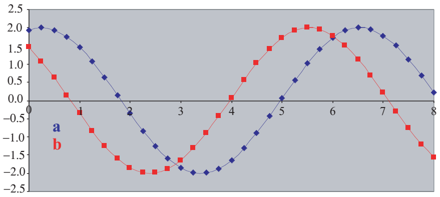
2



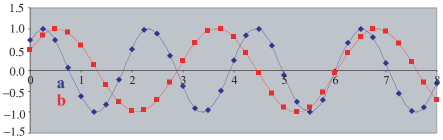
3



4



5



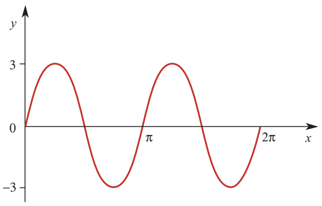
Exercise 7B

1 a 2π and 2 **b** π and 3 **c** $\frac{2\pi}{3}$ and $\frac{1}{2}$

d 4π and 3 **e** $\frac{2\pi}{3}$ and 4 **f** $\frac{\pi}{2}$ and $\frac{1}{2}$

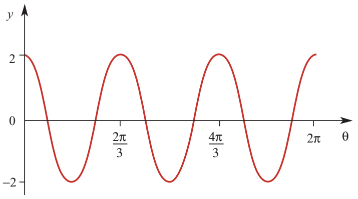
g 4π and 2

2 a



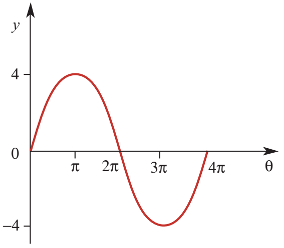
Amplitude $= 3$, Period $= \pi$

b



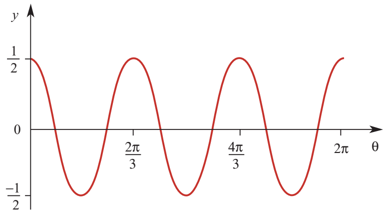
Amplitude $= 2$, Period $= \frac{2\pi}{3}$

c



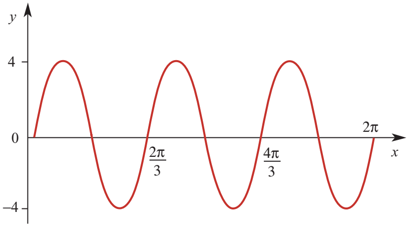
Amplitude $= 3$, Period $= 4\pi$

d



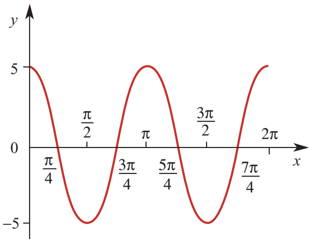
Amplitude $= \frac{1}{2}$, Period $= \frac{2\pi}{3}$

e



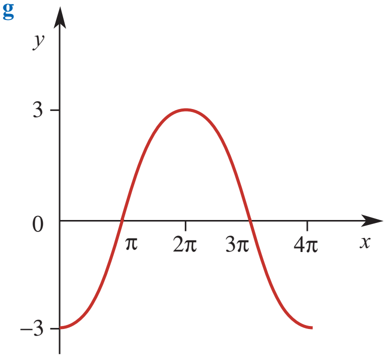
Amplitude $= 4$, Period $= \frac{2\pi}{3}$

f



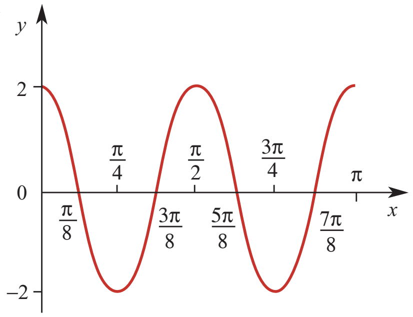
Amplitude $= 5$, Period $= \pi$

g



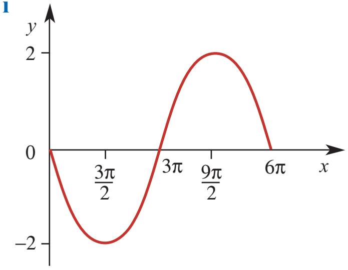
Amplitude $= 3$, Period $= 4\pi$

h



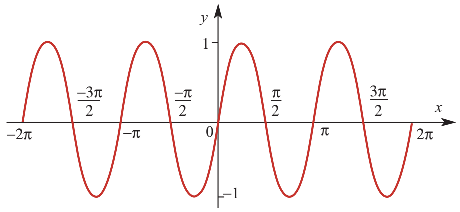
Amplitude $= 2$, Period $= \frac{\pi}{2}$

i

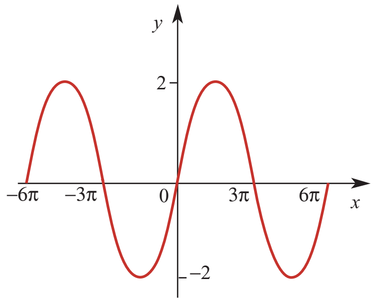


Amplitude $= 2$, Period $= 6\pi$

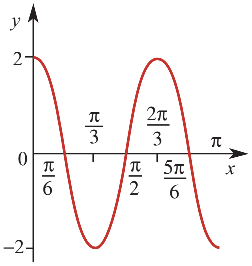
3 a



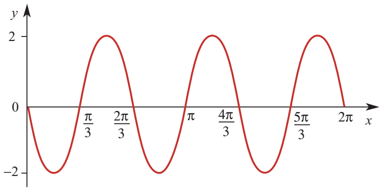
b



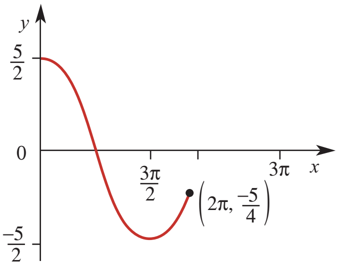
c



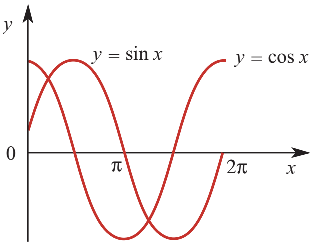
d



4

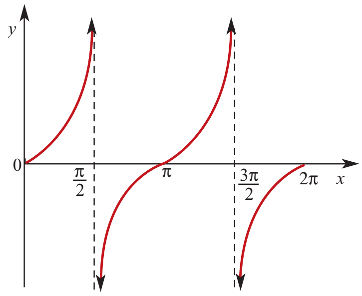


5 a

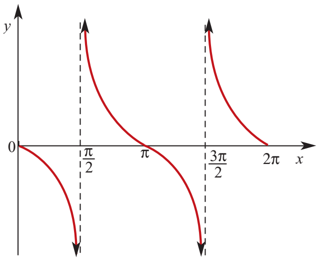


b $\frac{\pi}{4}$, $\frac{5\pi}{4}$

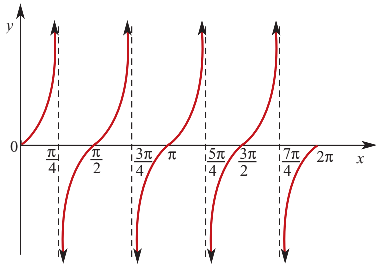
6 a $y = \tan x$, period $= \pi$



b $y = -2\tan x$, period $= \pi$



c $y = \tan 2x$, period $= \dfrac{\pi}{2}$



7 a Check with your teacher

b solutions are −4.49, 0, 4.49

8 a sine curve, $A = 3$, period $= 2\pi$, $B = 1$; hence, equation is $y = 3\sin x$

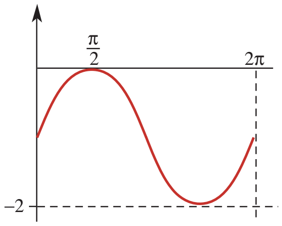
b inverted sine curve, $A = 1$, period $= \dfrac{\pi}{2}$, $B = 4$; hence, equation is $y = -\sin 4x$

c cosine curve, $A = 2.5$, period $= \dfrac{\pi}{3}$, $B = 6$; hence, equation is $y = 2.5\cos 6x$

d cosine curve, $A = 0.5$, period $= 24$, $B = \dfrac{2\pi}{24} = \dfrac{\pi}{12}$; hence, equation is $y = 0.5\cos\dfrac{\pi x}{12}$

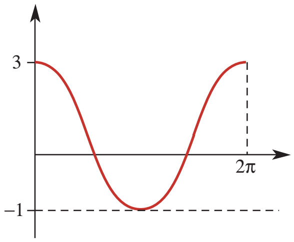
Exercise 7C

1 a



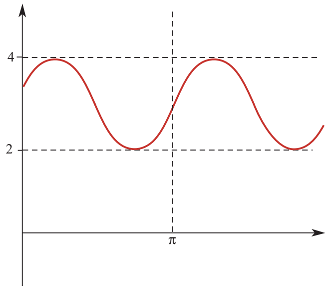
Period $= 2\pi$
Amplitude $= 1$
Greatest value $y = 0$
Least value $y = -2$

b



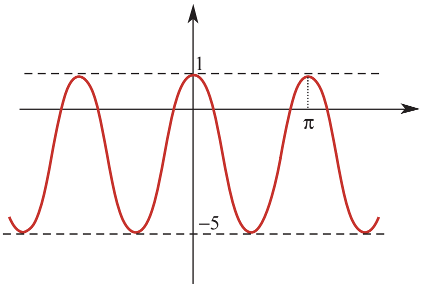
Period $= 2\pi$
Amplitude $= 2$
Greatest value $y = 3$
Least value $y = -1$

c



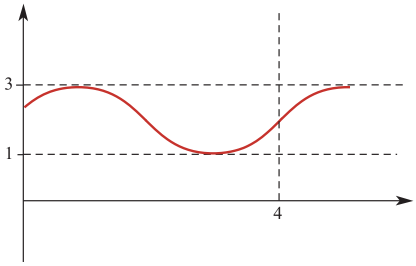
Period $= \pi$
Amplitude $= 1$
Greatest value $y = 4$
Least value $y = 2$

d



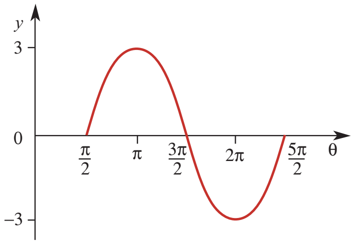
Period $= \pi$
Amplitude $= 3$
Greatest value $y = 1$
Least value $y = -5$

e



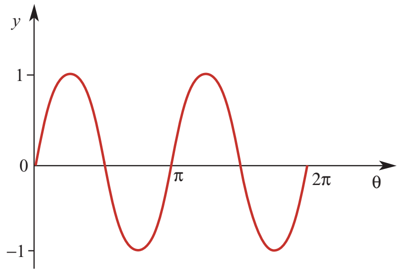
Period $= 4$
Amplitude $= 1$
Greatest value $y = 3$
Least value $y = 1$

2 a



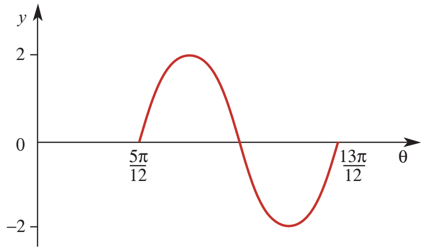
Period $= 2\pi$, Amplitude $= 3$, $y = \pm 3$

b



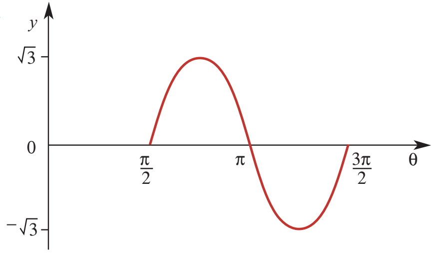
Period $= \pi$, Amplitude $= 1$, $y = \pm 1$

c



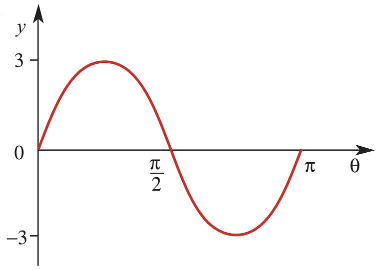
Period $= \frac{2\pi}{3}$, Amplitude $= 2$, $y = \pm 2$

d



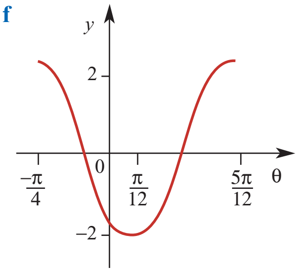
Period $= \pi$, Amplitude $= \sqrt{3}$, $y = \pm\sqrt{3}$

e



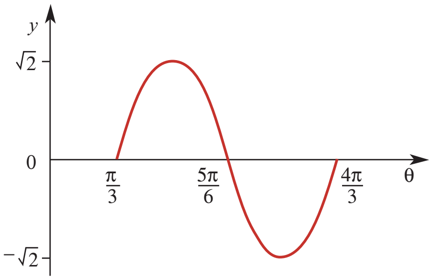
Period $= \pi$, Amplitude $= 3$, $y = -3, 3$

f



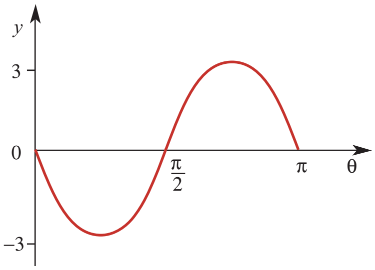
Period $= \frac{2\pi}{3}$, Amplitude $= 2$, $y = -2, 2$

g



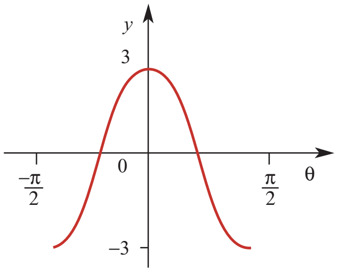
Period $= \pi$, Amplitude $= \sqrt{2}$, $y = \pm\sqrt{2}$

h



Period $= \pi$, Amplitude $= 3$, $y = -3, 3$

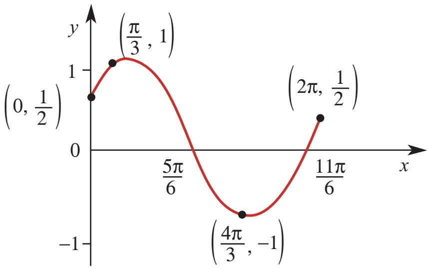
i



Period $= \pi$, Amplitude $= 3$, $y = 3, -3$

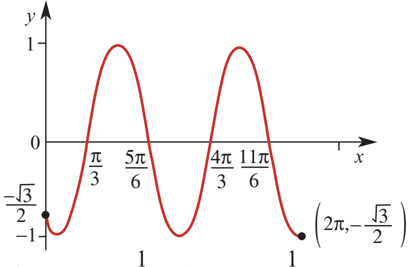
3 a $f(0) = \frac{1}{2}$, $f(2\pi) = \frac{1}{2}$

b



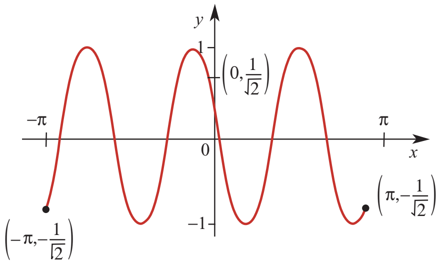
4 a $f(0) = -\frac{\sqrt{3}}{2}$, $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



5 a $f(-\pi) = -\frac{1}{\sqrt{2}}$, $f(\pi) = -\frac{1}{\sqrt{2}}$

b



6 Amplitude = 4, Period = π, Phase shift = $\frac{\pi}{2}$ to the right

Exercise 7D

1 a 30°, 330° **b** 210°, 330° **c** 60°, 240°
d 45°, 135° **e** 150°, 210° **f** 135°, 315°

2 a $x = \frac{\pi}{3}, \frac{2\pi}{3}$ **b** $x = \frac{\pi}{3}, \frac{5\pi}{3}$

c $x = \frac{5\pi}{4}, \frac{7\pi}{4}$ **d** $x = \frac{2\pi}{3}, \frac{5\pi}{3}$

e $\frac{\pi}{6}, \frac{11\pi}{6}$ **f** $\frac{\pi}{4}, \frac{5\pi}{4}$

3 a 180° **b** 90° **c** 0°, 180°, 360°
d 0°, 120° **e** 0°, 180°, 360° **f** 270°

4 a $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{2\pi}{3}$

c $\theta = -30, 150$

d $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

e no solution

f $-\frac{5\pi}{3}, -\frac{2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}$

5 a 0.3398, 2.8018 **b** 0.3805, 3.5221
c 1.8235, 4.4597 **d** 3.3226, 6.1022
e 0.6675, 2.4741 **f** 1.5958, 4.7374

6 a 33.69°, 213.69°

b $-\frac{15\pi}{8}, -\frac{7\pi}{8}, \frac{\pi}{8}, \frac{9\pi}{8}$

Exercise 7E

1 a 40°, 80°, 160°, 200°, 280°, 320°

b $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

c $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

d 10°, 70°, 130°, 190°, 250°, 310°

e $\frac{\pi}{12}, \frac{\pi}{6}, \frac{7\pi}{12}, \frac{2\pi}{3}$

f 14.96°, 165.04°, 194.96°, 345.04°

2 a $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b $\theta = 0, \frac{\pi}{6}, \pi$

c $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

d $\theta = -\pi, -\frac{3\pi}{4}, 0, \frac{\pi}{4}, \pi$

e $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

f $\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

g $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

3 a 55° **b** 63° **c** $\frac{\pi}{3}$

4 a $\sqrt{13}$ **b** $\frac{\sqrt{13}}{5}$ **c** $\frac{2\sqrt{3}}{\sqrt{13}}$

5 a i 0.8 **ii** $\frac{4}{3}$

b $-\frac{5}{13}$ **c** $-\frac{4}{3}$ **d** $\pm\frac{\sqrt{5}}{3}$

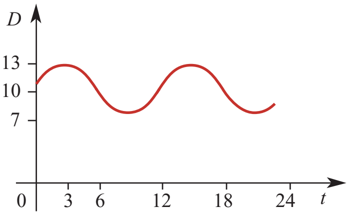
6 a $\frac{2}{\sqrt{5}}$ **b** $\frac{1}{2}$ **c** $\frac{2}{\sqrt{3}}$ **d** $\frac{1}{2}$

e $\frac{\sqrt{3}}{2}$ **f** $-\frac{2}{3}$ **g** $-\frac{1}{2}$ **h** $\frac{2}{\sqrt{3}}$

i $\frac{\sqrt{3}}{2}$ **j** $\frac{\sqrt{3}}{\sqrt{7}}$ or $\sqrt{\frac{3}{7}}$ **k** $-\frac{\sqrt{3}}{2}$ **l** $\frac{1}{\sqrt{3}}$

Exercise 7F

1 a



b 11.5 metres

c $t = 1.394, 4.606, 13.394, 16.606$ hours; i.e. at 1.24 a.m., 4.37 a.m., 1.24 p.m., 4.37 p.m.

d Between 1.24 a.m. and 4.37 a.m., and again between 1.24 p.m. and 4.37 p.m.

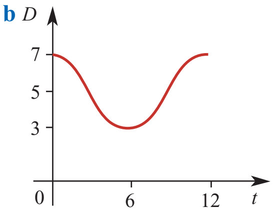
e 12.9 metres

2 a i 0.0018 hours
ii 11.79 hours

b 26 April ($t = 3.856$), 14 August ($t = 7.477$)

3 a $p = 5, q = 2$

b



c A ship can enter 2 hours after low tide.

4 a 5 **b** 1

c $t = 0.524$ s, 2.618 s, 4.712 s

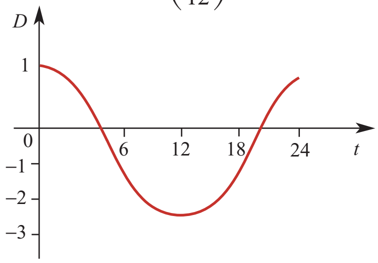
d $t = 0$s, 1.047 s, 2.094 s

e Particle oscillates about the point $x = 3$ from $x = 1$ to $x = 5$.

5 a 19.5° C

b $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

c



d $4 < t < 20$

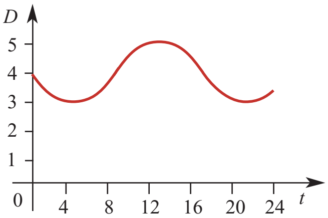
6 a 2 a.m. **b** 8 a.m. and 8 p.m.

7 a i $\frac{3}{2}$ **ii** 12

iii $d(t) = \frac{7}{2} - \frac{3}{2}\cos\frac{\pi}{6}t$ **iv** 1.5 metres

b between 9 p.m. and 3 a.m., and 9 a.m. and 3 p.m., each day

8 a



b The boat can enter 4 hours before noon and must leave by 4 p.m.

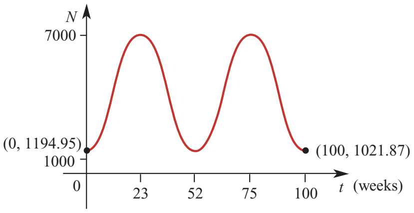
c The boat can enter at 6:40 a.m. and must leave by 5:20 p.m.

9 a i 52 weeks **ii** 3000

iii between 1000 and 7000

b i $N(0) = 1194.95$, $N(100) = 1021.87$

ii

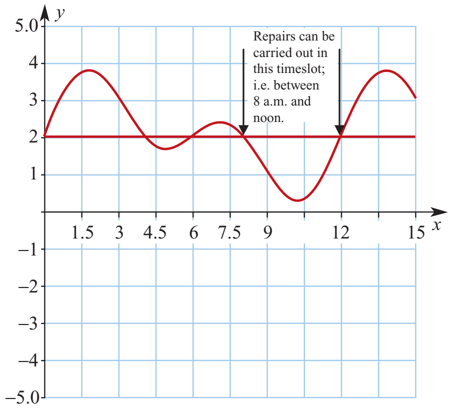


c i $t = 23, 75$ **ii** 49

d between $14\frac{1}{3}$ and $31\frac{2}{3}$ and also between $66\frac{1}{3}$ and $83\frac{2}{3}$

e $d = 25\,000$, $a = 15\,000$, $b = 10$, $c = 5$

10



Multiple-choice questions

1 B	**2** A	**3** D	**4** D	**5** C
6 D	**7** E	**8** E	**9** B	**10** B

Short-response questions

1 a $\frac{11\pi}{6}$ **b** $\frac{9\pi}{2}$ **c** 6π **d** $\frac{23\pi}{4}$

e $\frac{3\pi}{4}$ **f** $\frac{9\pi}{4}$ **g** $\frac{13\pi}{6}$ **h** $\frac{7\pi}{3}$

i $\frac{4\pi}{9}$

2 a 150° **b** 315° **c** 495° **d** 45°

e 1350° **f** −135° **g** −45° **h** −495°

i −1035°

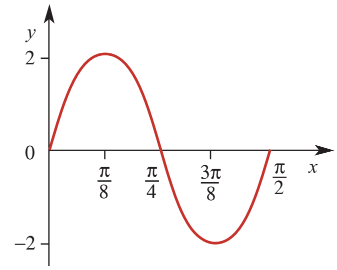
3 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{2}$ **d** $\frac{-\sqrt{3}}{2}$

e $\frac{\sqrt{3}}{2}$ **f** $-\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $\frac{-1}{\sqrt{2}}$

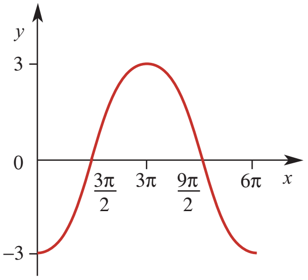
4 a $2, 4\pi$ **b** $3, \frac{\pi}{2}$ **c** $\frac{1}{2}, \frac{2\pi}{3}$

d $3, \pi$ **e** $4, 6\pi$ **f** $\frac{2}{3}, 3\pi$

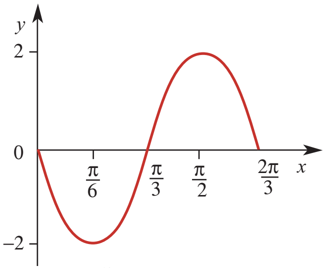
5 a $y = 2\sin 2(2x)$



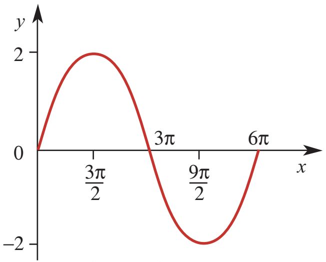
b $y = -3\cos\dfrac{x}{3}$



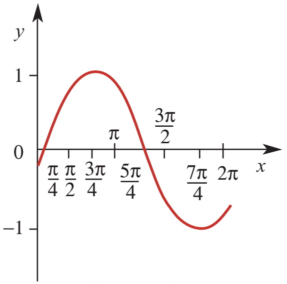
c $y = -2\sin 3x$



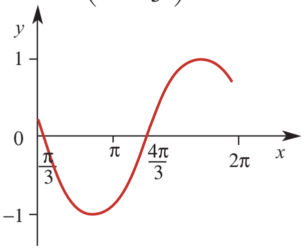
d $y = 2\sin\dfrac{x}{3}$



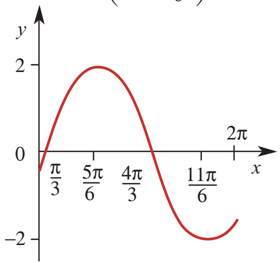
e $y = \sin\left(x - \dfrac{\pi}{4}\right)$



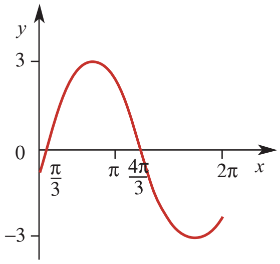
f $y = \sin\left(x + \dfrac{2\pi}{3}\right)$



g $y = 2\cos\left(x - \dfrac{5\pi}{6}\right)$



h $y = -3\cos\left(x + \dfrac{\pi}{6}\right)$

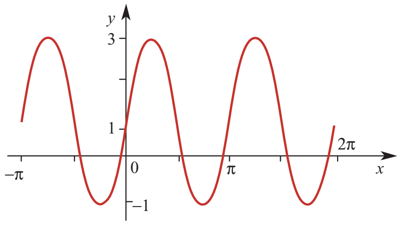


6 a $\dfrac{-2\pi}{3}, \dfrac{-\pi}{3}$ **b** $\dfrac{-\pi}{3}, \dfrac{-\pi}{6}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}$

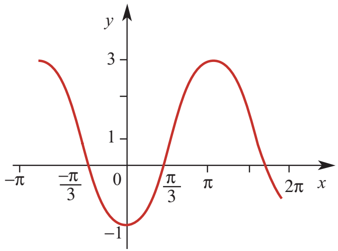
c $\dfrac{7\pi}{12}, \dfrac{11\pi}{12}, \dfrac{19\pi}{12}, \dfrac{23\pi}{12}$ **d** $\dfrac{3\pi}{2}$

e $\dfrac{\pi}{2}, \dfrac{7\pi}{6}$

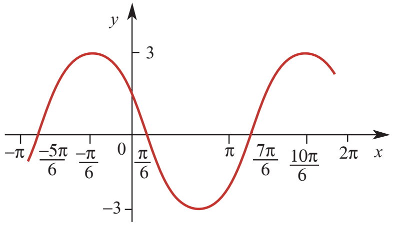
7 a $f(x) = 2\sin 2x + 1$



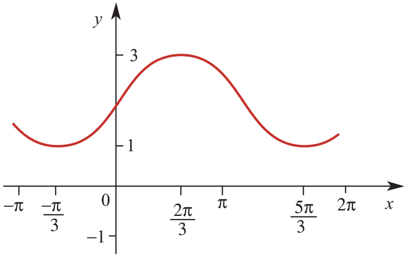
b $f(x) = 1 - 2\cos x$



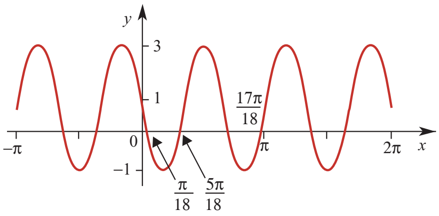
c $f(x) = 3\cos\left(x + \dfrac{\pi}{3}\right)$



d $f(x) = 2 - \cos\left(x + \dfrac{\pi}{3}\right)$



e $f(x) = 1 - 2\sin 3x$



8 a 0°, 30°, 150°, 180°

b $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

c 0°, 120°, 240°, 360°

9 a i 13.4 **ii** 2 **iii** 12

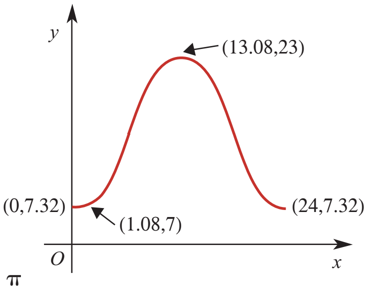
b 3 a.m., 9 a.m., 3 p.m., 9 p.m.

c $2 < t < 10$, $14 < t < 22$

10 a 7.3° **b** min = 7°, max = 23°

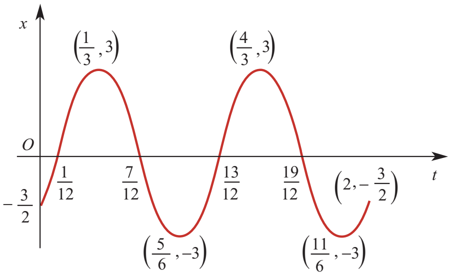
c between 9.40 a.m. and 4.30 p.m.

d



11 a $\dfrac{\pi}{6}$

b



c 3 metres

d $\dfrac{5}{6}$ seconds

e 1 second

f $\dfrac{1}{4}$ second

g i 24 metres **ii** 30 metres

12 a $p = 6, q = 4.2$

b 3 a.m., 3 p.m.

c 6 metres

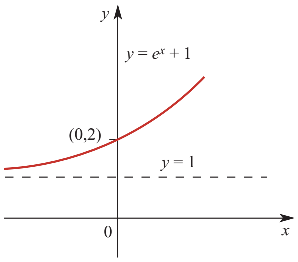
d 7 a.m., 11 a.m., 7 p.m., 11 p.m.

e 8 hours

Chapter 8

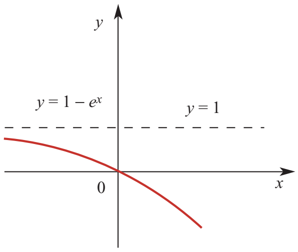
Exercise 8A

1 a



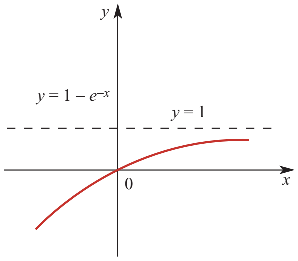
y translation +1 unit

b



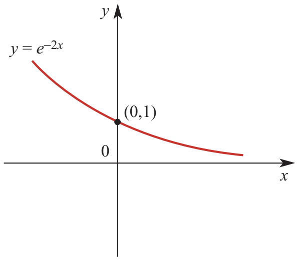
x-axis reflection, y translation +1 unit

c



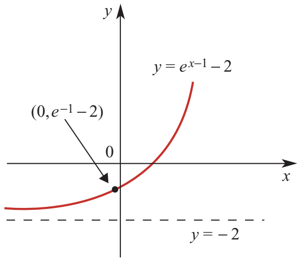
y-axis reflection, x-axis reflection, y translation +1 unit

d



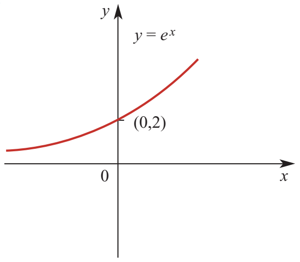
x dilation $-\frac{1}{2}$

e



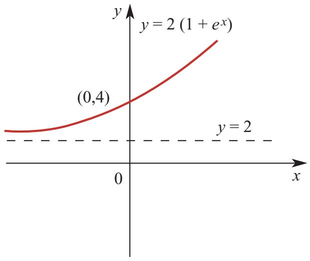
x translation $+1$ unit, y translation -2 units

f



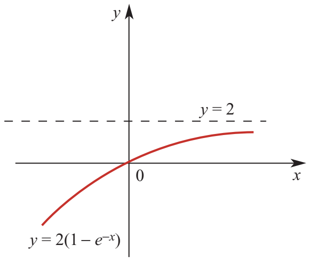
y dilation 2

g



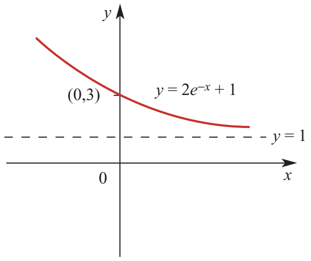
y dilation 2, y translation $+2$ units

h



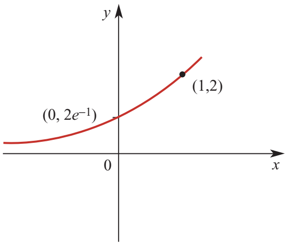
y-axis reflection, y dilation -2, y translation $+2$ units

i



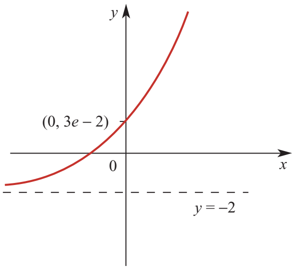
y-axis reflection, y dilation 2, y translation $+1$ unit

j



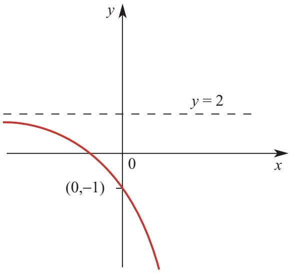
x translation $+1$ unit, y dilation 2

k



x translation -1 unit, y dilation 3, y translation -2 units

l



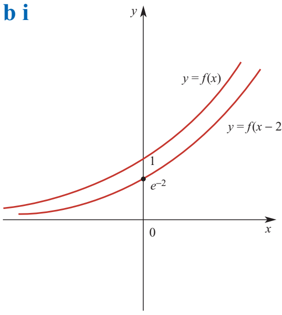
y dilation -3, y translation 2 units

2 a $2 + e^x$, y translation 2 units

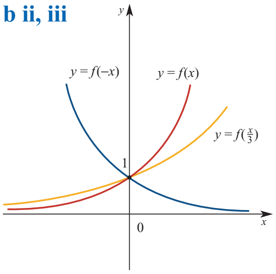
b $2\,e^{-x}$, y-axis reflection, y dilation 2

c $e^{(x+1)} - 1$, x translation -1 unit, y translation -1 unit

3a, b i



a b ii, iii

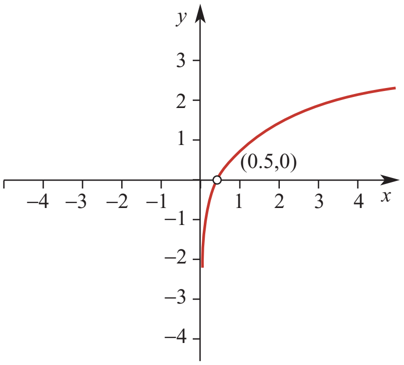


4 a $x = 1.146$ or $x = -1.841$
b $x = -0.443$
c $x = -0.703$
d $x = 1.857$ or $x = 4.536$

Exercise 8B

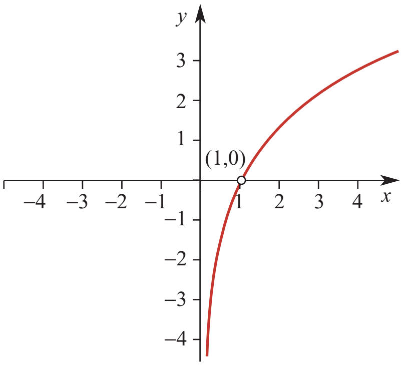
1 a 0.57 **b** 0.57

2 a



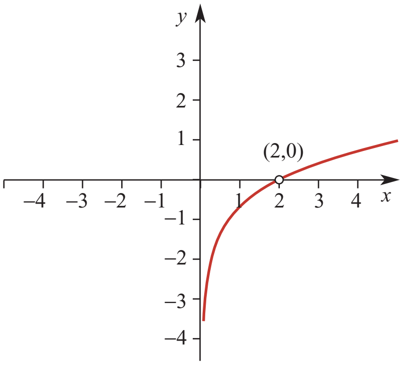
Domain: $x > 0$
Range: all real y

b



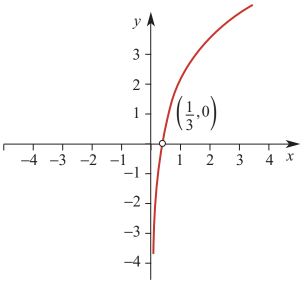
Domain: $x > 0$
Range: all real y

c



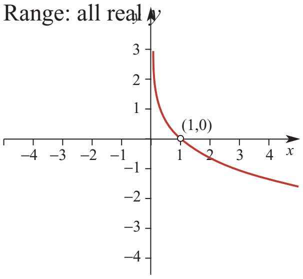
Domain: $x > 0$
Range: all real y

d



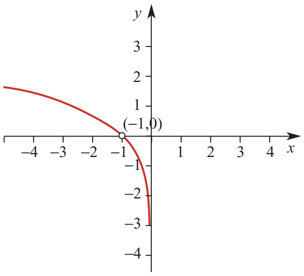
Domain: $x > 0$
Range: all real y

e



Domain: $x > 0$
Range: all real y

f



Domain: $x < 0$
Range: all real y

3 a $y = 2\ln x$ **b** $y = e^{\frac{x}{3}}$

c $y = \frac{1}{3}\ln x$ **d** $y = \frac{1}{3}e^{\frac{x}{2}}$

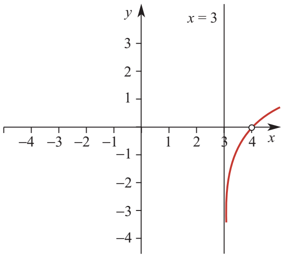
4 a $y = \ln(x-2)$ **b** $y = e^x + 3$

c $y = \ln\left(\frac{x-2}{4}\right)$ **d** $y = \ln(x+2)$

e $y = \frac{1}{3} \times e^x$ **f** $y = 3 \times e^x$

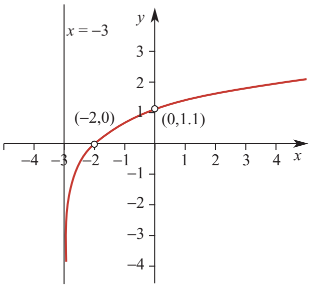
g $y = e^x - 3$ **h** $y = \ln\left(\frac{x+2}{5}\right)$

5 a



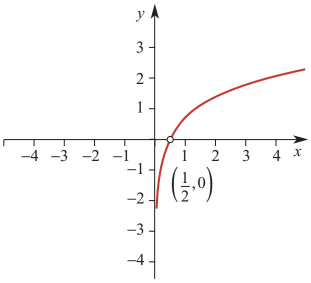
Domain: $x > 3$

b



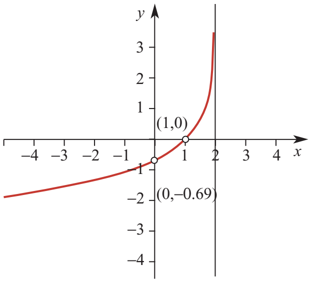
Domain: $x > -3$

c



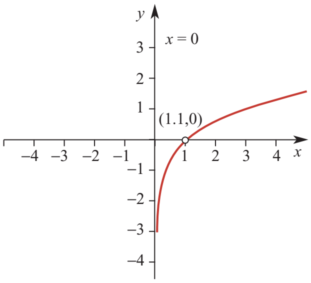
Domain: $x > 0$

d



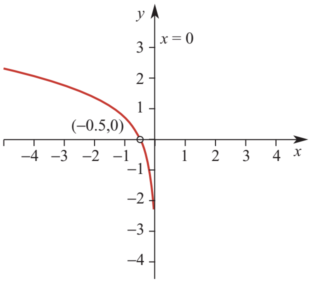
Domain: $x < 2$

e



Domain: $x > 0$

f



Domain: $x < 0$

6 Equivalent functions for $0 < x$. $\ln(x^2)$ includes $2\ln(x)$ *and* its y-axis reflection.

7 equivalent functions

8 Equivalent functions for $0 < x$. $\ln(6x^2)$ includes $\ln(2x) + \ln(3x)$ *and* its y-axis reflection.

9 $a = e,\ k = -1$

10 $a = -1,\ b = 2,\ c = 2$

Exercise 8C

1 A

2 $\frac{1}{3}\ln\left(\frac{287}{4}\right) \approx 1.42$

3 $a = \ln 5,\ b = 5,\ k = 2$

4 $a = 2,\ b = \frac{1}{3}\ln(5)$

5 $a = 250,\ b = \frac{1}{3}\ln(5)$

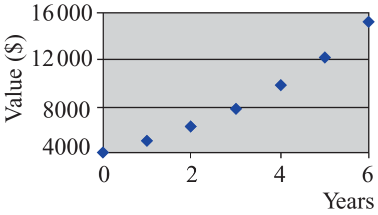
6 a $e^{\frac{y-5}{2}}$ **b** $-\frac{1}{6}\ \mathrm{In}\ \frac{P}{A}$

c $\dfrac{\ln\left(\frac{y}{a}\right)}{\ln(x)}$ **d** $\log_{10}\frac{y}{5}$

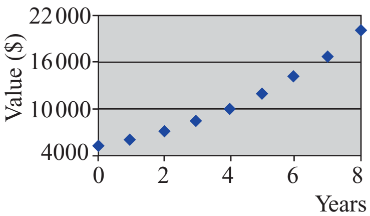
e $\frac{1}{2}e^{\frac{5-y}{3}}$ **f** $\frac{1}{2}\,\dfrac{\ln\left(\frac{y}{6}\right)}{\ln(x)}$

g $\frac{1}{2}(e^y + 1)$ **h** $-\ln\left(1 - \frac{y}{5}\right)$

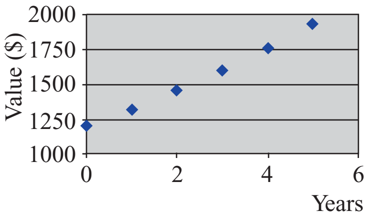
7 a



b



c

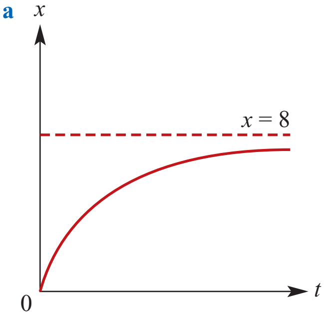


8 a \$11 485

b \$5010.58

c \$7042.69

9 a $k = 22\,497$, $\lambda = 0.22$
b \$11 523
10 a $A = 650\,000$, $p = 0.064$
b \$473 000
11 $m = 0.094$, $d_0 = 41.9237$
12 $k = 0.349$, $N_0 = 50.25$
13 a i $N_0 = 20\,000$ **ii** -0.223
b 6.22 years
14 a $M_0 = 10$, $k = 4.95 \times 10^{-3}$
b 7.07 g **c** 325 days
15 a 73.5366° C **b** 59.5946
16 a 75 **b** 2.37 **c** 0.646
17 a



b i 0 grams **ii** 2.64 grams
iii 6.92 grams
c 10.4 minutes

Exercise 8D

1 a 3, 6, 12, 24 **b** 3, −6, 12, −24
c 10 000, 1000, 100, 10
d 3, 9, 27, 81
2 a $\frac{5}{567}$ **b** $\frac{1}{256}$ **c** 32 **d** a^{x+5}
3 a $t_n = 3\left(\frac{2}{3}\right)^{n-1}$ **b** $t_n = 2(-2)^{n-1}$
c $t_n = 2(\sqrt{5})^{n-1}$
4 $\frac{2}{5}$ **5** t_9
6 a 6 **b** 9 **c** 9 **d** 6 **e** 8
7 a \$5397.31 **b** 48th year
8 a 21 870 m^2 **b** 10th day
9 47.46 cm
10 a 57.4 km **b** 14th day
11 \$5 369 000
12 a 24 **b** 12 288
13 a \$7092.60 **b** 12 years
14 $\frac{2}{3^5}$ **15** $16\sqrt{2}$
16 $t_{10} = 2048$ **17** $t_6 = 729$
18 \$3005.62 **19** 11.6% p.a. **20** 5 weeks
21 a 60 **b** 2.5 **c** 1 **d** x^4y^7

Exercise 8E

1 a 5115 **b** −182 **c** $-\frac{57}{64}$
2 a 1094 **b** −684 **c** 7812
3 a 1062.9 mL **b** 5692.23 mL
4 a 8 **b** $\{n : n > 19\}$
5 a \$18 232.59 **b** \$82 884.47
6 a 49 min (to nearest min)
b 164 min **c** Friday
7 $\frac{481\,835}{6561} = 73\frac{2882}{6561}$
8 Bianca's is worth \$3247.32, Andrew's is worth \$3000
9 a 155 **b** $\frac{15\sqrt{2}}{2} + 15$
10 $\frac{x^{2m+2}+1}{x^2+1}$

Exercise 8F

1 a $\frac{5}{4}$ **b** $\frac{3}{5}$
2 Perimeter of nth triangle $= \left(\frac{1}{2}\right)^{n-1} p$
Area of nth triangle $= \left(\frac{1}{4}\right)^{n-1} \frac{p^2\sqrt{3}}{36}$
Sum to infinity of perimeters $= 2p$
Sum to infinity of areas $= \frac{p^2\sqrt{3}}{27}$
3 $3333\frac{1}{3}$ m **4** yes
5 Yes, after ∞ hours, but problem becomes unrealistic after 4–5 h.
6 $S_\infty = 8$ **7** 50% **8** 12 m **9** 75 m
10 a $\frac{4}{9}$ **b** $\frac{1}{30}$ **c** $\frac{31}{3}$
d $\frac{7}{198}$ **e** 1 **f** $\frac{37}{9}$
11 $r = \frac{1}{2}$, $t_1 = 16$, $t_2 = 8$ or
$r = -\frac{1}{2}$, $t_1 = 48$, $t_2 = -24$
12 $\frac{5}{8}$ **13** $\frac{2}{3}$

Exercise 8G

1 a \$10 835 **b** \$11 305 **c** \$5827
2 a \$1000 **b** \$500 **c** \$647
3 a 7% **b** \$7.25% **c** 25%
4 a 15 **b** 20 **c** after 10 years
5 a 8% **b** 8.0060% **c** 0.075%
6 a 2.88 years, i.e. during the third year
b 3.11, i.e. during the fourth year
c > 77%

7 **a** 6581 **b** 14 292 **c** 4309 **d** 269 **e** 5.92% **f** 7.21% **g** 52 **h** 4 **i** 50 **j** 18

8 5.12%

9 7.19%

10 **a** 8.59% **b** 8.54%

11 10th

12 $\approx$ 9.57%

13 $\approx$ 17 years

Exercise 8H

1 **a** 9.42% **b** \$1568

2 **a** 6.72% **b** \$1745

3 **a** 6.18% **b** \$1036

4 **a** 8.25% **b** \$2500

5 **a** 1.040 81 **b** 4.081% **c** 4.00%

6 10.14%

7 $\approx$ 10 years

Exercise 8I

1 \$31 831 **2** \$100 **3** 3.6 years

4 $\approx$ \$1.13 **5** $\approx$ \$0.86 **6** 8.3%

7 \$3.26 **8** Class discussion

9 \$64.54

10 Much dearer, as today's cost should be \$158.71.

Exercise 8J

1 \$14 695

2 \$14 578

3 \$155

4 almost, she saves \$2428

5 \$3440

6 \$5420 quarterly, \$5409 weekly

7 5 years

8 **a** $>$ 7 years **b** 27% **c** about \$26 000

9 **a** \$8326 **b** \$1326

10 \$39.93

11 29.36 quarters, or after the second quarter of the 8th year

12 $>$ 7.52%

Exercise 8K

1 **a** \$3656, \$2877, \$2516, respectively

b \$46 223, \$72 608, \$101 286, respectively

c \$59 774, \$91 430, \$98 925, respectively

d **i** principal repaid \$641.66, interest repaid \$1874.40

ii principal repaid \$1945.01, interest repaid \$571

2 **a** \$101.34, \$53.30, \$24.68, respectively

b \$269.50, \$542.99, \$1416.80, respectively

c \$2564.92, \$384.45, \$4608.60, respectively

d **i** principal repaid \$92.26, interest repaid \$9.08

ii principal repaid \$43.73, interest repaid \$9.57

iii principal repaid \$14.83, interest repaid \$9.85

3 yes, repayments are \$2507.44 per month

4 \$274 953

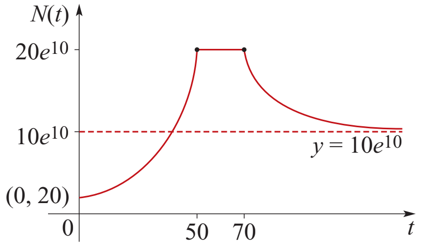
5 6%

6 8 years

Exercise 8L

1 0

2 **a**



b **i** $N(10) = 147.78$

ii $N(40) = 59\,619.16$

iii $N(60) = 20e^{10} = 440\,529.32$

iv $N(80) = 220\,274.66$

c **i** 25 days **ii** 35 days

3 $a = 1, k = 0$

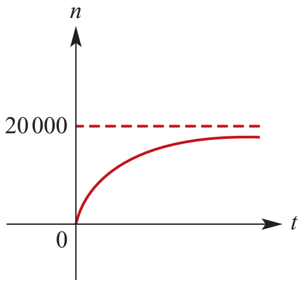
4 $y = \ln(2 - 2x)$

5 **a** **iii** $a = \frac{1}{2}$ or $a = 1$

iv If $a = 1, e^{-2B} = 1$ and $B = 0$, $A \in R^+$. If $a = \frac{1}{2}$, $B = \frac{1}{2}\log_e 2$.

v $A = 20\,000$

b

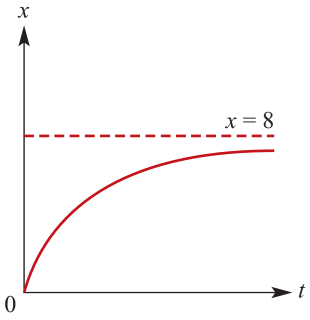


c $\dfrac{\log_e(0.1)}{\frac{1}{2}\log_e\left(\frac{1}{2}\right)} = \dfrac{2\log_e 10}{\log_e 2} \approx 6.644$

After 6.64 hours the population is 18 000.

6 $k = -0.5,\ A_0 = 100$

7 a



b i 0 grams **ii** 2.64 grams
iii 6.92 grams
c 10.4 min

8 a $k = 0.235$ **b** 22.7° C **c** 7.17 min

9 a 99.9999 mg **b** 100 mg

10 a $\frac{1}{729}$ m
b 1.499 m; no, the maximum height the water will reach is 1.5 m

11 a 27.49 **b** 1680.8

12 a $7\frac{1}{9}$ m **b** 405 m

13 $2^{64} - 1 \approx 1.845 \times 10^{19}$

14 Approximately, as $\sqrt{72} = 8.49$ and solving $(1 + 0.01x)^x = 2$ gives $x = 8.50$; alternatively, $\left(1 + 0.01\sqrt{72}\right)^{\sqrt{72}} = 1.996.$

15 $\approx 6.40\%$, $\approx 12.80\%$

16 11.16%

17 1.2%

18 3.8%

19 10%

20 15 years and 9 months

21 7.95%

Multiple-choice answers

1 C	**2** A	**3** D	**4** A	**5** E
6 B	**7** B	**8** C	**9** A	**10** D

Short-response answers

1 a $y = e^2x$ **b** $y = 10x$
c $y = 16x^3$ **d** $y = \frac{x^5}{10}$
e $y = \frac{e^3}{x}$ **f** $y = e^{2x-3}$

2 a $x = 2.183$ **b** $x = -0.322$
c $x = -2.709$

3 1.42

4 $a = \log_e 5$, $b = 5$, $k = 2$

5 $a = 2$, $b = 4$

6 $a = \frac{14}{e-1}$, $b = \frac{14}{1-e}$ $(a \approx 8.148, \quad b \approx -8.148)$

7 $b = 1$, $a = \frac{2}{\ln 2}$, $c = 8$ $(a \approx 2.885)$

8 $a = \frac{2}{\ln 2}$, $b = 4$

9 1

10 a $1410.60 **b** 21 years

11 $t_2 = 6$, $t_4 = \frac{8}{3}$ or $t_2 = -6$, $t_4 = -\frac{8}{3}$

12 96

13 199

14 D

15 a $r = \frac{-x}{x+1}$; $x \neq -1$
b i $S_\infty = \frac{4}{3}$ **ii** $S_\infty = 18$ **iii** $S_\infty = \frac{9}{20}$
c $x > \frac{-1}{2}$ and $x \neq 0$

16 $12 217.62

17 $620.92

18 7.4%

19 12 years

20 $8501.52

21 $3500

22 3 years

23 fortnightly

24 b is the better investment

25 a 8.33% **b** $1112.77

26 a 7.5% **b** $9997

27 $6.25

28 16

29 $1.13

30 €544

31 3.38%

32 $3875

33 c is the better investment

34 a $13 4849 **b** $49 849

35 a $3005.69, $1780.53, $1208.39, respectively
b $30 341.53, $63 663.18, $140 013.55, respectively
c $96 626.73, $128 244.25, $143 009.22, respectively
d i principal repaid $2987.02, interest paid $18.46
ii principal repaid $1217.55, interest repaid $562.98
iii principal repaid $391.24, interest repaid $817.15

Chapter 9

Exercise 9A

1 0.4 m/year

2 0.12 L/min

3 a 12 meals/h **b** 5 min **c** 50 min

4 a 0.4 **b** 2.5 **c** £200 **d** $1250

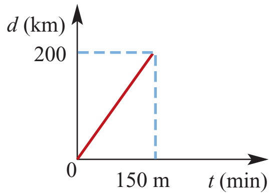
5 a 11.1 **b** 9000 km/tyre

6 a 0.125 **b** 8 **c** 9.375 tonne

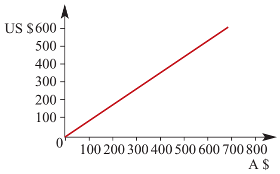
7 a 8.5 infections/100 individuals
b 102 individuals

Exercise 9B

1 $\frac{4}{3}$ km/min = 80 km/h



2



3 a 60 km/h **b** 3 m/s

c 400 m/min = 24 km/h = $6\frac{2}{3}$ m/s

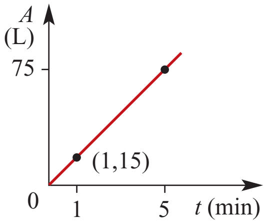
d 35.29 km/h **e** 20.44 m/s

4 a 8 litres/minute **b** 50 litres/minute

c $\frac{200}{17}$ litres/min **d** $\frac{135}{13}$ litres/min

5

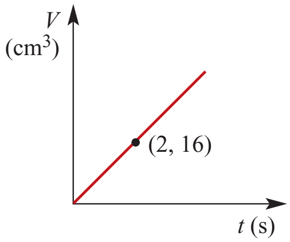
t	0	0.5	1	1.5	2	3	4	5
A	0	7.5	15	22.5	30	45	60	75



6 $\frac{\$200}{13}$ per hour = $15.38 per hour

7 $208\frac{1}{3}$ m/s

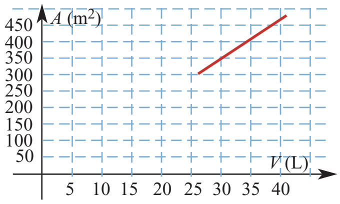
8



9 −3000

10 a 12

b



c no physical meaning

d $A = 12(V - 1)$

11 a 6 **b** 1.5 **c** −7 **d** 0.5 **e** 0.5

12 a 0 **b** undefined

13 a 0.155 **b** 8.88 kWh/day

Exercise 9C

1 a $\frac{-25}{7}$ **b** $\frac{-18}{7}$ **c** 4 **d** $\frac{4b}{3a}$

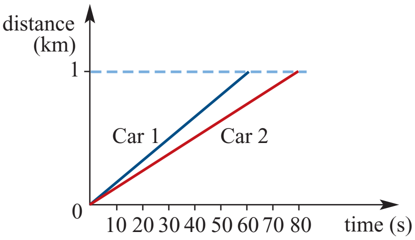
2 a 2 **b** 7 **c** $\frac{-1}{2}$ **d** $\frac{1-\sqrt{5}}{4}$

3 a 4 m/s **b** 32 m/s

4 a $2450.09 **b** $150.03 per year

5 3.125 cm/min **6** C

7

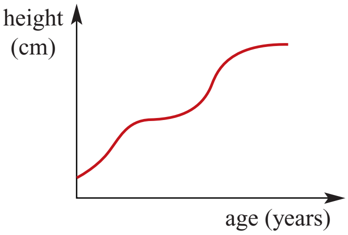


Exercise 9D

Note: For questions 1–4 there may not be a single correct answer.

1 *C* is the most likely. Scales should come into your discussion.

2



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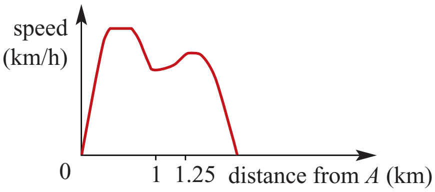
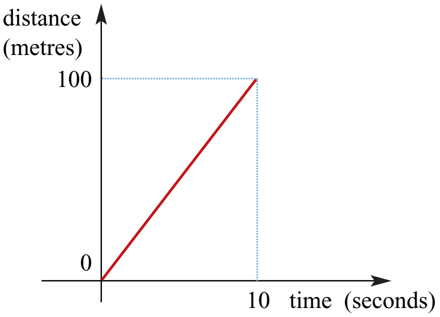
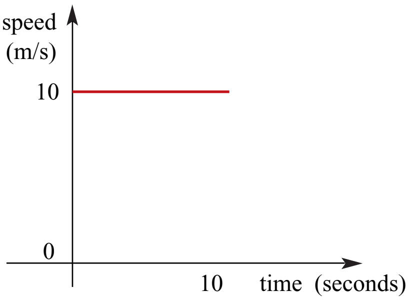


4 B, C or D are the most likely.

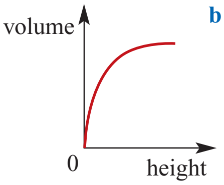
5 a



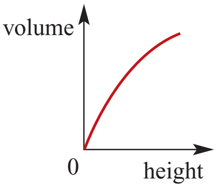
b



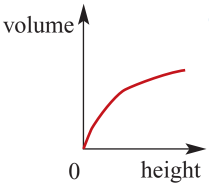
6 a



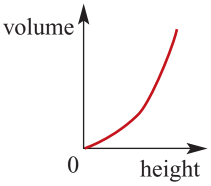
b



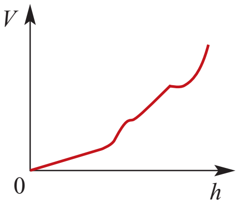
c



d



7



8 d **9** c

10 a $-7 \le x \le -4, \quad 0 < x \le 3$
b $-7 \le x \le -4, \quad 0 < x \le 3$

11 a $-5 \le x \le -3, \quad 0 < x \le 2$
b $-5 \le x \le -3, \quad 0 < x \le 2$

Exercise 9E

1 a $\frac{1}{3}$ kg/month (answers will vary)
b $\frac{1}{2}$ kg/month (answers will vary)
c $\frac{1}{5}$ kg/month (answers will vary)

2 a ≈ 0.004 m^3/s (answers will vary)
b ≈ 0.01 m^3/s (answers will vary)
c ≈ 0.003 m^3/s (answers will vary)

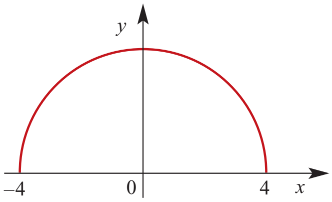
3 a $\frac{1}{80} = 0.0125$ litres/kg m
b $\frac{1}{60} \approx 0.0167$ litres/kg m

4 a $\approx$ 8 years **b** $\approx$ 7 cm/year

5 a 25° C at 1600 hours
b $\approx$ 3° C/h **c** -2.5° C/h

6 -0.5952

7



a 0 **b** -0.6 **c** -1.1

8 4

9 a 16 m^3/min **b** 10 m^3/min

10 a 18 million/min **b** 8.3 million/min

11 a 620 m^3/min flowing out
b 4440 m^3/min flowing out
c 284 000 m^3/min flowing out

12 7.19

13 a 7 **b** 9 **c** 2 **d** 35

14 28 **b** 12

15 a 10 **b** 4

16 a i $\frac{2}{\pi} \approx 0.637$ **ii** $\frac{2\sqrt{2}}{\pi} \approx 0.9003$
iii 0.959 **iv** 0.998
b 1

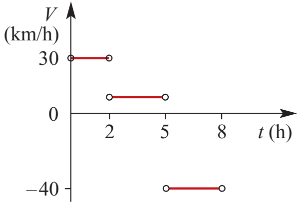
17 a i 9 **ii** 4.3246 **iii** 2.5893 **iv** 2.3293
b 2.30

Exercise 9F

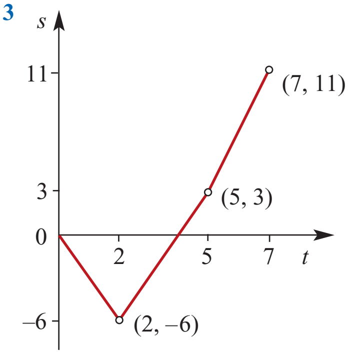
1 a 4 m/s **b** 1.12 m/s **c** 0 m/s
d $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3}\,)$ **e** $(-1, 1)$

2 a i 30 km/h **ii** $\frac{20}{3}$ km/h **iii** -40 km/h

c



3



4 a C **b** A **c** B

5 a +ve slowing down **b** +ve speeding up
c −ve slowing down **d** −ve speeding up

6 a gradually increasing speed
b constant speed (holds speed attained at **a**)
c final speeding up to finishing line

7 a $t = 6$ **b** 15 m/s **c** 17.5 m/s
d 20 m/s **e** −10 m/s **f** −20 m/s

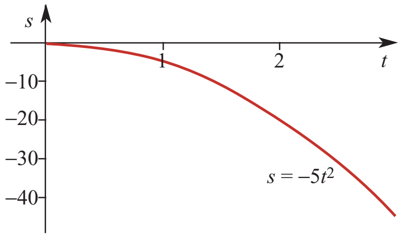
8 a $t = 2.5$ **b** $0 \leq t < 2.5$
c 6.25 m **d** 5 seconds **e** 3 m/s

9 a 11 m/s **b** 15 m **c** 1 s **d** 2.8 s **e** 15 m/s

10 a $t = 2, t = 3, t = 8$ **b** $0 < t < 2.5$ and $t > 6$
c $t = 2.5$ and $t = 6$

11 a aCi
b bAii
c cBiii

12



13 $t \geq 8$

Exercise 9G

1 a $5 + 3h$ **b** 5.3 **c** 5
2 a $2x + 2$ **b** 13 **c** $3x^2 + 4x$
3 a 1 **b** $3x^2 + 1$ **c** 20 **d** $30x^2 + 1$ **e** 5
4 1476 m/s **5** 7 per day
6 a $\frac{-1}{2+h}$ **b** −0.48 **c** $\frac{-1}{2}$
7 a $6 + h$ **b** 6.1 **c** 6

Exercise 9H

1 a $6x$ **b** 4 **c** 0 **d** $6x + 4$
e $6x^2$ **f** $8x - 5$ **g** $-2 + 2x$

2 a $2x + 4$ **b** 2 **c** $3x^2 - 1$
d $x - 3$ **e** $15x^2 + 6x$ **f** $-3x^2 + 4x$

3 a $12x^{11}$ **b** $21x^6$ **c** 5
d 5 **e** 0 **f** $10x - 3$
g $50x^4 + 12x^3$ **h** $8x^3 + x^2 - \frac{1}{2}x$

4 a i 3 **ii** $3a^2$ **b** $3x^2$

5 a −1 **b** 0 **c** $12x^2 - 3$ **d** $x^2 - 1$
e $2x + 3$ **f** $18x^2 - 8$ **g** $15x^2 + 3x$

6 a $\frac{dy}{dx} = 3(x-1)^2 \geq 0$ for all $x \in R$ and gradient of graph ≥ 0 for all x
b $\frac{dy}{dx} = 1$ for all $x \neq 0$ **c** $18x + 6$

7 a 1, gradient = 2 **b** 1, gradient = 1
c 3, gradient = −4 **d** −5, gradient = 4
e 28, gradient = −36 **f** 9, gradient = −24

8 a i $4x - 1, 3, \left(\frac{1}{2}, 0\right)$
ii $\frac{1}{2} + \frac{2}{3}x, \frac{7}{6}, \left(\frac{3}{4}, \frac{25}{16}\right)$
iii $3x^2 + 1, 4, (0, 0)$
iv $4x^3 - 31, -27, (2, -46)$
b coordinates of the point where gradient = 1

9 a $6t - 4$ **b** $-2x + 3x^2$ **c** $-4z - 4z^3$
d $6y - 3y^2$ **e** $6x^2 - 8x$ **f** $19.6t - 2$

10 a (4, 16) **b** (2, 8) and (−2, −8)
c (0, 0) **d** $\left(\frac{3}{2}, -\frac{5}{4}\right)$
e (2, −12) **f** $\left(-\frac{1}{3}, \frac{4}{27}\right)$, (1, 0)

Exercise 9I

1 a $\frac{1}{3}x^{-\frac{2}{3}}$ **b** $\frac{3}{2}x^{\frac{1}{2}}; x > 0$
c $\frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}}; x > 0$ **d** $x^{-\frac{1}{2}} - 5x^{\frac{2}{3}}$
e $-\frac{5}{6}x^{-\frac{11}{6}}$ **f** $-\frac{1}{2}x^{-\frac{3}{2}}; x > 0$

2 a i $\frac{4}{3}$ **ii** $\frac{4}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{3}$

3 a $\{x : 0 < x < 1\}$ **b** $\left\{x : x > \left(\frac{2}{3}\right)^6\right\}$

4 a $-5x^{-\frac{1}{2}}\left(2 - 5\sqrt{x}\right)$ **b** $3x^{-\frac{1}{2}}\left(3\sqrt{x} + 2\right)$
c $-4x^{-3} - \frac{3}{2}x^{-\frac{5}{2}}$ **d** $\frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$
e $\frac{15}{2}x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

5 $x = -1$

6 $x = -\frac{1}{4}$

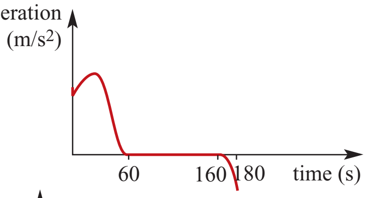
Exercise 9J

1 a yes, the relation is linear
b 0.05 ohm/° C

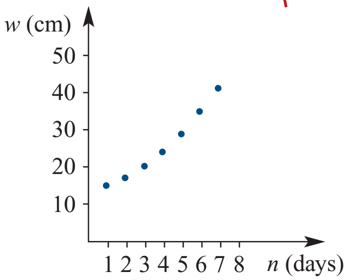
2 a **i** 9.8 m/s **ii** 29.4 m/s
b **i** $4.9(8h - h^2)$ **ii** $4.9(8 - h)$ **iii** 39.2 m/s

3 a **i** $\frac{1}{4}$ m/s^2 **ii** 0.35 m/s^2

b

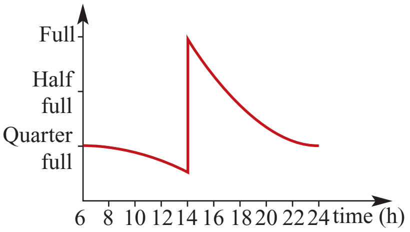


4 a



b gradient $= 5\frac{1}{4}$; Average rate of growth of the watermelon is $5\frac{1}{4}$ cm/day
c $\approx$ 4.5 cm/day

5



6 a $b + a\ (a \neq b)$ **b** 3 **c** 4.01

7 a $2\frac{2}{3}, 1\frac{3}{5}$; gradient $= -1\frac{1}{15}$
b 2.1053, 1.9048; gradient $= -1.003$
c $-1.000\,025$ **d** $-1.000\,000\,3$ **e** gradient is -1

8 6

9 a $\approx 3\frac{1}{3}$ kg/year **b** $\approx$ 4.4 kg/year
c $\{t : 0 < t < 5\} \cup \{t : 10 < t < 12\}$
d $\{t : 5 < t < 7\} \cup \{t : 11 < t < 17\frac{1}{2}\}$

10 a **i** 2.5×10^8 **ii** 5.3×10^8
b 0.007 billion/year
c **i** 0.004 billion/year **ii** 0.015 billion/year
d 25 years after 2020

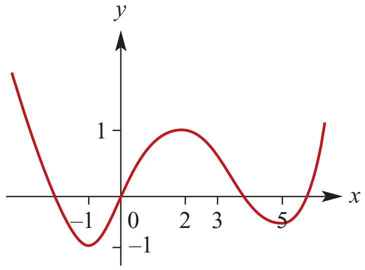
11 a **i** 1049.1 **ii** 1164.3 **iii** 1297.7 **iv** 1372.4
b Check with your teacher
c Check with your teacher

12 a $a^2 + ab + b^2$ **b** 7 **c** 12.06 **d** $3b^2$

13 a B **b** A **c** 35 m **d** 45 s
e 0.95 m/s, 1.7 m/s, 1.1 m/s

14 a m **b** cm **c** $-c$ **d** results are the same

15



16 $y = \frac{7}{36}x^3 + \frac{1}{36}x^2 - \frac{20}{9}x$

17 a **i** 71°34′ **ii** 89°35′ **b** 2 km

18 a 0.12, -0.15
b $x = 2, y = 2.16$. The height of the pass is 2.16 km.

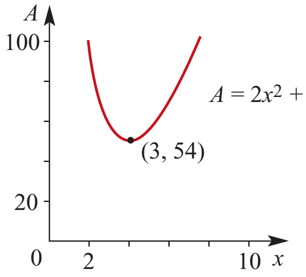
19 a At $x = 0$, gradient $= -2$; at $x = 2$, gradient $= 2$
Angles of inclination to positive directions of x-axis are supplementary.

20 a $h = \frac{400}{\pi r^2}$ **c** $\frac{dA}{dr} = 4\pi r - \frac{800}{r^2}$
d $r = \left(\frac{200}{\pi}\right)^{\frac{1}{3}} \approx 3.99$

21 a $y = \frac{16}{x}$ **c** $x = 2, P = 20$

22 a $A(4, 0), B(0, -2)$ **b** $\frac{1}{25x}$
c **i** $\frac{1}{2}$ **ii** $y = \frac{1}{2}x - \frac{3}{2}$
d $x > \frac{1}{4}$

23 a $h = \frac{18}{x^2}$ **c** $x = 3, h = 2$
d



24 a $y = \frac{250}{x^2}$
c $\frac{dS}{dx} = 24x - \frac{3000}{x^2}$ **d** $x = 5$

25 a 36; $\frac{36}{1} = 36$ **b** $48 - 12h$ **c** 48

26 a $1200t - 200t^2$ **b** 1800 dollars/month
c At $t = 0$ and $t = 6$

27 a -3 cm/s **b** $2\sqrt{3}$ s

28 a $15 - 9.8t$ m/s **b** -9.8 m/s^2

29 a $30 - 4P$ **b** 10; -10

c For $P < 7.5$ revenue is increasing as P increases.

30 a 50 people/year **b** 0 people/year

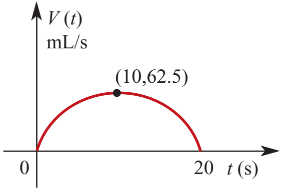
c decreasing by 50 people/year

31 a i 0 mL

ii $833\frac{1}{3}$ mL

b $V'(t) = \frac{5}{8}\left(20t - t^2\right)$

c



32 a 64 m/s **b** 32 m/s **c** 0 m/s

33 a 0 s, 1 s, 2 s

b 2 m/s, -1 m/s, 2 m/s; -6 m/s^2, 0 m/s^2, 6 m/s^2

c 0 m/s^2

34 a $\frac{1}{2}$ m/s **b** $2\frac{1}{2}$ m/s

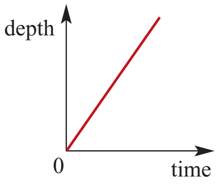
35 a 100 **b** $100x$ **c** undefined

Multiple-choice answers

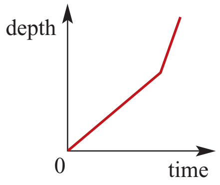
1 C **2** B **3** D **4** E **5** D
6 B **7** C **8** E **9** A **10** A
11 D **12** B **13** E **14** B **15** C
16 C **17** A **18** E **19** A **20** D
21 D

Short-response answers

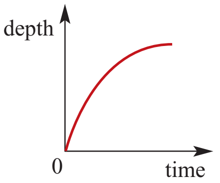
1 a



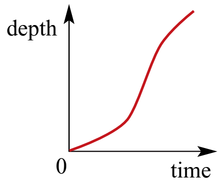
b



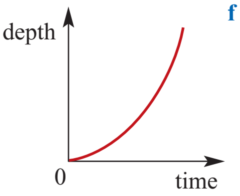
c



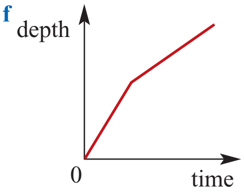
d



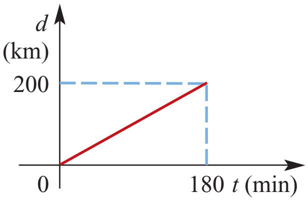
e



f

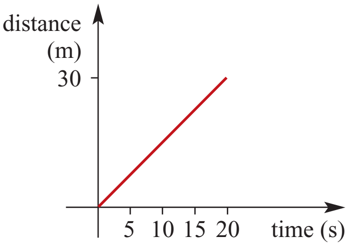


2 a

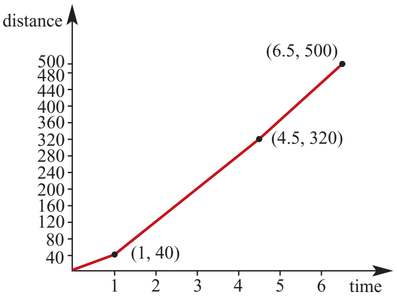


$$\text{constant speed} = \frac{200}{3}\text{ km/h}$$
$$= \frac{200}{180}\text{ km/min}$$
$$= \frac{10}{9}\text{ km/min}$$

b



c



3 36 cm^2/cm

4 a i -0.05 **ii** -0.005

b 0

5 a -2 m/s **b** -12.26 m/s **c** -14 m/s

6 a $6x - 2$ **b** 0 **c** $4 - 4x$

d $4(20x - 1)$ **e** $6x + 1$ **f** $-6x - 1$

7 a -1 **b** 0 **c** $\frac{4x + 7}{4}$

d $\frac{4x - 1}{3}$ **e** x

8 a 1; 2 **b** 3; -4 **c** -5; 4 **d** 28; -36

9 a $\left(\frac{3}{2}, -\frac{5}{4}\right)$ **b** $(2, -12)$

c $\left(-\frac{1}{3}, \frac{4}{27}\right)$ $(1, 0)$ **d** $(-1, 8)(1, 6)$

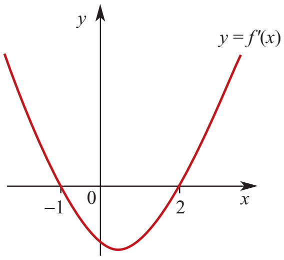
e $(0, 1)$ $\left(\frac{3}{2}, -\frac{11}{16}\right)$ **f** $(3, 0)(1, 4)$

10 a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x > \frac{1}{2}$

d $x < \frac{1}{2}$ **e** $x \neq \frac{1}{2}$ **f** $x = \frac{5}{8}$

11 a $a = 2, b = -1$ **b** $\left(\frac{1}{4}, -\frac{1}{8}\right)$

12



13 a $-1 < x < 4$

b $-\infty < x < -1, 3 < x < \infty$

c $-1, 4$

14 a $-4x^{-5}$ **b** $-6x^{-4}$ **c** $\frac{2}{3x^3}$

d $\frac{4}{x^5}$ **e** $\frac{-15}{x^6}$ **f** $\frac{-2}{x^3} - \frac{1}{x^2}$

g $\frac{-2}{x^2}$ **h** $10x + \frac{2}{x^2}$

15 a $\frac{1}{2x^{\frac{1}{2}}}$ **b** $\frac{1}{3x^{\frac{2}{3}}}$ **c** $\frac{2}{3x^{\frac{4}{3}}}$

d $\frac{4}{3}x^{\frac{1}{3}}$ **e** $-\frac{1}{3x^{\frac{4}{3}}}$ **f** $-\frac{1}{3x^{\frac{4}{3}}} + \frac{6}{5x^{\frac{2}{5}}}$

16 a $8x + 12$ **b** $24(3x + 4)^3$

17 a $\frac{1}{6}$ **b** $-\frac{1}{16}$ **c** -2

18 $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$

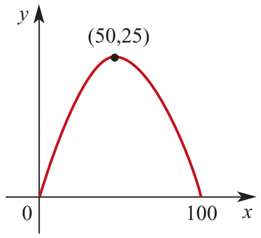
19 $\left(\frac{1}{16}, \frac{1}{4}\right)$

20 a 100

b $\frac{dy}{dx} = 1 - 0.02x$

c $x = 50, y = 25$

d



e (25, 18.75) (75, 18.75)

21 a i $r = \sqrt{1 - x^2}$

ii $h = 1 + x$

b $V = \frac{\pi}{3}\left(1 + x - x^2 - x^3\right)$

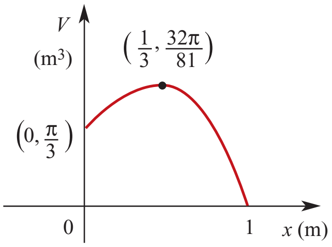
c $0 < x < 1$

d i $\frac{dV}{dx} = \frac{\pi}{3}\left(1 - 2x - 3x^2\right)$

ii $\left\{\frac{1}{3}\right\}$

iii $\frac{32\pi}{81}$ m^3

e



22 a 1000 insects

b 1366 insects

c i $t = 40$

ii $t = 51.70$

d 63.64

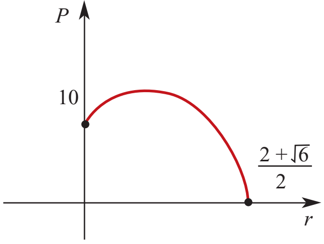
e i $\frac{1000 \times 2^{\frac{3}{4}}(2^{\frac{h}{20}} - 1)}{h}$

ii consider h decreasing and approaching zero; instantaneous rate of change ≈ 58.286 insects/day

23 a 10 000 people/km^2

b $0 < r < \frac{2 + \sqrt{6}}{2}$

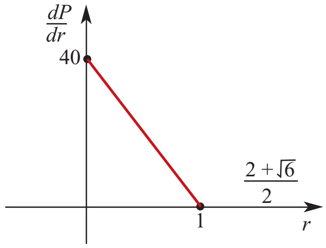
c



d i $40 - 40r$

ii 20, 0, -40

iii



e $r = 1$

24 a $t = \sqrt[3]{250}$, 11.9 cm/s

b 3.97 cm/s

25 a $OA = \frac{120}{x}$

b $OX = \frac{120}{x} + 7$

c $OZ = x + 5$

d $A = 7x + \frac{600}{x} + 155$

e $x = 9.26$ cm